Smits 5.1 & 5.2

Lecture #1

user name: me431/599
password: fluid$

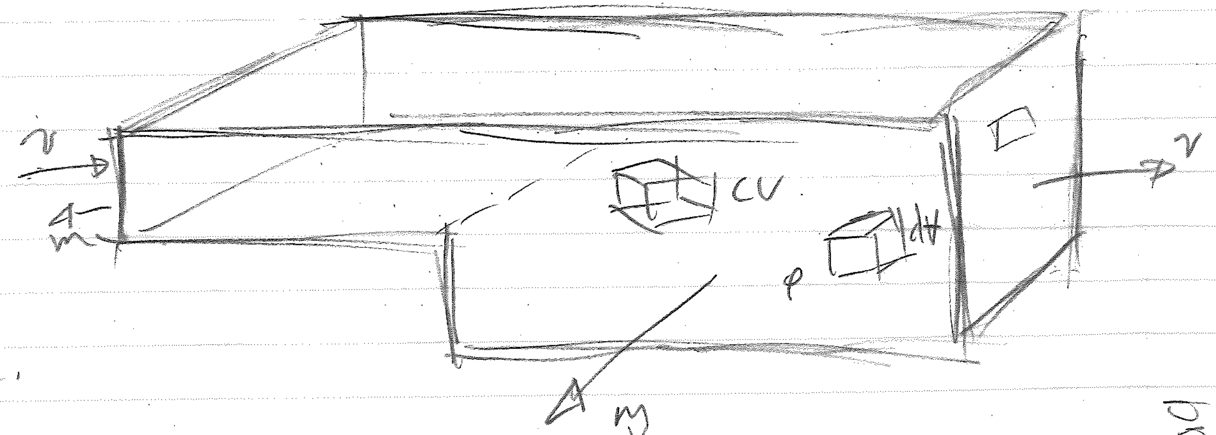
want to solve equations to get the flow field.

I/E ~~Equation~~ of motion

solve
1) conservation of mass
2) momentum balance
3) conservation of energy
4) fluid/gas law.

1) Conservation of mass

Consider a control volume, fixed volume in space through which fluid flows.



small v = velocity
big V = volume.

[rate of increase of mass in cv] = [net rate at which mass enters / leaves]

valid for any control volume

$$\Rightarrow \frac{d}{dt}\iiint \rho \, dV = -\iint \rho \, v \cdot n \, dA$$

minus negative to indicate outward normal

$d\forall$: differential volume
$dx\,dy\,dz$ or
$dr\,d\theta\,dz$

$dA = dx\,dy$

$\rho = \rho(\underbrace{x, y, z}_{\underset{\sim}{x}}, t) = \rho(\underset{\sim}{x}, t)$ fluid density

↳ can write with position vector

fluid velocity $\underset{\sim}{v} = [u(\underset{\sim}{x}, t), v(\underset{\sim}{x}, t), w(\underset{\sim}{x}, t)]$

$\underset{\sim}{n}$: unit normal vector.

⇒ Consider small differential volume element:

CV fixed in space ⇒ does not depend on time.

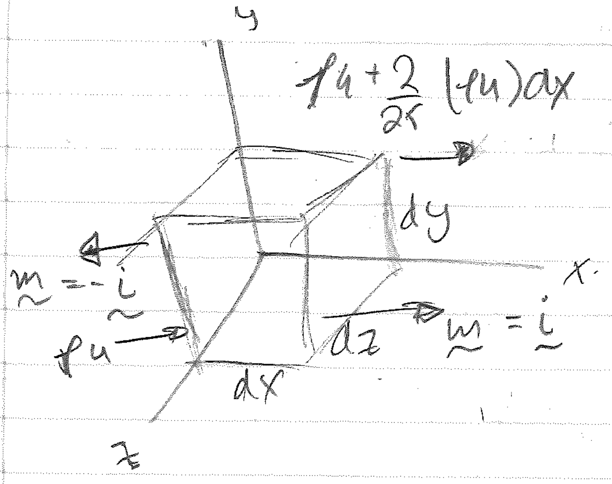
↳ $$\frac{d}{dt}\iiint_{CV} \rho(\underset{\sim}{x}, t)\,dv = \iiint_{CV} \frac{\partial \rho}{\partial t}(x, t)\,d\forall$$

(Le theorm)

$\frac{\partial \rho}{\partial t} = \frac{\partial \rho}{\partial t}\Big|_{\underset{\sim}{x}\ \text{fix}}$ time dependent
t point $\sim$ fixed.

CV is very small ⇒ $\frac{\partial \rho}{\partial t}$ is constant.

$$\iiint \frac{\partial \rho}{\partial t}\,d\forall = \frac{\partial \rho}{\partial t}\Big|_{xyz} dx\,dy\,dz$$



the net mass flux has contribution from all 6 faces.

Surfaces moved along the x-direction

left control surface.

$$-\iint_{cs} \rho(\underset{\sim}{v}\cdot\underset{\sim}{m})\,dA = +\rho u\Big|_x \,dz\,dy$$

R control surface

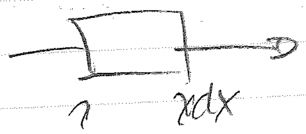
$$-\iint \rho(\underset{\sim}{v}\cdot\underset{\sim}{m})\,dA = -\rho u\Big|_{x+dx}\,dz\,dy.$$

↳ complete Taylor series expansion

$$(\rho u)\Big|_{x+dx} = (\rho u)\Big|_x + \frac{\partial}{\partial x}(\rho u)_x dx + HOT$$

The net contribution from L/R surfaces.

$$-\frac{\partial}{\partial x}(\rho u)\,dx\,dy\,dz + HOT$$

e.g. 

$$\frac{\partial}{\partial x}(\rho u) > 0$$

$$(\rho u)_{x+dx} > (\rho u)_x$$

then

$$\frac{d}{dt}(\ldots) = -\partial$$

same argument for front + back faces.

$$-\frac{\partial}{\partial y}(\rho u)\,dz\,dy\,dt$$

top + bottom

$$-\frac{\partial}{\partial y}(\rho v)\,dx\,dy\,dt.$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0$$

Differential form of the conservation of mass

≡ continuity equation

↳ for polar continuity see moodle pdf

introduce del operator $\nabla \equiv$

$\underset{\sim}{i}\frac{\partial}{\partial x} + \underset{\sim}{j}\frac{\partial}{\partial y} + \underset{\sim}{k}\frac{\partial}{\partial z}$

$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \underset{\sim}{v} = 0$

makes eq more compact.

Material derivative: $\frac{D}{Dt} = \frac{\partial}{\partial t} + \underset{\sim}{v} \cdot \nabla$

inner product?

$\nabla \cdot (\rho \underset{\sim}{v}) = (\underset{\sim}{v} \nabla)\rho + \rho(\nabla \cdot \underset{\sim}{v})$

show that $\boxed{\frac{D\rho}{Dt} + \rho(\nabla \cdot v) = 0}$

---

consider incompressible flow $\rightarrow$ flow for which ρ = constant independent of x and time

$\Rightarrow \nabla \cdot \underset{\sim}{v} = 0$ or $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$

Mach number $M = \frac{v}{a} < 0.2$

v = characteristic velocity
a = velocity of sound

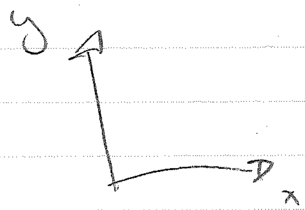
→ e.g. air $a = 356$ m/s $= 760$ mph

$M = 0.13$.

hurricane 150 mph $M = 0.2$ marginal

→ e.g. 2D incompressible flow

neglect z variation $\left(\frac{\partial}{\partial z} = 0\right)$

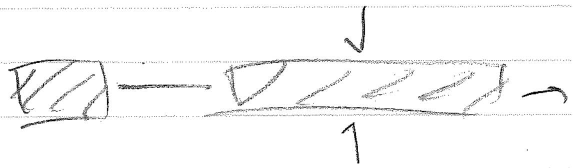


$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \Rightarrow \frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y}$$ [illegible]

if $\frac{\partial u}{\partial x} > 0$ locally then $\frac{\partial v}{\partial y} < 0$

to conserve mass

$\rightarrow \; \rightarrow \quad \frac{\partial u}{\partial x} > 0$



→ e.g. $u = \alpha x^2$ ($\alpha > 0$, constant)

$$\frac{\partial u}{\partial x} = 2\alpha x = -\frac{\partial v}{\partial y} \Rightarrow v = -2\alpha xy + f(x)$$

$$\frac{\partial \rho}{\partial t} + \rho \nabla \cdot \underset{\sim}{v} = 0 \qquad \rho(x, y, z, t)$$

$$\nabla \cdot \underset{\sim}{v} = 0$$

For 2D flow (x, y) (r, θ) Smits 6.3

also useful for stream lines

Incompressible

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

introducing stream function $\psi(x, y)$

definition → $u = \frac{\partial \psi(x,y)}{\partial y}$ $\quad v = \frac{-\partial \psi}{\partial x}$ ← definition

take continuity equation

$$\frac{\partial u}{\partial x}\left(\frac{\partial \psi}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial \psi}{\partial x}\right)$$

$$\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial y \partial x} = 0$$

as long as you are solving string don't have to check that continuity holds if ψ exists:

1) Solve 1 unknown ψ instead of 2 (u, v)

2) Garuntee the conservation of mass.

Streamline: line which is tangent to the velocity vector.

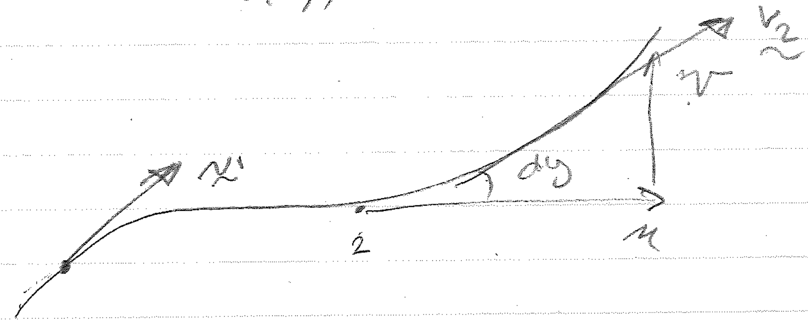
Streamline is at a fixed time (t).

line // to the local velocity

$$\frac{dy}{dx} = \frac{v(x,y)}{u(x,y)}$$

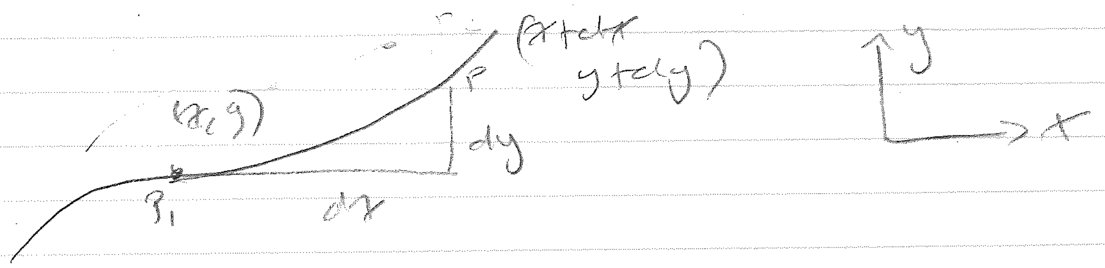
Smits 4.1.1

(chapter 4.1)



Consider 2D, incompressible such as ψ exist.

- 2 points close together along a stream line.



Taylor expansion

The value @ P_2 $\psi(x+dx, y+dy) =$

$$\psi(x,y) + \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy$$

$$= \psi(x,y) - v\,dx + u\,dy$$

the streamline is such that

$$\frac{dy}{dx} = \frac{v}{u} \Rightarrow u\,dy = v\,dx$$

$\psi(x+dx, y+dy) = \psi(x,y)$ then

$\psi = \text{const}$

(If we can find this then we can just plot constant values of the streamline.)

if $\psi(x,y)$ const, $\psi = \text{const} \equiv$ streamline

↳ in matlab could do countour plot

The velocity can change magnitude the stream function doesn't change.

↳ this is only about the direction of the field NOT the magnitude e.g. $u = -\alpha x = \frac{\partial \psi}{\partial y}$

$v = +\alpha x = -\frac{\partial \psi}{\partial x}$

→ integrate u over y

$\psi(x,y) = \int u\,dy = \int -\alpha x\,dy$

$= -\alpha x y + f(x)$

$\frac{\partial(-\alpha x y + f(x))}{\partial y} = -\alpha x + 0$

$\psi(x,y) = -\int v\,dx$

$= -\int \alpha y\,dx$

$= -\alpha x y + g(y)$

↓ put them together.

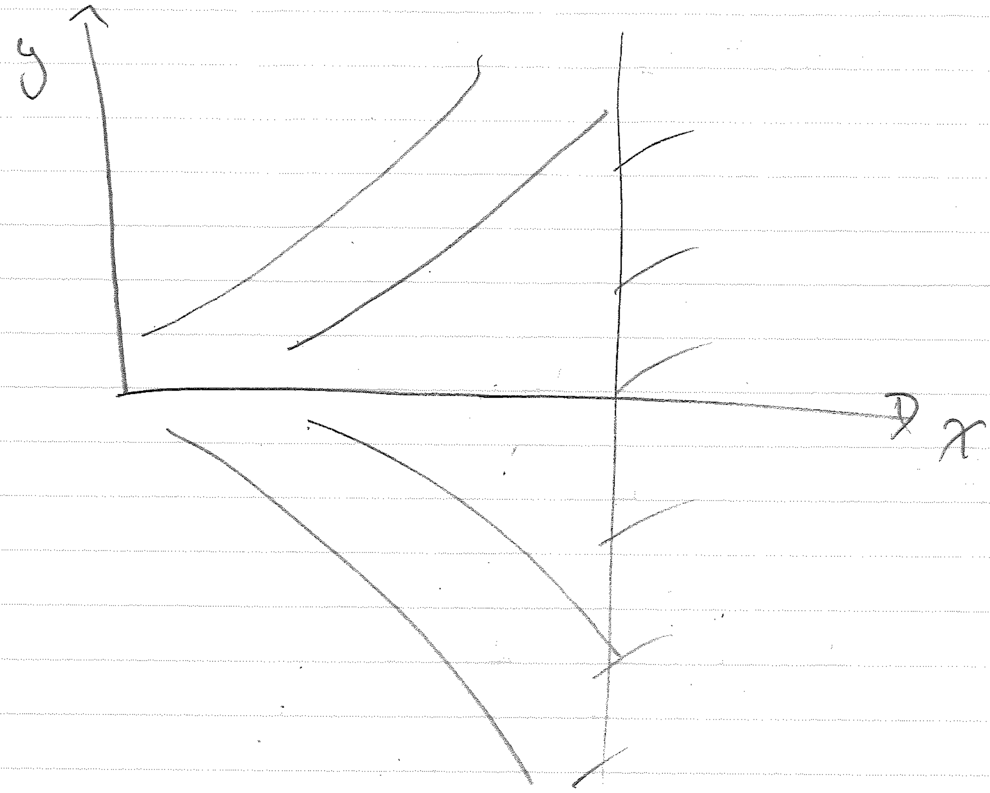
$f(x) = g(y) = c_1$

Streamline $\psi = \text{const} = c_2$

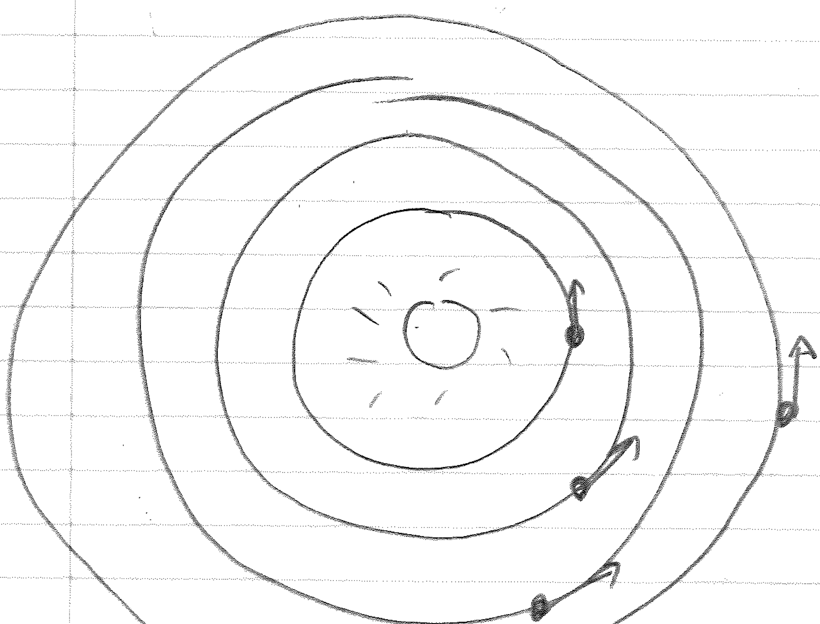
$-\alpha x y + c_1 = c_2$

$\alpha x y = c_1 - c_2 = c$ → $\boxed{y = \frac{c}{\alpha}\frac{1}{x}}$

$y(x)$



recirculation



[Streamlines don't cross]

"An Album of Fluid Motion"
↳ Van Dyke

We will use Star-CCM+
↳ download.

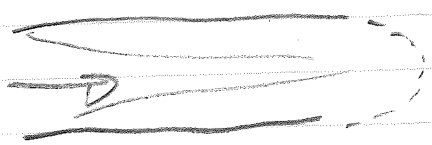
user: me431538@gmail.com
pass: Combu$tion1

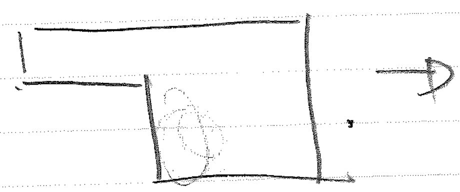
Outline of computation portion

- 5 guided projects
- Final project
  - 2-D
  - steady
  - laminar

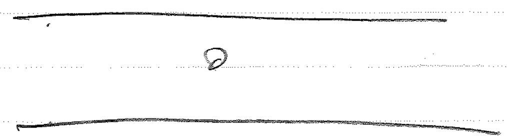
  } this will make your life better.

* ① Flow through a pipe



② 

③ flow past a circular cylinder



→ various iterations each with increasing complexity

Perfect's colleague:

Anyone who tells you that CFD software can accurately tackle any problem in a user friendly way is a liar or an idiot.

How to ask for help

1. what causes the bug?
2. What type of bug is it?
3. What you have tried?
4. what problems you have ruled out

Licensing

→ Shared access key

uUq219tXihiSnl/Jy5ydWA

↳ class code need to run + open

Cant access?

1. check internet need this to work
2. make sure the key is right
3. check the remote desktop.
4. Ask a friend.
5. Restart! STAR / computer.

---

off campus? → Husky OnNet

- Star CCM tutorials
- Discussing past simulations
- Bonus lecture
- Practice problems
- mini lecture to clarify

Be able to open Star CCM & complete basic assignment.

Smits 5.3

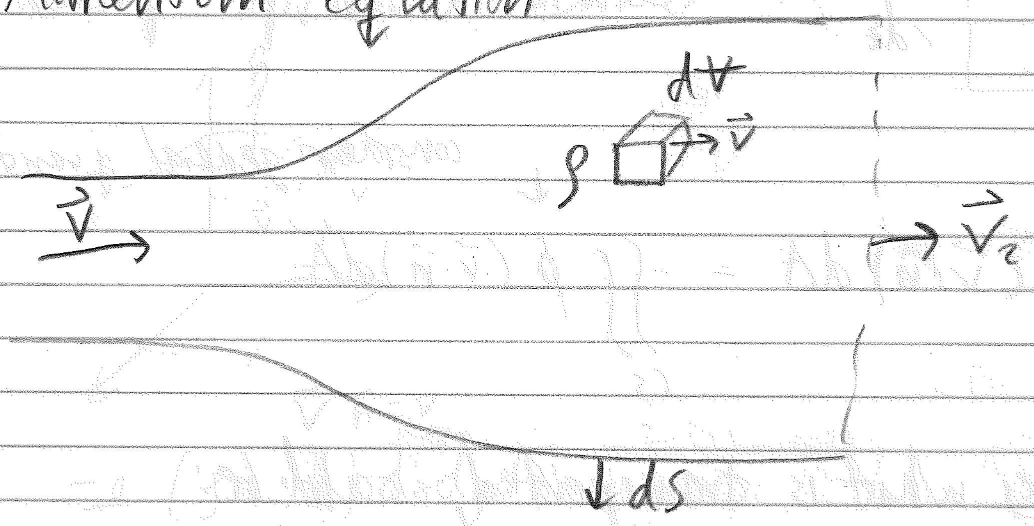
lecture 3

Oct 1, 2018

Review: $u = \frac{\partial \Psi}{\partial y}$ $v = -\frac{\partial \Psi}{\partial x}$

$\frac{dy}{dx} = \frac{v(x,y)}{u(x,y)}$ ⇒ derivative of the streamline, which is tangent the velocity field

Momentum equation



Newton's 2nd law applied to fluids

$\begin{bmatrix}\text{rate of change of} \\ \text{momentum in a CV}\end{bmatrix} = \begin{bmatrix}\text{sum of the} \\ \text{forces applied to CV}\end{bmatrix} + \begin{bmatrix}\text{net rate of} \\ \text{momentum in}\end{bmatrix}$ control surface

$$\frac{d}{dt}\iiint_{CV} \rho \vec{v}\, d\forall = \sum_i F_i - \iint_{CS} \rho \vec{v} \underbrace{(\vec{v}\cdot\hat{n})}_{\text{scalar}} dA$$ 3 eqn $(\hat{i}\hat{j}\hat{k})$

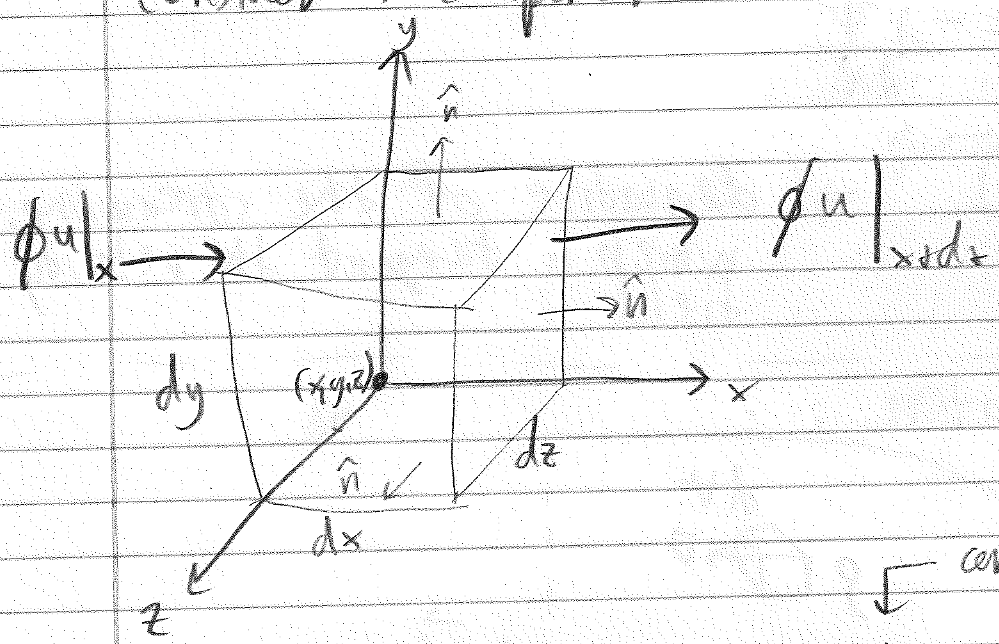
Evaluate for infinitesimal CV

$$\frac{d}{dt}\iiint \rho \vec{v}\, d\forall = \iiint \frac{d}{dt} \rho \vec{v}\, d\forall = \frac{\partial}{\partial t} \rho \vec{v}\, dx\,dy\,dz$$

CV indep. of time

$\frac{\partial(\rho\vec{v})}{\partial t}$ = constant in CV

Consider x-component:



consider general quantity ϕ

$$-\iint_{CS} \rho u (\vec{v}\cdot\hat{n})\, dA = -\iint_{CS} \phi (\vec{v}\cdot\hat{n})\, dA$$

ϕ describes what is transported, could be:

$\phi = \rho$ mass
$\phi = \rho u$ momentum
$\phi = \rho e$ energy where e = internal energy/vol.

at x: $-\phi(\vec{v}\cdot\hat{n}) = -\phi(-u) = \phi u$

at x+dx: $-\phi(\vec{v}\cdot\hat{n}) = -\phi(u) = -\phi u$

If $\phi u|_{x+dx} = \phi u|_x \Rightarrow$ no change w/in C.V.

Net effect: left/right $-\iint_{CS} \phi(\vec{v}\cdot\hat{n})\, dA = \phi u|_x\, dy dz - \phi u|_{x+dx}\, dy dz$

$f(x) = f(a) + \frac{df}{dx}(a)\delta x$

1st term: $\frac{df}{dx} = \frac{f(a+\delta x) - f(a)}{\delta x}$

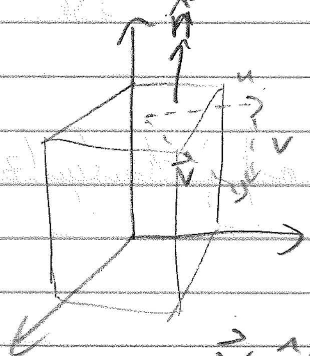
Taylor expansion

$= -\frac{\partial(\phi u)}{\partial x}\,dx\,dy\,dz$

x-comp. momentum $\phi = \rho u$

$-\frac{\partial}{\partial x}(\rho u \cdot u)\,dx\,dy\,dz$

x-comp top/bottom surfaces:



$\vec{V}\cdot\hat{n} = v$

$-\frac{\partial}{\partial y}(\rho u \cdot v)\,dx\,dy\,dz$

x-comp back + front surfaces:

$-\frac{\partial}{\partial z}(\rho u \cdot w)\,dx\,dy\,dz$

Total for 6 surfaces:

$-\left[\frac{\partial}{\partial x}(\rho u \cdot u) + \frac{\partial}{\partial y}(\rho u \cdot v) + \frac{\partial}{\partial z}(\rho u \cdot w)\right]dx\,dy\,dz$

y-momentum: $\phi = \rho v$

y comp, x surfaces:

$$-\frac{\partial}{\partial x}\left(\rho v \cdot u\right) dx\,dy\,dz$$

$$\Rightarrow -\left[\frac{\partial}{\partial x}\left(\rho v u\right) + \frac{\partial}{\partial y}\left(\rho v \cdot v\right) + \frac{\partial}{\partial z}\left(\rho v \cdot w\right)\right] dx\,dy\,dz$$

z-momentum: $\phi = \rho w$

$$\Rightarrow -\left[\frac{\partial}{\partial x}\left(\rho w \cdot u\right) + \frac{\partial}{\partial y}\left(\rho w \cdot v\right) + \frac{\partial}{\partial z}\left(\rho w \cdot w\right)\right] dx\,dy\,dz$$

In vector form, the final result is:

$$-\left[\frac{\partial}{\partial x}\left(\rho \vec{V} \cdot u\right) + \frac{\partial}{\partial y}\left(\rho \vec{V} \cdot v\right) + \frac{\partial}{\partial z}\left(\rho \vec{V} \cdot w\right)\right] dx\,dy\,dz$$

Moving this term to the LHS of mom. balance:

$$\left[\frac{\partial}{\partial t}\left(\rho \vec{V}\right) + \frac{\partial}{\partial x}\left(\rho \vec{V} \cdot u\right) + \frac{\partial}{\partial y}\left(\rho \vec{V} \cdot v\right) + \frac{\partial}{\partial z}\left(\rho \vec{V} \cdot w\right)\right] d\forall$$

chain rule

$$= \vec{V}\left[\cancel{\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w)}\right]^{\to 0} d\forall$$ ← conservation of mass

$$+ \rho\left[\frac{\partial \vec{V}}{\partial t} + u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}\right] d\forall$$ ← material derivative

$$= \rho \frac{D\vec{V}}{Dt}\, d\forall \quad \text{where} \quad \frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$$

Note! $\frac{\partial}{\partial t} = \frac{\partial}{\partial t}\Big|_{\vec{x}\text{ fixed}}$

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial x}\Big|_{y,z,t,\text{ fixed}}$$

Momentum balance:

$$\rho \frac{D\vec{v}}{Dt}\, d\forall = \sum_i F_i$$

Lecture 4

Oct 3, 2018

$$m\vec{a} = \sum F_i$$

$$\rho \frac{D\vec{V}}{Dt} d\forall = \sum_i F_i$$

1) Body forces → proportional to mass

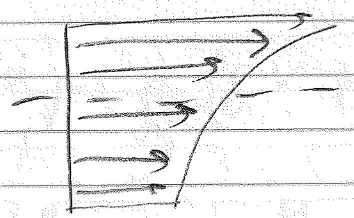
e.g. gravity, magnetic / electric

only gravity $\vec{F}_B = m\vec{g}$ $\vec{g} = (0, 0, -g)$

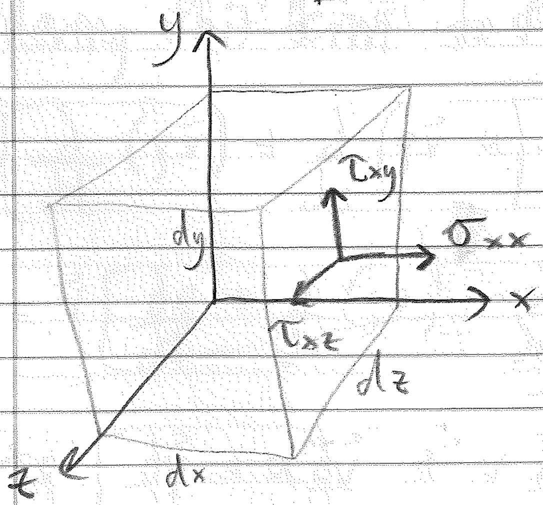
C.V. $F_B = \iiint_{C.V.} \rho \vec{g} \, d\forall = \rho \vec{g} \, dx dy dz$

2) Surface forces → proportional to surface area

e.g.



"sliding" ⇒ shear forces



x - surface : τ_{xi} ← unit normal of surface ; ← direction of stress

σ_{xx}

Each surface
3 components
↳ 1 normal σ
↳ 2 tangential τ

- 9 component matrix (tensor)

$$\underline{\underline{\sigma}} = \begin{pmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{pmatrix} \leftarrow x \text{ surface}$$

surface stress

x direction

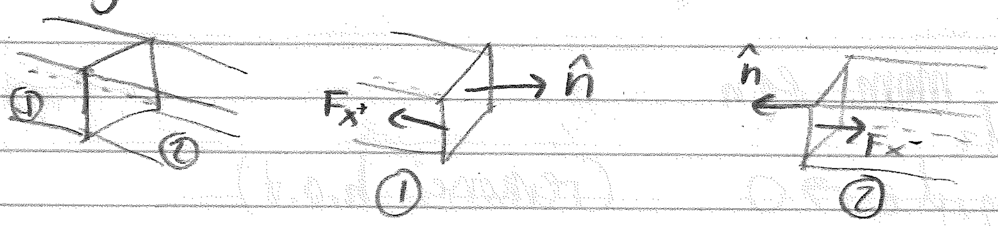
(1) $\tau_{xy}\, dy dz$: force in y direction on surface of area $dy dz$ with normal $\hat{n} = \hat{x}$

(2) diagonal : $\sigma_{xx}, \sigma_{yy}, \sigma_{zz} \Rightarrow$ normal stresses

off diagonal: $\tau_{ij} \Rightarrow$ shear stresses

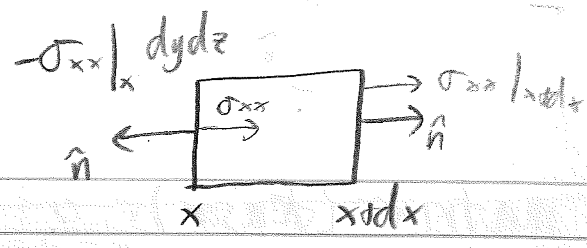
(3) sign convention: a component of $\underline{\underline{\sigma}}$ is positive if the force vector component and the area normal ($\hat{n}$) are in the same direction (both pos. or both neg.)

e.g. $\sigma_{xx} < 0$



$$F_{x^-} = -F_{x^+} \qquad 3^{rd} \text{ law}$$

<u>Note</u>: pressure applies inward, normal is outward $\Rightarrow -p$ always



6 surfaces:

x direction

The force on the left face:

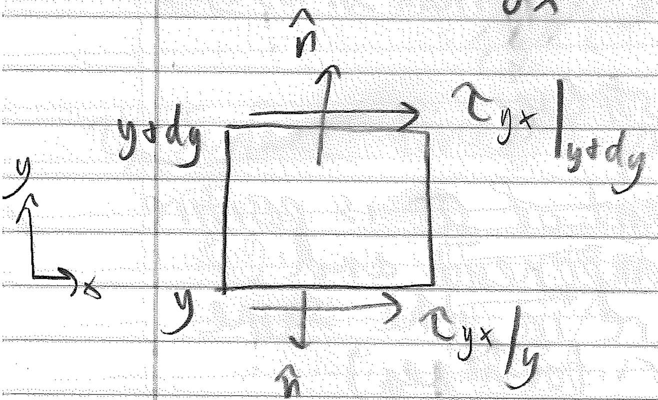
$$-\sigma_{xx}\big|_x \, dydz$$

" " right face:

$$\sigma_{xx}\big|_{x+dx} \, dydz = \left[\sigma_{xx}\big|_x + \frac{d\sigma_{xx}}{dx}dx\right] dydz$$

Combining:

$$\frac{\partial \sigma_{xx}}{\partial x} dxdydz$$



$$\left(\frac{\partial \tau_{yx}}{\partial y} dy\right) dxdz$$

$$\left(\frac{\partial \tau_{zx}}{\partial z} dz\right) dxdy$$

x-comp of mom. eqn

$\div \, dxdydz$

$dx, dy, dz \to 0$ (removes h.o.t)

$$\underbrace{\rho \frac{Du}{Dt}}_{\substack{\text{rate of change} \\ \text{that follows} \\ \text{a fluid particle}}} = \underbrace{\rho g_x}_{\substack{\text{body} \\ \text{force}}} + \underbrace{\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}}_{\text{surface forces}}$$

$$\rho \frac{Dv}{Dt} = \rho g_y + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z}$$

$$\rho \frac{Dw}{Dt} = \rho g_z + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z}$$

Equations

$$\left.\begin{array}{ll} \text{conservation of mass} & 1\text{ eqn} \\ \text{momentum balance} & 3\text{ eqn} \\ \text{conservation of energy} & 1\text{ eqn} \\ \text{(angular momentum balance)} & 3\text{ eqns} \\ \quad \hookrightarrow \text{leads to symmetry of stress tensor} & \end{array}\right\} 8\text{ eqns}$$

Unknowns: $u, v, w, \rho, \sigma_{xx}, \tau_{xy}, \tau_{xy} \dots \Rightarrow 13$ unknowns

Properties of a fluid

1) Constitutive eqn (mechanical properties)

2) Gas law - thermodynamics

lecture 5 Oct 5, 2018

→ Fluids cannot support a shear stress without continuously deforming.

→ Static fluids can only support normal stresses

Normal stress

1) compressive (opposite to the outward normal)

2) isotropic (same for any surface oriented in any direction)

3) usual static pressure

$\therefore \quad \sigma_{xx} = \sigma_{yy} = \sigma_{zz} = -P$ when the fluid is static

$\tau_{xy} = \tau_{yz} = \dots = 0$ i.e. shear stresses are zero

The same pressure is assumed to exist for a moving fluid.

take it at

$$\underline{\sigma} = \begin{pmatrix} -P+\tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & -P+\tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & -P+\tau_{zz} \end{pmatrix}$$

$\sigma_{xx} = -P + \tau_{xx}$

$\underline{\tau}$ is the remainder of the surface stresses that are due to viscosity

For solid ⤳ strain e.g. Hooke's law: stress $\propto$ strain

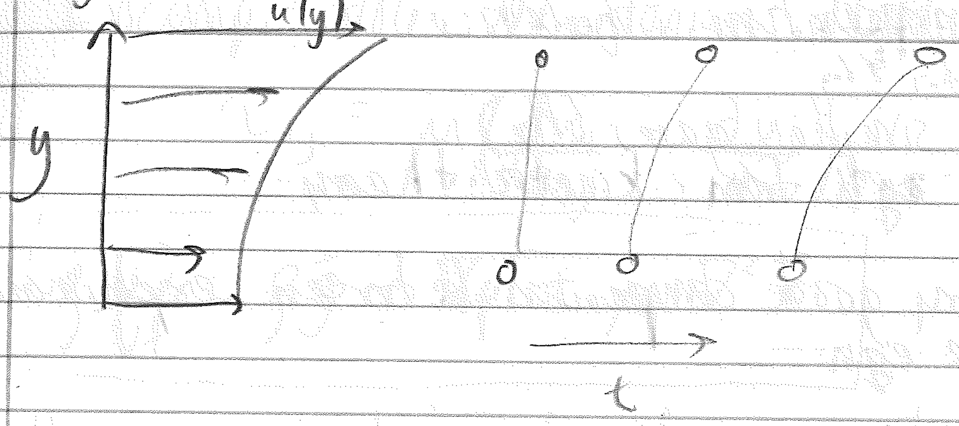
For fluid ⤳ stress $\propto$ rate of strain

i.e. rate of change of relative displacement

$\iint \vec{V}\cdot d\vec{A}$ $\quad \frac{m^3}{s}$

$d\vec{A} = dA\cdot\hat{n}$

e.g. shear flow



rate of strain is related to $\frac{du}{dy}$

Newtonian fluids: stress is a linear fn of rate of strain

For incompressible fluids → relationship between $\underline{\tau}$ and $\vec{V}$

$$\underbrace{\tau_{xx} = 2\mu \frac{\partial u}{\partial x}}_{\text{normal rate of strain}} \qquad \tau_{yy} = 2\mu\frac{\partial v}{\partial y} \qquad \tau_{zz} = 2\mu\frac{\partial w}{\partial z}$$

$$\tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

$$\tau_{xz} = \tau_{zx} = \mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)$$

$$\tau_{yz} = \tau_{zy} = \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)$$

$\underline{\tau}$ is a symmetric tensor

$$\mu\frac{\partial}{\partial x}\left(2\frac{\partial u}{\partial x}\right) = 2\mu\frac{\partial^2 u}{\partial x^2}$$

$$\mu\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial x\partial y}$$

$$\mu\frac{\partial}{\partial z}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) = \frac{\partial^2 w}{\partial x\partial z} + \frac{\partial^2 u}{\partial z^2}$$

$$\mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) + \cancel{\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)}$$

$\nabla\cdot V = 0$

* symmetric nature of τ comes from angular momentum balance

* in case of simple gases (H_2)
  ↳ derive eqn from kinetic theory

* generally, very good comparison between experiments and the eqn

⇒ plug that into momentum eqn
↳ $Ma < 0.2$

$\rho = const.$
$\mu = const.$

$$\rho \frac{Du}{Dt} d\forall = \rho g_x + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}$$

$$x: \rho \frac{Du}{Dt} = \rho g_x - \frac{\partial p}{\partial x} + \mu \underbrace{\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right)}_{\nabla^2 u}$$

$$\nabla \cdot (\nabla u) = \nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

$$y: \rho \frac{Dv}{Dt} = \rho g_y - \frac{\partial p}{\partial y} + \mu \nabla^2 v$$

$$z: \rho \frac{Dw}{Dt} = \rho g_z - \frac{\partial p}{\partial z} + \mu \nabla^2 w$$

Lecture 6

Oct 8, 2018

Incompressible Newtonian fluids

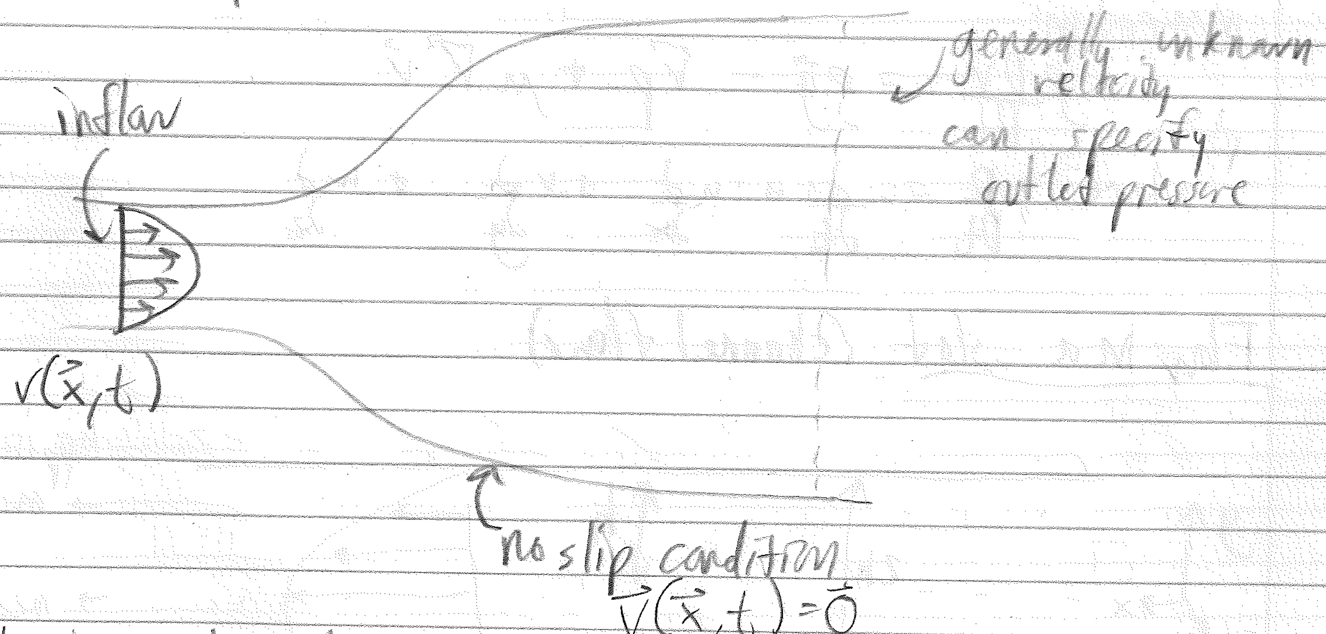
$$\tau_{ij} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

$$\boxed{\rho \frac{D\vec{v}}{Dt} = \rho\vec{g} - \nabla p + \mu \nabla^2 \vec{v}} \qquad \boxed{\nabla \cdot \vec{u} = 0}$$

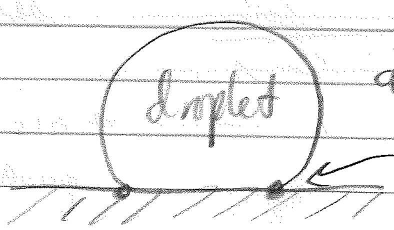
4 eqns + initial + boundary conditions

* Initial conditions → needed for unsteady flow $\frac{d}{dt} \neq 0$
  → guess for steady flow simulations

* Boundary conditions



When is no slip not true?



air capillary effects → $\vec{v}_w \neq 0$

triple point (3 phases)

Challenges in simulations:

→ setting proper BCs

→ nonlinear terms i.e. unknown × unknown

e.g. $u\frac{\partial u}{\partial x}$, $v\frac{\partial u}{\partial y}$, $w\frac{\partial v}{\partial z}$ etc.

Review

continuity:

$$\frac{\partial \rho}{\partial t} + \nabla\cdot\vec{V} = 0 \qquad \text{incompressible} \Rightarrow \nabla\cdot\vec{V}=0$$

stream function Ψ:

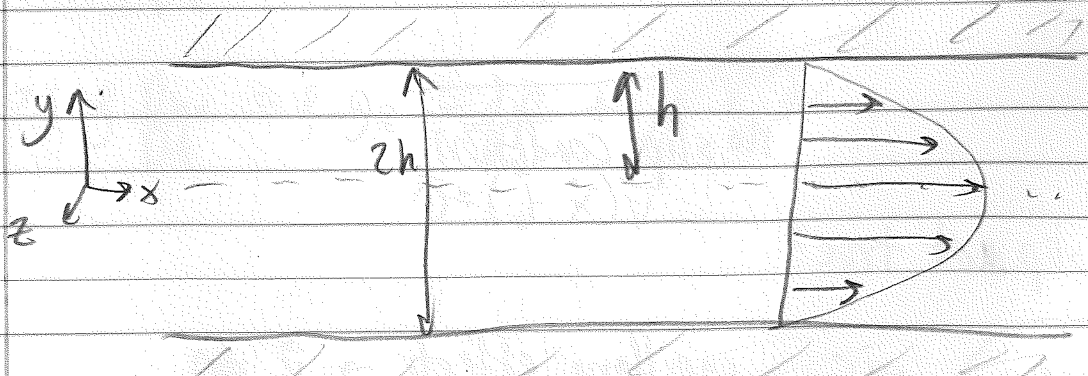
$$u = \frac{\partial \Psi}{\partial y} \qquad v = -\frac{\partial \Psi}{\partial x}$$

momentum:

$$\rho\frac{D\vec{V}}{Dt} = \rho\vec{g} - \nabla p + \mu\nabla^2\vec{V}$$

where $\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$

Flow in a slot (channel flow) Smits 8.5.1



Applications

→ microfluidics

→ mechanical electromagnetic devices

→ oil between piston + cylinder

Pressure driven flow

Two dimensional: $\frac{\partial}{\partial z} = 0$, $w = 0$

Steady: $\frac{\partial}{\partial t} = 0$

Laminar: $Re < 2000$

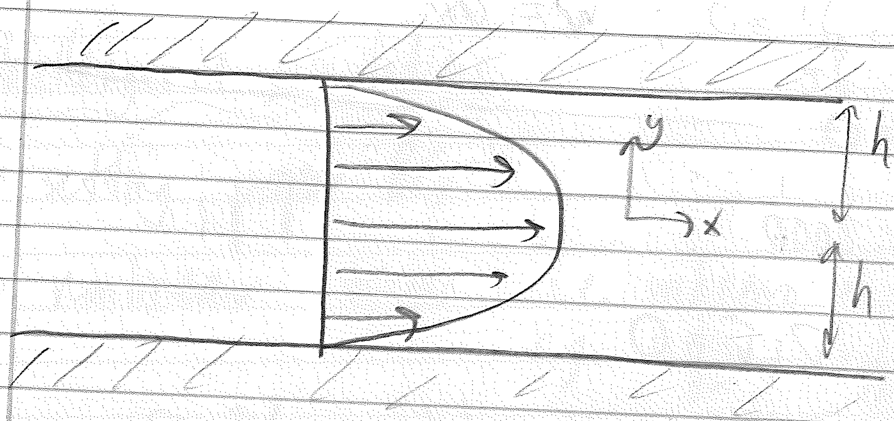
Incompressible: $\nabla \cdot \vec{V} = 0$

Fully developed: far from the inlet $\therefore \frac{\partial \vec{V}}{\partial x} = 0$

Neglecting gravity: $\rho \vec{g} = 0$

Lecture 7

Oct 10, 2018



Continuity

$$\underbrace{\cancel{\frac{\partial u}{\partial x}}}_{\text{fully developed}} + \frac{\partial v}{\partial y} + \underbrace{\cancel{\frac{\partial w}{\partial z}}}_{2D} = 0$$

$$\frac{\partial v}{\partial y} = 0 \Rightarrow v(y) = C$$

no slip BC: $v(y = \pm h) = 0$

$$\vec{V} = (u(y), 0, 0)$$

x: $\rho\left(\underbrace{\cancel{\frac{\partial u}{\partial t}}}_{\text{steady}} + \underbrace{\cancel{u\frac{\partial u}{\partial x}}}_{\text{fully dev}} + \underbrace{\cancel{v\frac{\partial u}{\partial y}}}_{v=0} + \underbrace{\cancel{w\frac{\partial u}{\partial z}}}_{w=0}\right) = -\frac{\partial p}{\partial x} + \mu\left(\underbrace{\cancel{\frac{\partial^2 u}{\partial x^2}}}_{\text{fully dev}} + \frac{\partial^2 u}{\partial y^2} + \underbrace{\cancel{\frac{\partial^2 u}{\partial z^2}}}_{2D}\right) + \underbrace{\cancel{\rho g_x}}_{\text{no body force}}$

y: $\rho\left(\cancel{\frac{\partial v}{\partial t}} + \cancel{u\frac{\partial v}{\partial x}} + \cancel{v\frac{\partial v}{\partial y}} + \cancel{w\frac{\partial v}{\partial z}}\right) = -\frac{\partial p}{\partial y} + \mu\left(\cancel{\frac{\partial^2 v}{\partial x^2}} + \cancel{\frac{\partial^2 v}{\partial y^2}} + \cancel{\frac{\partial^2 v}{\partial z^2}}\right) + \cancel{\rho g_y}$

$v = 0$

$$\frac{\partial p}{\partial y} = 0$$

pressure is uniform along cross section

$$\therefore \; p = p(x)$$

x:

$$\frac{dp}{dx} = \mu \frac{d^2u}{dy^2} = \text{constant}$$

$$\frac{dp}{dx} = \text{constant} < 0 \quad \text{for flow in } +x \text{ direction}$$

i.e. P must linearly decrease

$$\frac{d^2u}{dy^2} = \frac{1}{\mu}\frac{dp}{dx}$$

$$\frac{du}{dy} = \frac{1}{\mu}\frac{dp}{dx}y + A$$

$$u(y) = \frac{1}{2\mu}\frac{dp}{dx}y^2 + Ay + B$$

no slip: $u(y=\pm h) = 0$

$$\frac{1}{2\mu}\frac{dp}{dx}h^2 + Ah + B = 0$$

$$-\left(\frac{1}{2\mu}\frac{dp}{dx}(-h)^2 - Ah + B = 0\right)$$

$$2Ah = 0$$

$$A = 0$$

Σ eqns: $\frac{1}{\mu}\frac{dp}{dx}h^2 + 2B = 0$

$$B = -\frac{h^2}{2\mu}\frac{dp}{dx}$$

$$u(y) = \frac{h^2}{2\mu}\frac{dP}{dx}\left(\frac{y^2}{h^2} - 1\right)$$

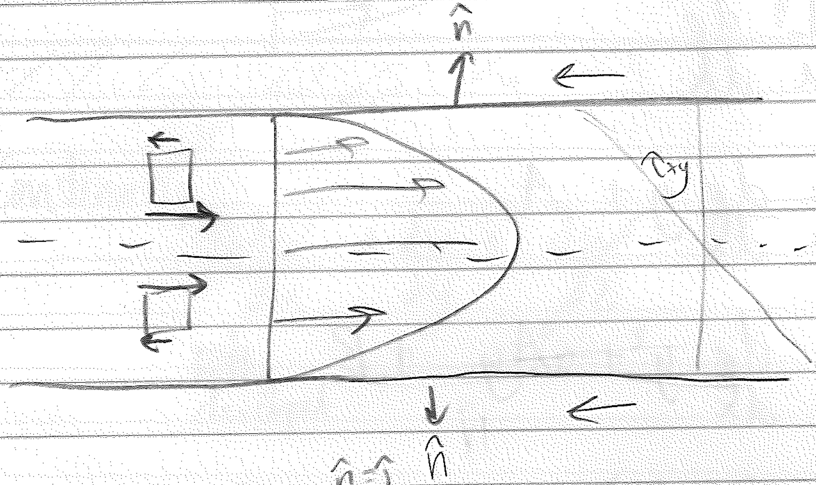
$$= u_m\left(1 - \frac{y^2}{h^2}\right) \qquad u_m = -\frac{h^2}{2\mu}\frac{dP}{dx} > 0$$

There is only one shear stress:

$$\tau_{xy} = \tau_{yx} = \mu\left(\frac{du}{dy} + \frac{dv}{dx}\right) = -2\mu\, U_m \frac{y}{h^2}$$

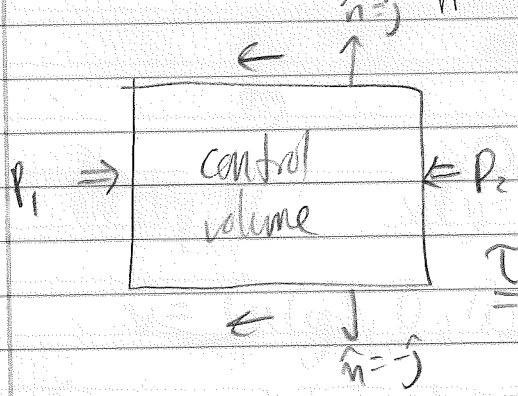
$$= \frac{dP}{dx}\, y \qquad \frac{dP}{dx} < 0$$

☆



$y > 0,\ \tau_{xy} < 0$

$y < 0,\ \tau_{xy} > 0$



$$\vec{F}_\mu = \iint_A \underline{\underline{\tau}} \cdot \hat{n}\, dA$$

$$\underline{\underline{\tau}} \cdot \hat{j} = \tau_{yx}\hat{i} + \cancel{\tau_{yy}}\hat{j} + \cancel{\tau_{yz}}\hat{k}$$

$$\vec{F}_\mu = \iint_A (\tau_{yx}\, dA)\,\hat{i}$$

At the wall: $\tau_{yx}(h) = -2\mu \frac{U_m h}{h^2}$

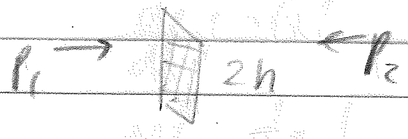
upper: $\int \tau_{yx}\, dx = -\frac{2\mu U_m l}{h}$

lower: $\tau_{yx}(-h) = -2\mu \frac{U_m(-h)}{h^2} = \frac{2\mu U_m h}{h^2}$

$$\tau_{yx} \cdot -\hat{j} = -\frac{2\mu U_m l}{h}$$

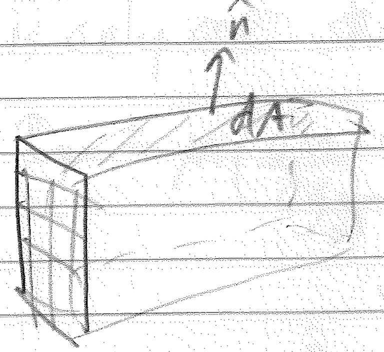
total: $= -\dfrac{4\mu U_m \ell}{h}$

⊛ pressure



$$(P_1 - P_2)\, 2h = \left(-\frac{dp}{dx}\,\ell\right) 2h = \left(\frac{2\mu U_m}{h^2}\right)\ell\, 2h = \frac{4\mu U_m \ell}{h}$$

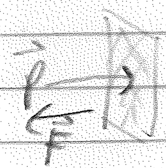
lecture 8 Oct 11, 2018



shear force:

$$\vec{F}_v = -\iint_A \overset{\text{tensor rank 2}}{\underline{\underline{\tau}}} \cdot \overset{\text{vector}}{\hat{n}}\, dA$$ gives a vector

$$\vec{F}_p = \left(\iint -p\, dA\right)\hat{n}$$



$\sum F = 0$ ⇒ no acceleration, steady state

$$Q = \iint_A \vec{v}\cdot\hat{n}\, dA$$

Channel: $Q = \int_{-h}^{h} u\, dy$

$$= V_m \int_{-h}^{h}\left(1-\frac{y^2}{h^2}\right) dy$$ $s = \frac{y}{h}$

$h s = dy$

$$= V_m h \int_{-h}^{h} (1-s^2)\, dy$$

$$= V_m h \left[s^2 - \frac{s^3}{3}\right]_{-h}^{h} = V_m h \left(1-(-1) + \frac{1}{3} - \left(-\frac{1}{3}\right)\right)$$

$$= \frac{4}{3} V_m h = \boxed{\frac{2}{3}\frac{h^3}{\mu}\left|\frac{dP}{dx}\right|}$$

$$\left|\frac{dP}{dx}\right| = \frac{3}{2} Q \frac{\mu}{h^3}$$

$\frac{dP}{dx} \propto Q$ $\qquad$ $\frac{dP}{dx}\uparrow$ $Q\uparrow$

$\frac{dP}{dx} \propto \mu$ $\qquad$ $\mu\uparrow$ $\frac{dP}{dx}\uparrow$

$\frac{dP}{dx} \propto \frac{1}{h^3}$ $\qquad$ $h\downarrow$ $\frac{dP}{dx}\uparrow$

average velocity: $\langle u \rangle = \frac{Q}{2h} = \frac{1}{3}\frac{h^2}{\mu}\left|\frac{dP}{dx}\right| = \frac{2U_m}{3}$

$$= \frac{\int_{-h}^{h} u\,dy}{h-(-h)}$$

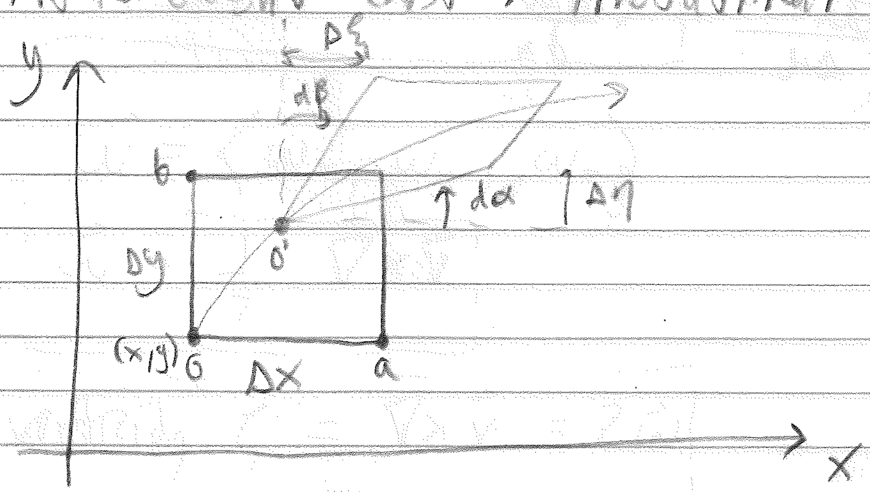
Vorticity

Smits 6.1

→ describes rotation of a fluid element (vortex)

→ can be used in turbulent flow

→ if it doesn't exist → irrotational



1. rotation
2. translation
3. deformation (strain rate)

At time t, velocity at $(x,y) = (u_0, v_0)$

During Δt, the fluid particle moves from O to O' with coordinates $(x + u_0 \Delta t,\ y + v_0 t)$

Simultaneously, the lines $\overline{OA}$ and $\overline{OB}$ rotates around O

The rotation rate of $\overline{OA}$ is

$$\omega_{OA} = \lim_{\Delta t \to 0} \frac{\Delta \alpha}{\Delta t}$$

$$\frac{\Delta \eta}{\Delta x} = \tan(\Delta \alpha) \approx \Delta \alpha \qquad \text{(small angle)}$$

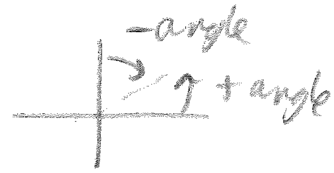
$$V_A = V_0 + \frac{\partial v}{\partial x} \Delta x$$

$$\Delta \eta = \frac{\partial v}{\partial x} \Delta x \Delta t$$

$$\Delta \eta = (V_A - V_0) \Delta t$$

$$\Delta \alpha = \frac{\Delta \eta}{\Delta t} = \frac{\partial v}{\partial x} \Delta t$$

$$\omega_{OA} = \lim_{\Delta t \to 0} \frac{\frac{\partial v}{\partial x} \Delta t}{\Delta t} = \frac{\partial v}{\partial x}$$



$$\omega_{OB} = \lim_{\Delta t \to 0} \frac{-\Delta \beta}{\Delta t}$$

$$\frac{\Delta \xi}{\Delta y} = \tan(\Delta \beta) \approx \Delta \beta$$

$$u_p = u_0 + \frac{\partial u}{\partial y} \Delta y$$

$$\Delta \xi = (u_p - u_0)\Delta t$$

$$\Delta \xi = \frac{\partial u}{\partial y} \Delta y \Delta t$$

$$\lim_{\Delta t \to 0} \frac{-\frac{\partial u}{\partial y}\Delta t}{\Delta t} = -\frac{\partial u}{\partial y}$$

$$\boxed{\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)}$$

average of two terms
xy plane → z comp. ω

$$\boxed{\omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)}$$

yz plane → x comp. ω

$$\boxed{\omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)}$$

xz plane → y comp. ω

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ u & v & w \end{vmatrix}$$

$$\vec{\omega} = \langle \omega_x, \omega_y, \omega_z \rangle$$

$$\boxed{\vec{\omega} = \frac{1}{2} \nabla \times \vec{V}}$$

$$\boxed{\text{vorticity } \zeta = \nabla \times \vec{V} = 2\vec{\omega}}$$

The fluid element

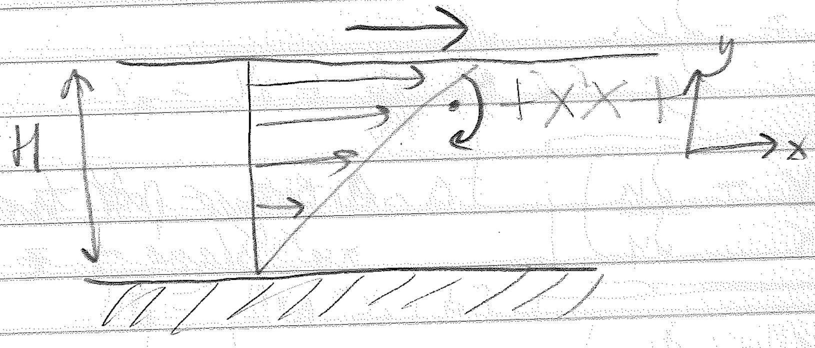
1) advects (u_0, v_0)

2) rotates $\frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$

3) deforms $\frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)$

<u>Examples</u>

Plane Couette flow

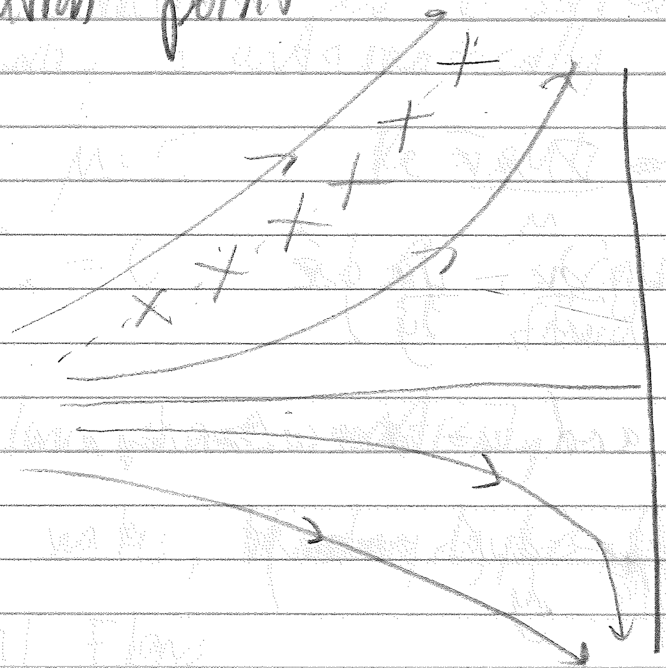


$$u = \frac{U}{h} y \qquad v = 0$$

$$\zeta = \zeta_z = \cancel{\frac{\partial v}{\partial x}} - \frac{\partial u}{\partial y} = -\frac{U}{h}$$

clockwise rotation at a rate of $\frac{1}{2}\frac{U}{h}$

stagnation point



$u = -ax$ $\qquad$ $\zeta_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$ irrotational

$v = ay$

$$-\frac{\partial}{\partial y}\left(\rho\frac{Du}{Dt} = \cancel{\rho g} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \mu\frac{\partial^2 v}{\partial y^2}\right) - \cancel{\frac{\partial p}{\partial x}}\right)$$

$$\frac{\partial}{\partial x}\left(\rho\frac{Dv}{Dt}\right) = \cancel{\rho g} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) - \cancel{\frac{\partial p}{\partial y}}$$

Lecture 9 — Oct 15, 2018

$\vec{\omega} = \frac{1}{2}\nabla \times \vec{V}$

3D flow: $\zeta = \nabla \times \vec{V}$

2D flow: $\zeta_z = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\hat{z}$

In 2D, for x- and y- momentum eqn:

$$\frac{\partial}{\partial x}(y\text{-mom.}) - \frac{\partial}{\partial y}(x\text{-mom.})$$

Continuity for incompressible flow:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\Rightarrow \boxed{\rho\frac{D\zeta_z}{Dt} = \mu\left(\frac{\partial^2 \zeta_z}{\partial x^2} + \frac{\partial^2 \zeta_z}{\partial y^2}\right) = \mu\nabla^2\zeta_z}$$

heat equation!

Stream function:

$$u = \frac{\partial \Psi}{\partial y} \qquad v = -\frac{\partial \Psi}{\partial x}$$

$$\zeta_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

$$\boxed{\zeta_z = -\nabla^2\Psi}$$

vorticity can be determined by Ψ and vice versa

these combined can be solved instead of N-S
Continuity is automatically satisfied.

Assume $\mu = 0$ $\quad Re = \frac{\rho U L}{\mu} \to \infty$

$\frac{D\xi_z}{Dt} = 0$ $\quad$ so ξ_z is conserved following a fluid particle

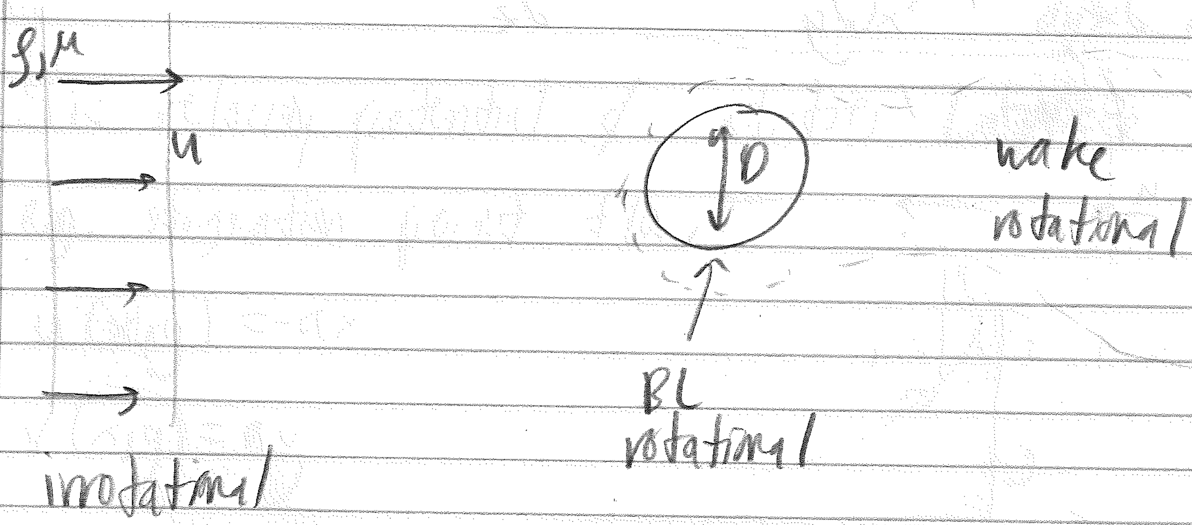
Many flows have vorticity

e.g. wakes, boundary layers, shear layers (jets)

Potential Flows

Smits 6.2

Consider the flow past a cylinder (Smits Chap 6)
$Re \gg 1$



$\nabla \times \vec{v} = 0$

Vorticity is generated by viscous effects i.e. boundary layer, and swept into wake.

The rest of the flow is irrotational.

Advantageous to split into two parts:
rotational + irrotational

$\vec{\zeta} = \nabla \times \vec{v} = \vec{0}$

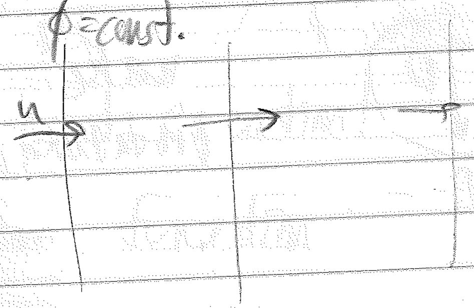
$\nabla \times (\nabla \phi) = 0$ from calculus

ϕ = any scalar
$\phi = (\vec{x}, t)$

Suggest $\vec{v} = \nabla \phi$ gradient of "velocity potential"

$\vec{\zeta} = \nabla \times (\nabla \phi) = 0$

$u = \dfrac{\partial \phi}{\partial x}, \quad v = \dfrac{\partial \phi}{\partial y}, \quad w = \dfrac{\partial \phi}{\partial z}$



Lecture 10

Oct 17, 2018

* $\rho \frac{D\xi_z}{Dt} = \mu \nabla^2 \xi_z$ in 2D, incompressible

* $\xi_z = -\nabla^2 \psi$

* $\vec{V} = \nabla\phi$ velocity potential

$u = \frac{\partial\phi}{\partial x}$ $v = \frac{\partial\phi}{\partial y}$ $w = \frac{\partial\phi}{\partial w}$ irrotational

$\vec{\xi} = \nabla\times\vec{v} = \nabla\times(\nabla\phi) = \vec{0}$

For incompressible flow, continuity becomes

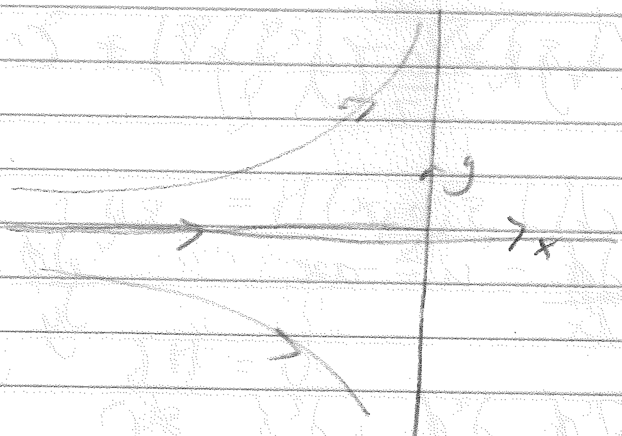
$$\nabla\cdot\vec{V} = \nabla\cdot(\nabla\phi) = \boxed{\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0}$$

The velocity potential ϕ satisfies Laplace eqn.

e.g. stagnation point flow

$u(x,y) = -ax$

$v(x,y) = ay$



$\nabla\cdot\vec{v} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = -a + a = 0$ ✓

$\xi_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$

Incompressible, irrotational → a velocity potential ϕ exists, satisfying: $u = \frac{\partial \phi}{\partial x} = -ax$

$$v = \frac{\partial \phi}{\partial y} = ay$$

$\int u\,dx \Rightarrow \phi = \int u\,dx + f(y)$

$$= -\frac{ax^2}{2} + f(y)$$

$\int v\,dy \Rightarrow \phi = \int v\,dy + g(x)$

$$= \frac{ay^2}{2} + g(x)$$

Solutions must be the same

$$f(y) = \frac{ay^2}{2} + C_1$$

$$g(x) = \frac{-ax^2}{2} + C_2$$

$$\phi(x,y) = \frac{a}{2}(y^2 - x^2) + C$$

Check

$\frac{\partial \phi}{\partial x} = u = -ax$ $\qquad$ $\frac{\partial \phi}{\partial y} = v = ax$

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

Bernoulli's eqn Smits 4.2

Consider incompressible, irrotational flow

$$\nabla^2 \phi = 0$$

Consider momentum eqn. for incompressible, inviscid flow ($\mu = 0$)

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\vec{v} = -\frac{1}{\rho}\nabla p + g + \cancel{\frac{\mu}{\rho}\nabla^2 \vec{v}}$$

Irrotational + inviscid go hand in hand. $\Rightarrow \nabla^2 \vec{v} = 0$

Irrotational: $\vec{v} = \nabla \phi$

Vector identity: $\nabla \cdot \vec{v} = 0$

? $$(\vec{v} \cdot \nabla)\vec{v} = \nabla \frac{|\vec{v}|^2}{2} - \vec{v} \times \xi$$

Gravity: $\vec{g} = \nabla(-gz) = \langle 0, 0, -g \rangle$

$$\Rightarrow \frac{\partial (\nabla \phi)}{\partial t} + \nabla \frac{|v^2|}{2} - \vec{v} \times \xi = \nabla\left(\frac{p}{\rho}\right) - \nabla(gz)$$

$$H = \frac{\partial \phi}{\partial t} + \frac{|\vec{v}|^2}{2} + \frac{p}{\rho} + gz = H(x, y, z, t)$$

$$\frac{\partial H}{\partial x} = 0 \quad \frac{\partial H}{\partial y} = 0 \quad \frac{\partial H}{\partial z} = 0$$

$$\boxed{\frac{\partial \phi}{\partial t} + \frac{|\vec{v}|^2}{2} + \frac{p}{\rho} + gz = H(t)}$$ unsteady Bernoulli

Steady flow $\frac{\partial \phi}{\partial t} = 0$

H_0 = constant

$$\frac{|\vec{v}^2|}{2} + \frac{p}{\rho} + gz = 0$$

Notes

→ assumed irrotational, incompressible, inviscid

→ dimensions of energy although it comes from momentum

→ other versions only require inviscid

→ Bernoulli's ⇒ p from $\vec{v}$ or ϕ

e.g. stagnation point

$u = -ax \quad v = ay$

steady, incomp., neglect gravity

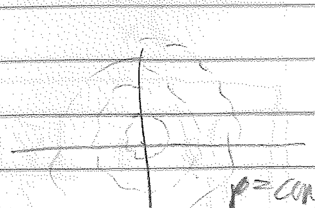
$$\frac{1}{2}(u^2+v^2) + \frac{p}{\rho} = \frac{1}{2}a(x^2+y^2) + \frac{p}{\rho} = H_0$$

Evaluate H_0 at stagnation point (0,0)

$p = p_0$

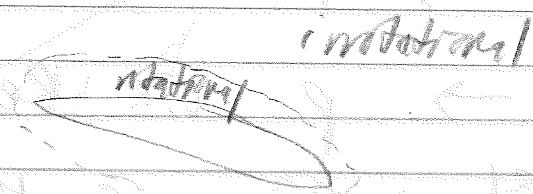
$$\frac{1}{2}a^2(x^2+y^2) + \frac{p}{\rho} = \frac{p_0}{\rho}$$

$$p(x,y) = p_0 - \frac{1}{2}\rho a^2(x^2+y^2)$$



Pressure is highest @ stagnation pt, decreases as $r^2 = x^2 + y^2$

A general approach to flows w/ substantial irrotational region

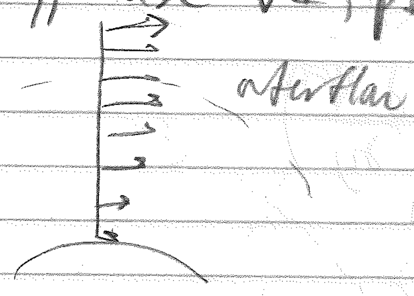


1) Solve for $\nabla^2 \phi = 0$ + BC → gives $\vec{v}_I = \nabla \phi_I$

2) Use Bernoulli's → $p_I(\vec{x})$

3) Compute pressure force on object

4) If necessary, use $\vec{v}_I, p_I$ to compute BL flow



5) Compute viscous force

6) If necessary, go back to #1 with the object "changed" due to BL

Lecture 11 Oct 19, 2018

$\vec{\zeta} = \vec{0}$

$\vec{v} = \nabla\phi$

$\nabla^2\phi = 0 \longrightarrow \vec{v}$

$\longrightarrow p$ via Bernoulli

$$\frac{\partial\phi}{\partial t} + \frac{|\vec{v}|^2}{2} + \frac{p}{\rho} + gz = H_0$$

Geometrical interpretation of ϕ (2D, incompressible)

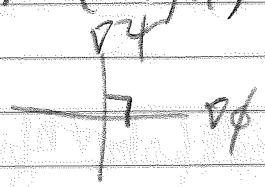
$$\vec{v} = \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}\right)$$

$$= \left(\frac{\partial\psi}{\partial y}, -\frac{\partial\psi}{\partial x}\right)$$

So $\nabla\psi = \left(\frac{\partial\psi}{\partial x}, \frac{\partial\psi}{\partial y}\right)$

$= (-v, u)$

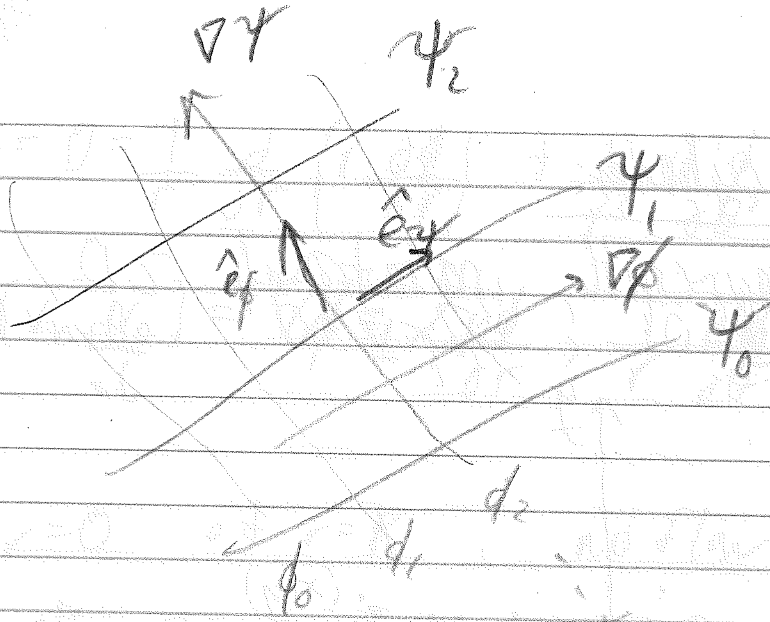
$\nabla\psi\cdot\nabla\phi = (u, v)\cdot\underbrace{(-v, u)}_{\nabla\psi} = -uv + uv = 0$

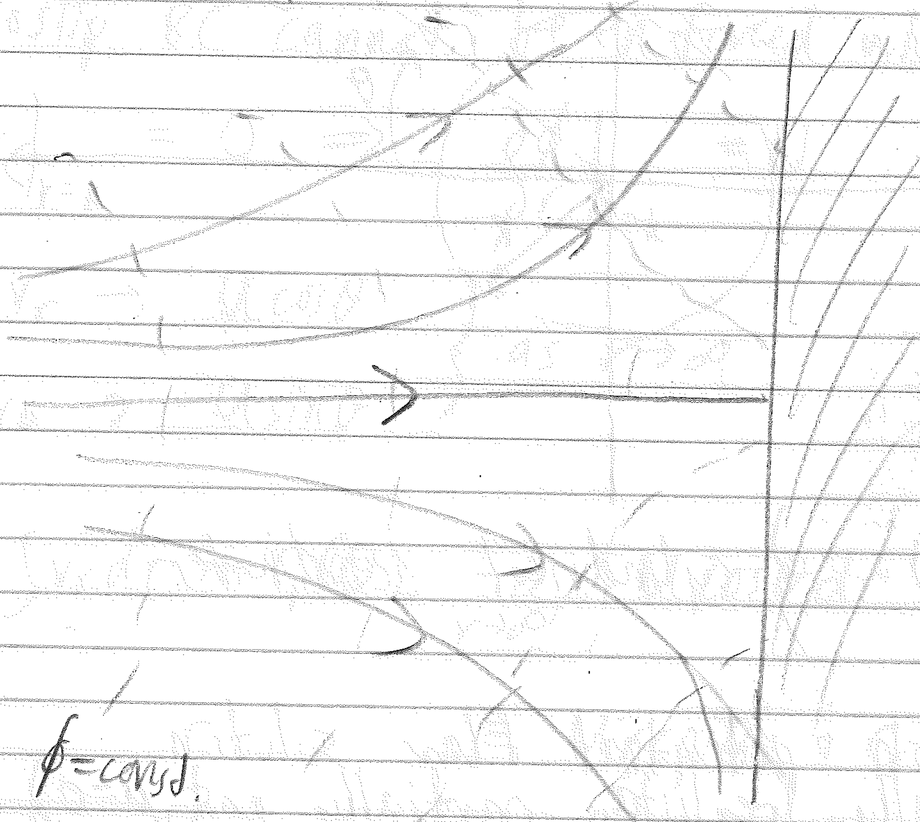


Lines of constant ψ are streamlines, and $\nabla\psi$ is $\perp$ to streamlines

$\nabla\psi\cdot\hat{e}_\psi = 0 \qquad \therefore \hat{e}_\psi\cdot\hat{e}_\phi = 0$

$\nabla\phi\cdot\hat{e}_\psi = 0$





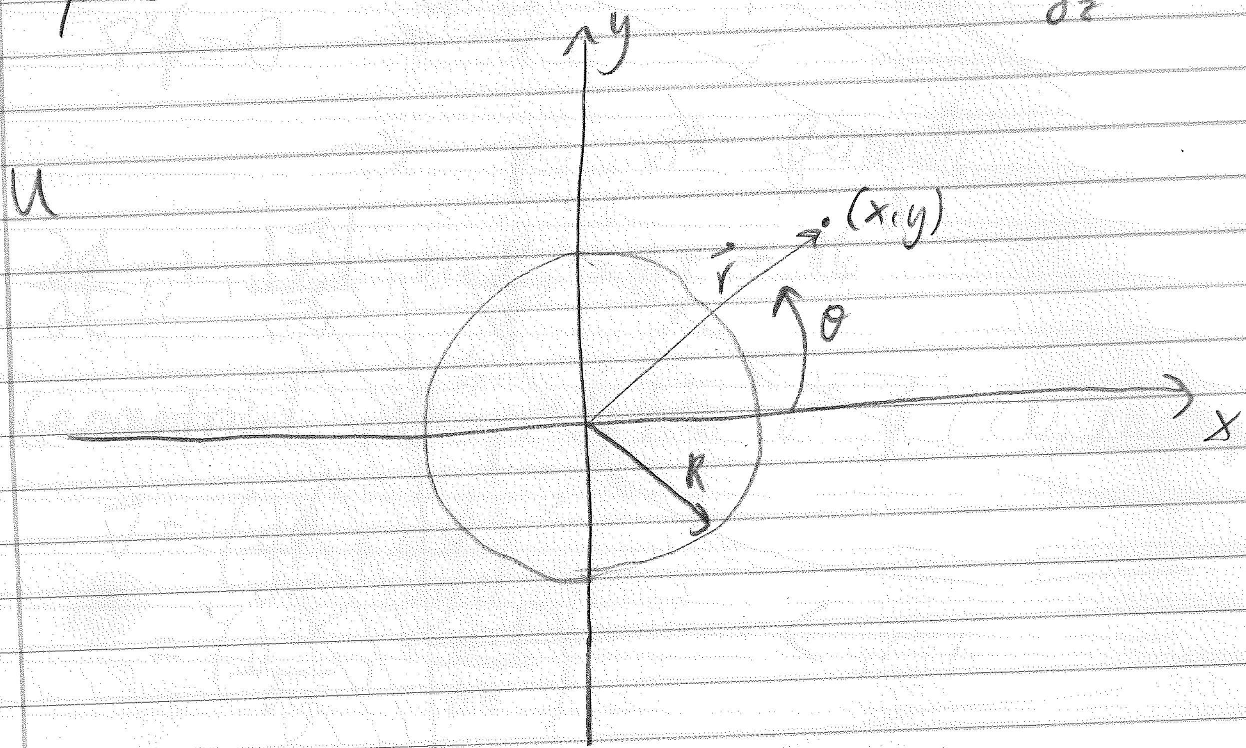
$\nabla\phi$ ‖ streamlines

ϕ ⊥ streamlines

Flow past a cylinder

Smits 6.9

inviscid, incompressible, irrotational, 2D
$\mu = 0$, $\nabla \cdot \underline{v} = 0$, $\nabla \times \underline{v} = 0$, $\frac{\partial}{\partial z} = 0,\ w = 0$



$r^2 = x^2 + y^2$
$\tan\theta = \frac{y}{x}$
} cylindrical coordinates

Work with potential flow solution
(partially realistic, partially not $\mu = 0$)

$$\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2}$$

$$\nabla f = \frac{\partial f}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\theta}$$

$$\nabla^2\phi = 0 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}$$

$$\vec{V} = (v_r, v_\theta) = \nabla\phi = \left\langle \frac{\partial\phi}{\partial r}, \frac{1}{r}\frac{\partial\phi}{\partial\theta} \right\rangle$$

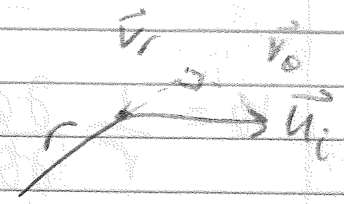
BC

① $v_r = 0$ at $r = R$ no flow $\perp$ cylinder

noslip BC cannot be enforced when $\mu = 0$

$$v_r\Big|_{r=R} = 0 \Rightarrow \frac{\partial\phi}{\partial r}\Big|_{r=R} = 0$$

② $\left.\begin{array}{l} v_r \to U\cos\theta \\ v_\theta \to -U\sin\theta \end{array}\right\}$ as $r \to \infty$



$\frac{\partial\phi}{\partial r} = v_r$

$$\phi = \int v_r\,dr + c_1(\theta) \xrightarrow[r\to\infty]{} \int U\cos\theta\,dr + c_1(\theta)$$

$\frac{1}{r}\frac{\partial\phi}{\partial\theta} = v_\theta$

$$= Ur\cos\theta + \cancel{c_1(\theta)}$$

$$\phi = r\int v_\theta\,d\theta + c_2(r) \xrightarrow[r\to\infty]{} r\int -U\sin\theta\,d\theta + c_2(r)$$

$$= Ur\cos\theta + \cancel{c_2(r)}$$

$$\phi(r,\theta) = Ur\cos\theta\left[1 + \frac{R^2}{r^2}\right]$$

Show $\nabla^2\phi = 0$

Solve for velocity with ϕ

$$v_r = \frac{\partial \phi}{\partial r} = U\cos\theta\left(1+\frac{R^2}{r^2}\right)$$

$$+\,Ur\cos\theta\left(-\frac{2R^2}{r^3}\right)$$

$$= U\cos\theta\left(1-\frac{R^2}{r^2}\right)$$

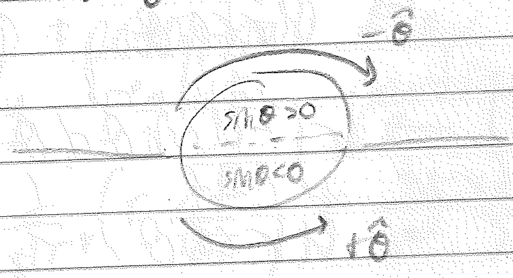
✓ $v_r = 0$ at $r = R$

$$v_r \xrightarrow[r\to\infty]{} U\cos\theta$$

$$v_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = \left(-U\sin\theta\left(1+\frac{R^2}{r^2}\right)\right)$$

$$v_\theta \xrightarrow[r\to\infty]{} -U\sin\theta$$

$v_\theta = -2U\sin\theta$ at $r = R$

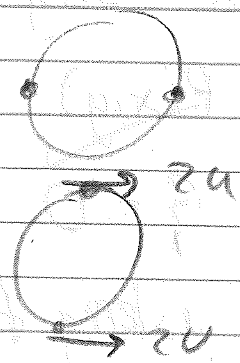


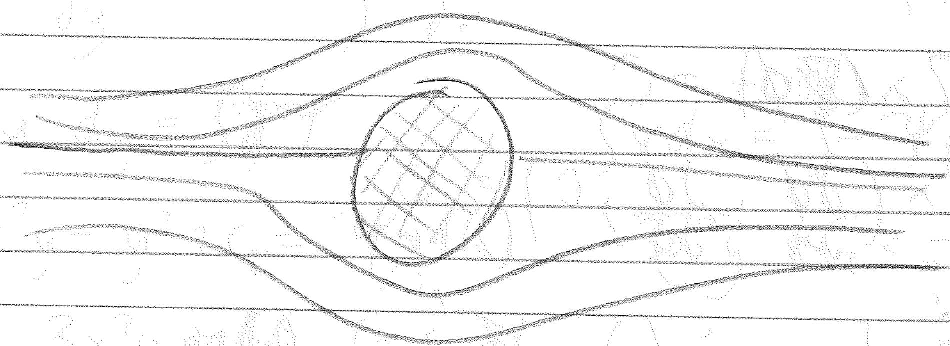
Stagnation point $(\vec{v} = \vec{0})$

$\theta = 0, \pi$ $\qquad r = R$

Max velocity

$\theta = \frac{\pi}{2}, \frac{3\pi}{2}$ $\qquad r = R$





Recitation

Oct 19, 2018

Given: $\phi = x^3 - 3xy^2$

Find: Ψ, $P(x,y)$

Assumptions: $\zeta = 0$ irrotational

asking for $\Psi \rightarrow$ incompressible

$$\vec{u} = \nabla \phi$$

$$\nabla \times u = \nabla \times (\nabla \phi) = 0$$

$$\nabla \times \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, 0\right) = 0$$

$$\begin{array}{ccc} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & 0 \end{array} \quad \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & 0 \end{vmatrix} = \hat{j}\left(\cancel{\frac{\partial}{\partial z}\frac{\partial \phi}{\partial x}}^{\,0} - \cancel{\frac{\partial}{\partial x} 0}\right) + \hat{k}\left(\cancel{\frac{\partial}{\partial z}\frac{\partial \phi}{\partial y}} - \frac{\partial}{\partial y}\frac{\partial \phi}{\partial x}\right)$$

$$+ \hat{i}\left(\cancel{\frac{\partial}{\partial y} 0} - \cancel{\frac{\partial}{\partial z}}\frac{\partial \phi}{\partial y}\right)$$

$$= \frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial^2 \phi}{\partial x \partial y} = 0 \quad \checkmark$$

$$u = \frac{\partial \phi}{\partial x} = 3x^2 - 3y^2$$

$$v = \frac{\partial \phi}{\partial y} = -6xy$$

Verify incompressibility:

$\nabla \cdot \vec{V} = 0$

$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 6x - 6x = 0$ ✓

Reconstruct Ψ

$\int u = \frac{\partial \Psi}{\partial y}$ $\quad$ $v = \int -\frac{\partial \Psi}{\partial x}$

$\int u \, dy = \int \partial \Psi$ $\quad$ $\int v dx = \int -\Psi dx$

$3x^2y - y^3 + c_1 = \Psi + f(x)$ $\quad$ $-3x^2y + C_2 = -\Psi + g(y)$

$\Psi = 3x^2y + \underline{g(y)} + C_2$

$\Psi = 3x^2y \underline{- y^3} + f(x) + c_1$

$g(y) = -y^3$

$f(x) = C_3 = C_2 - C_1$

$\Psi = 3x^2y - y^3 + \cancel{C_2} \rightarrow 0$

$\Psi = 0$ at stagnation points
$\hookrightarrow$ $\vec{u} = 0$ at $x = 0, y = 0$

$\Psi(0,0) = C_2 = 0$

Pressure → use Bernoulli

$$p + \frac{\rho}{2}|V|^2 = \text{const.}$$

$$p + \frac{\rho}{2}(u^2 + v^2) = \text{const.}$$

$$p + \frac{\rho}{2}\left(9x^4 - 18x^2y^2 + 9y^4 + \underline{36xy}\right) = \text{const.}$$

$$p + \frac{9}{2}\rho\left(\left(x^2 - y^2\right)^2 + 4xy\right)$$

$$(x^2 - y^2)(x^2 - y^2)$$

$$x^4 + 2x^2y^2 + y^4$$

$$p(0,0) = p_{gauge} = 0$$

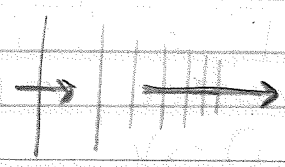
$$p = p_0 - \frac{9\rho}{2}\left(\left(x^2 - y^2\right)^2 + 4xy\right)$$

Lecture 12 Oct 22, 2018

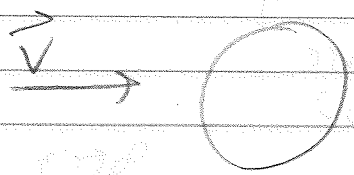
$\phi \perp \psi$

$\phi \to$ magnitude of $|\vec{v}|$

$\psi \to$ direction of $\vec{v}$



Pressure field



$$\frac{1}{2}\left(v_r^2 + v_\theta^2\right) + \frac{p}{\rho} = H_0$$

$$v_r^2 + v_\theta^2 = U^2\cos^2\theta\left[1 - \frac{2R^2}{r^2} + \frac{R^4}{r^4}\right] + U^2\sin^2\theta\left[1 + \frac{2R^2}{r^2} + \frac{R^4}{r^4}\right]$$

$$= U^2\left(1 + \frac{R^4}{r^4}\right) + 2U^2 \underbrace{\left(\sin^2\theta - \cos^2\theta\right)}_{-\cos(2\theta)} \frac{R^2}{r^2}$$

$$= U^2\left[\left(1 + \frac{R^4}{r^4}\right) - 2\cos 2\theta \frac{R^2}{r^2}\right]$$

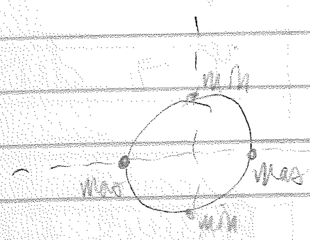
$$\frac{p}{\rho} = H_0 - \frac{U^2}{2}\left[\left(1 + \frac{R^4}{r^4}\right) - 2\cos 2\theta \frac{R^2}{r^2}\right)\Big]$$

$r \to \infty$, $p \to p_\infty$ $H_0 = \text{const.} = \frac{p_\infty}{\rho} - \frac{U^2}{2}$

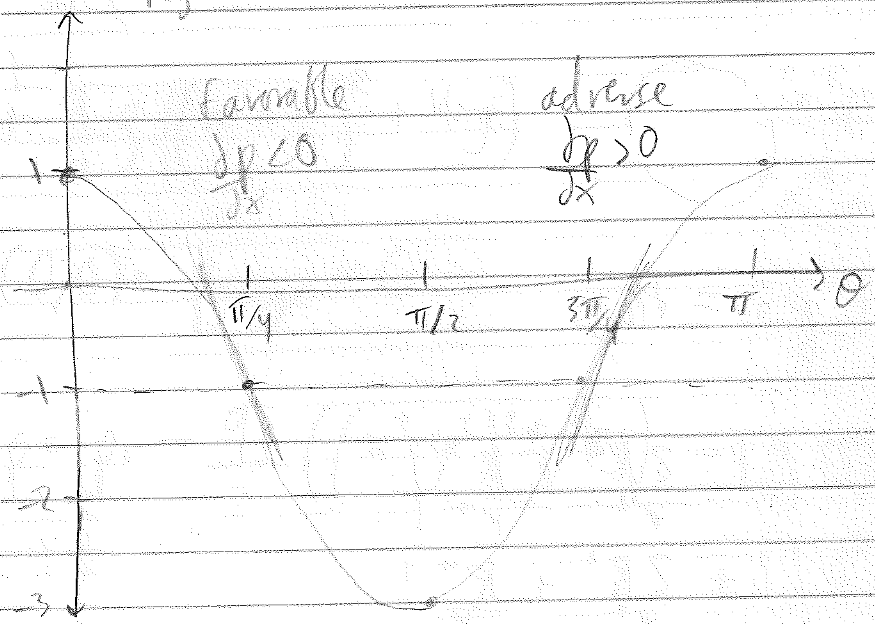
$$p = p_\infty + \frac{1}{2}\rho U^2 \left(2\cos(2\theta)\frac{R^2}{r^2} - \frac{R^4}{r^4}\right)$$

Cylinder surface, $r = R$

$$p(r,\theta)\Big|_{r=R} = p_\infty + \frac{1}{2}\rho U^2 \left(2\cos(2\theta) - 1\right)$$



$$C_p = \frac{p - p_\infty}{\frac{1}{2}\rho U^2}$$

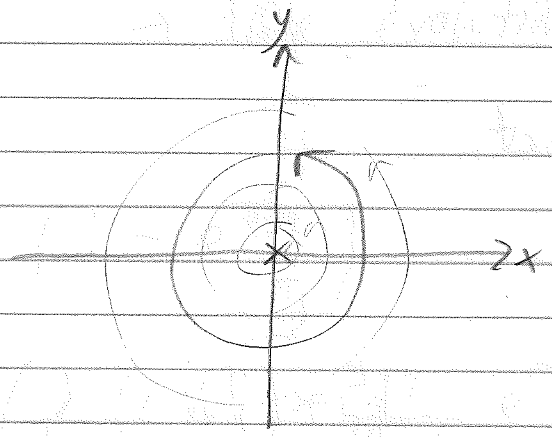


inviscid + symmetric $\longrightarrow$ no force (d'Alembert's paradox)

Smits 6.7.3

Line vortex (a line in the z-direction)

→ flow near vortex



$v_r = 0$

$v_\theta = \dfrac{\Gamma}{2\pi r}$

$v_\theta \to 0 \qquad r \to \infty$

$v_\theta \to \infty \qquad r \to 0$

$$\zeta_z = \frac{1}{r}\frac{\partial}{\partial r}(r v_\theta) - \cancel{\frac{1}{r}\frac{\partial v_r}{\partial \theta}}^{\to 0}$$

$$= \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\cancel{r}\Gamma}{2\pi \cancel{r}}\right) = 0 \quad \therefore \text{ irrotational}$$

All vorticity is concentrated at the origin (singularity)

Γ = strength of vortex (circulation)

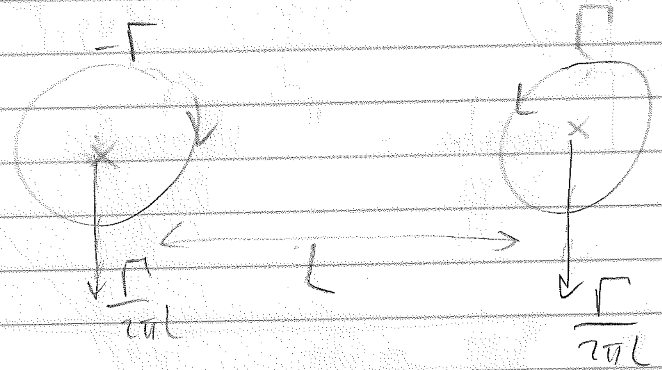
$$\Gamma = \oint \vec{v} \cdot d\vec{\ell} \qquad \left[\frac{m}{s}\right][m] = \left[\frac{m^2}{s}\right]$$

$d\vec{\ell} = r\,d\theta\,\hat{e}_\theta$

$$\Gamma = \int_0^{2\pi} v_\theta\, r\, d\theta = \int_0^{2\pi} \frac{\Gamma}{2\pi r}\, r\, d\theta = \Gamma \quad \checkmark$$

$$\left.\begin{aligned} \zeta &= \nabla \times \vec{V} \\ \Gamma &= \iint_S \zeta \cdot n \, ds \end{aligned}\right\} \text{stokes thm}$$

Vortices can interact...



vortices get advected at $v = \frac{\Gamma}{2\pi L}$

Superposition $\nabla^2 \phi = 0$

$$\nabla^2(\phi_1 + \phi_2) = \nabla^2 \phi_1 + \nabla^2 \phi_2 = 0$$

Lecture 13 Smits 6.9.2 Oct 24, 2018

Point (line) vortex = model

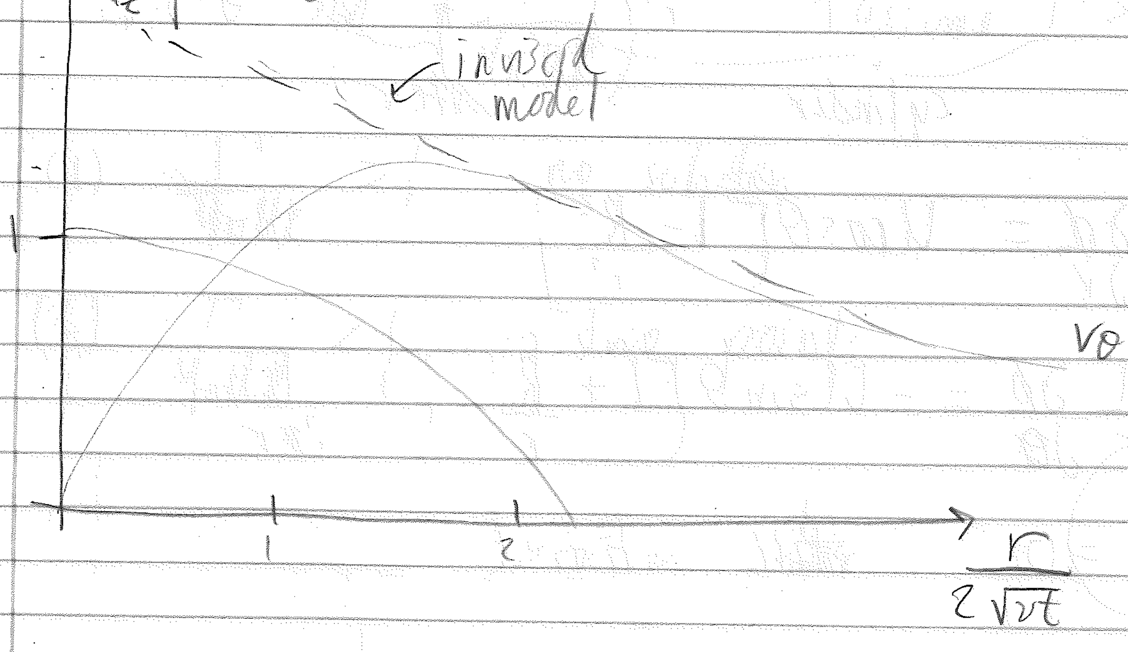
More realistic → viscosity
→ time evolution

$v_r = 0$

$$v_\theta = \frac{\Gamma}{2\pi r}\left(1 - e^{\frac{-r^2}{4\nu t}}\right) \qquad \nu = \frac{\mu}{\rho} \quad [m^2/s] \quad \text{"ponder's rate"}$$

$$\xi_z = \frac{1}{r}\left(\frac{\partial}{\partial r}(r v_\theta)\right) = \frac{\Gamma}{4\pi\nu t} e^{-\frac{r^2}{4\nu t}}$$

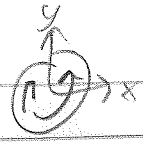
$$\xi_z \frac{4\pi\nu t}{\Gamma} = e^{-\left(\frac{r}{2\sqrt{\nu t}}\right)^2}$$



Note: there is often an associated v_z velocity

- airplane → core velocity can be very high
  ↳ trade off between safety + inter-takeoff time

high velocity → low pressure → condensation → ice crystals
↳ contrails!

<u>Lift</u> $\rightarrow$ 

e.g. flow around a cylinder + vortex

$$\nabla^2(\phi_1+\phi_2)=\nabla^2\phi_1+\nabla^2\phi_2 \quad + \text{B.C.s}$$

Laplace is linear $\rightarrow$ superposition

doesn't work for pressure

$$p \sim |v|^2$$

$$|v_1|^2+|v_2|^2 \neq |v_1+v_2|^2$$

So $$\phi = \underbrace{Ur\cos\theta\left[1+\frac{R^2}{r^2}\right]}_{\text{cylinder}} + \underbrace{\frac{\Gamma}{2\pi}\theta}_{\text{vortex}}$$

$$v_r=\frac{\partial\phi}{\partial r}=U\cos\theta\left[1-\frac{R^2}{r^2}\right]$$

$$v_\theta=\frac{1}{r}\frac{\partial\phi}{\partial\theta}=-U\sin\theta\left[1+\frac{R^2}{r^2}\right]+\frac{\Gamma}{2\pi r}$$

$v_r\Big|_{r=R}=0$ still satisfied

$v_r \rightarrow U\cos\theta$ as $r\rightarrow\infty$ still satisfied

$v_\theta \rightarrow -U\sin\theta$

Stagnation point @ surface $(r=R)$

Without vortex $\rightarrow$ $\theta = 0, \pi$

$$v_\theta\Big|_{r=R} = -U\sin\theta\,(1+1) + \frac{\Gamma}{2\pi R} = 0$$

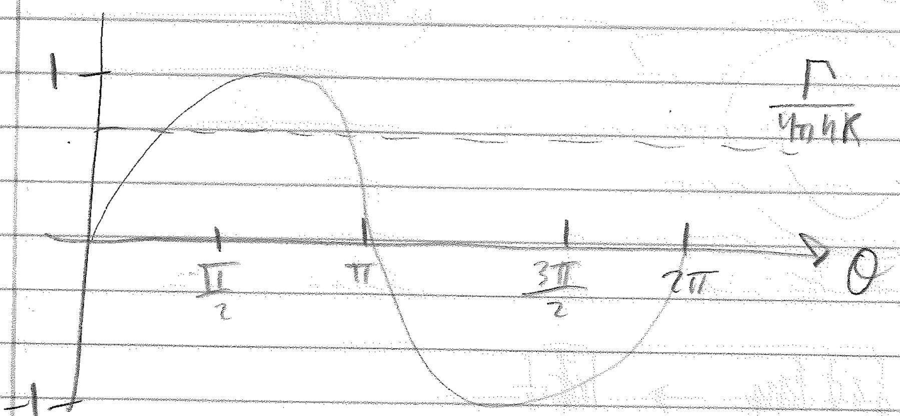
$$= -2U\sin\theta + \frac{\Gamma}{2\pi}R = 0$$

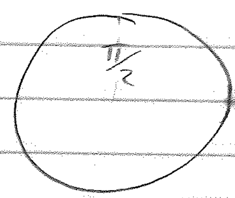
$$\sin\theta = \frac{\Gamma}{4\pi U R}$$

$$\theta = \sin^{-1}\left(\frac{\Gamma}{4\pi U R}\right) \qquad \text{assume } \Gamma > 0$$

(1) $\frac{\Gamma}{4\pi U R} > 1$ no solution

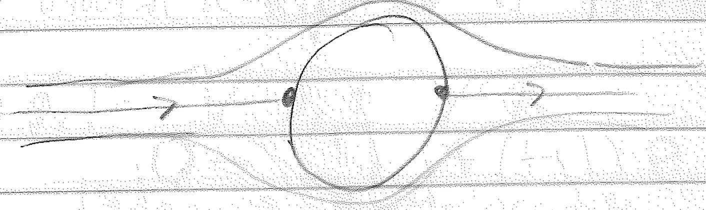
(2) $\frac{\Gamma}{4\pi U R} < 1$ $\rightarrow$ two points





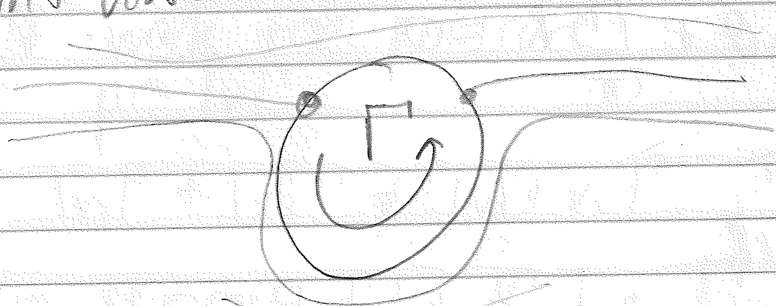
Uniform

$$\frac{\Gamma}{4\pi U R} = 0$$

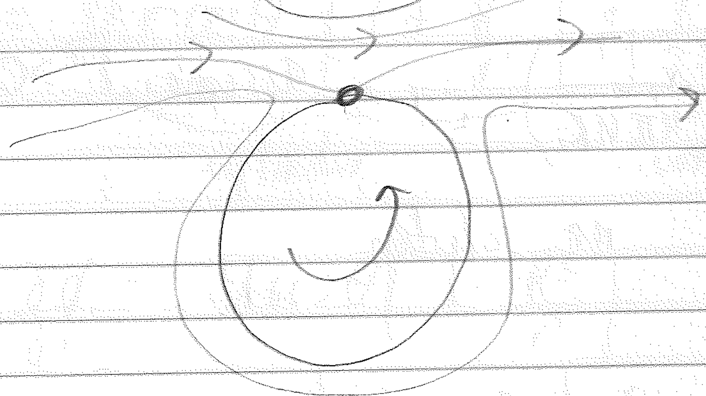


Uniform + vortex

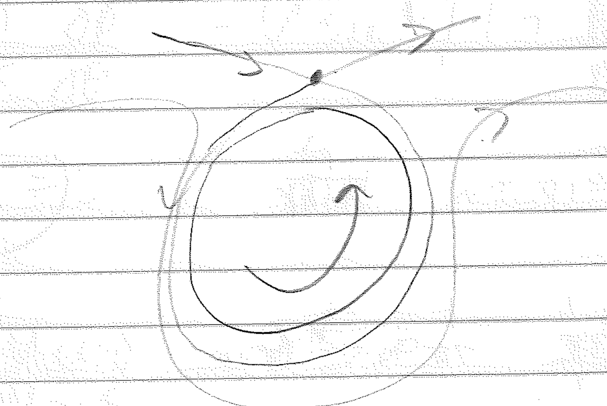
$$\frac{\Gamma}{4\pi U R} < 1$$



$$\frac{\Gamma}{4\pi U R} = 1$$



$$\frac{\Gamma}{4\pi U R} > 1$$



asymmetry top/bottom → lift

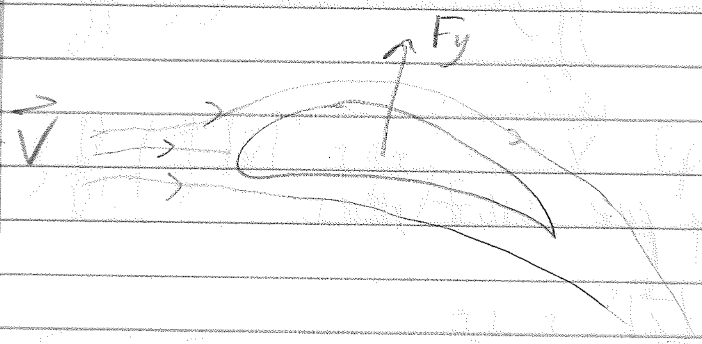
symmetry front/back → no drag

Smits 6.10

Pressure from Bernoulli → compute force

$$F_y = -\rho U \Gamma$$

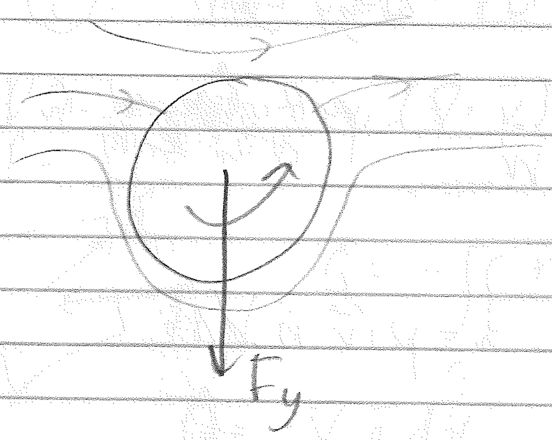
(1) Related to 2D flow over airfoil



$F_y = \rho U \Gamma$ $(F_y < 0)$

(2) Related to flow over a spinning cylinder

→ Magnus effect



$\Gamma > 0$

$F_y = -\rho U \Gamma < 0$

Review

Conservation of mass (continuity)

$$\frac{D\rho}{Dt} + \rho(\nabla\cdot\vec{v}) = 0 \qquad \frac{\partial\rho}{\partial t} + \vec{v}\cdot\nabla\rho$$

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\vec{v}) = 0$$

ρ is not a fn of space

$$\frac{\partial\rho}{\partial t} + \rho\,\nabla\cdot\vec{v} = 0$$

incompressible

$$\nabla\cdot\vec{v} = 0$$

ψ exists is $\nabla\cdot\vec{v} = 0$

$$u = \frac{\partial\psi}{\partial y} \qquad v = -\frac{\partial\psi}{\partial x}$$

ψ = constant $\rightarrow$ streamline $\parallel \vec{v}$

Conservation of momentum (NS)

Incompressible NS

$$\rho\frac{D\vec{v}}{Dt} = \rho\vec{g} - \nabla p + \mu\nabla^2\vec{v}$$

$$\nabla p = \left(\frac{\partial p}{\partial x}, \frac{\partial p}{\partial y}, \frac{\partial p}{\partial z}\right)$$

$$\nabla\cdot\vec{v} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\cdot(u, v, w)$$

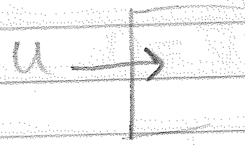
$$\nabla^2\vec{v} = \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)\hat{i} + \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right)\hat{j} + \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right)\hat{k}$$

$$(\vec{V}\cdot\nabla)\vec{V} = \left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right)\hat{i} + \left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right)\hat{j} + \left(u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right)\hat{k}$$

BCs

no slip $\vec{V}\big|_{wall} = \vec{0}$

$\vec{V}\big|_{inlet} = (U,0,0)$



Heat equation

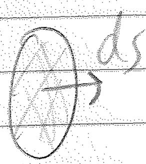
$$\frac{DT}{Dt} = K\nabla^2 T$$

Flow rate

$$Q = \iint_S \vec{v}\cdot\hat{n}\, dS$$

in cylindrical coordinates: $r\,dr\,d\theta$

Pressure force



$$F = -\iint p\,\hat{n}\, dS$$

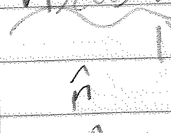
Viscous force

1) shear stress $\underline{\underline{\tau}} = \begin{bmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{bmatrix}$

symmetric

$\tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$

$\tau_{xx} = 2\mu\frac{\partial u}{\partial x}$



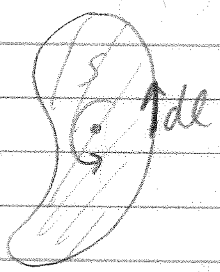
$$\vec{F}_v = \iint \underline{\underline{\tau}}\cdot\hat{n}\, dS$$

e.g. $\hat{n} = \hat{j}$

$$\underline{\underline{\tau}} \cdot \hat{j} = \tau_{yx}\hat{i} + \tau_{yy}\hat{j} + \tau_{yz}\hat{k}$$

Circulation

$$\Gamma = \oint \vec{V} \cdot d\vec{l} = \iint_S \xi \cdot \hat{n}\, dS$$



Vorticity

$$\xi = \nabla \times \vec{V} = 2\vec{\omega}$$

$\vec{\omega}$ rotation rate

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ u & v & w \end{vmatrix}$$

in 2D: $\xi = \xi_z \hat{k}$

$$= \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

Irrotational flow: $\nabla \times \vec{V} = 0$

$$\vec{V} = \nabla \phi$$

+ incompressible $\rightarrow$ $\nabla^2 \phi = 0$

ϕ = constant $\perp$ ψ = constant

Bernoulli: incompressible, irrotational (inviscid)

$$\frac{\partial \phi}{\partial t} + \frac{|\vec{v}|^2}{2} + \frac{p}{\rho} = H(t) = H_0$$

Lecture 15 Oct 29, 2018

$$\phi = A\,(x^3y - xy^3)$$

a) Dimensions of A?

$$\vec{V} = \nabla\phi$$

$$\left[\frac{m}{s}\right] = \frac{1}{m}\,\phi \qquad \phi = \left[\frac{m^2}{s}\right]$$

$$\left[\frac{m^2}{s}\right] = [A][m^4]$$

$$A = \left[\frac{1}{m^2 \cdot s}\right]$$

b) $\vec{V} = ?$

$$\vec{V} = \nabla\phi$$

$$u = \frac{\partial\phi}{\partial x} = 3Ax^2y - Ay^3$$

$$v = \frac{\partial\phi}{\partial y} = Ax^3 - 3Axy^2$$

c) incompressible?

$$\nabla \cdot v = 0$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$6Axy + (-6Axy) = 0 \checkmark$$

d) $u = \frac{\partial\psi}{\partial y} \qquad v = -\frac{\partial\psi}{\partial x} \qquad \int u\,dy = \frac{3Ax^2y^2}{2} - \underbrace{\frac{Ay^4}{4}}_{g(y)} + f(x)$

* don't forget c! $\boxed{\psi = \frac{3Ax^2y^2}{2} - \frac{Ay^4}{4} - \frac{Ax^4}{4} + C}$ $\qquad -\int v\,dx = -\left(\underbrace{\frac{Ax^4}{4}}_{f(x)} - \frac{3Ax^2y^2}{2}\right) + g(y)$

e) $p(0,0) = p_0$, neglect gravity

Find pressure.

$\nabla \times \vec{v} = 0$ irrotational

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = (3Ax^2 - 3Ay^2) - (3Ax^2 - 3Ay^2) = 0 \checkmark$$

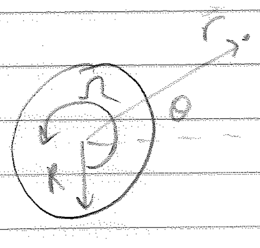
$$H(t) = \frac{p}{\rho} + \frac{1}{2}|\vec{v}|^2 + \cancel{gz}^{\,0} + \cancel{\frac{\partial \phi}{\partial t}}^{\,0} = H_0$$

$$|\vec{v}|^2 = \left(\sqrt{u^2+v^2}\right)^2 = u^2+v^2$$

$\vec{v}(0,0) = 0$ $\qquad \frac{p_0}{\rho} + 0 = H_0$

$$p = p_0 - \frac{1}{2}\rho(u^2+v^2)$$

$$p = p_0 + \frac{1}{2}\rho\left(3x^2y^2(x^2+y^2) + x^6 + y^6\right)$$



2D, viscous, incompressible, far from cylinder $\vec{v} = \vec{0}$, steady

$v_\theta(r,\theta) = ?$ $\qquad$ BCs: $v_\theta\big|_{r\to\infty} = 0$, $v_r\big|_{r\to\infty} = 0$

no slip $\begin{cases} v_\theta\big|_{r=R} = R\Omega \\ v_r\big|_{r=R} = 0 \end{cases}$