Modeling the Reynolds Stresses

Consider a laminar shear flow of a Newtonian fluid in incompressible flow, where the flow field is given by

$$\mathbf{v} = [u(y), 0, 0]\,,$$

where u is some arbitrary function of y. The important shear stress for this problem is given by

$$\tau_{xy} = \mu\left\{\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right\} = \mu\frac{\partial u}{\partial y} = \rho\nu\frac{\partial u}{\partial y}\,. \tag{1}$$

Here μ is the dynamic (molecular) viscosity, and $\nu = \mu/\rho$ is the kinematic (molecular) viscosity.

The kinetic theory of gases for simpler monatomic gases, e.g., hydrogen, can be used to give an exact prediction of ν, which is

$$\nu = \mathcal{C}\,\ell\,\mathcal{U}\,, \tag{2}$$

where $\mathcal{C}$ is a constant of order one, predicted by the theory, ℓ is the mean free path between the gas the molecules, and $\mathcal{U}$ is the sound speed. A major assumption in this theory is that

$$\frac{\ell}{\ell_u} \ll 1\,, \tag{3}$$

where ℓ_u is a length scale of the flow, for example the jet diameter for a jet flow, and the boundary layer thickness for a boundary layer flow. This is related to the assumption required for the continuum approximation to hold.

Now consider a turbulent, incompressible shear flow, where the average flow field is given by

$$\bar{\mathbf{v}} = (\bar{u}(y), 0, 0)\,,$$

where $\bar{u}$ is an arbitrary function of y. From the notes and the handout on the time-averaged equations of motion, the important shear stress for this case is the combination

$$\bar{\tau}_{xy} - \rho\overline{u'v'}\,,$$

where

$$\bar{\tau}_{xy} = \mu\frac{\partial \bar{u}}{\partial y}\,,$$

and $-\rho\overline{u'v'}$ is the Reynolds stress. The problem is how to model this Reynolds stress.

In analogy with laminar flow and with Equation (1), it is often hypothesized that the Reynolds stress is given by

$$-\rho\overline{u'v'} = \mu_T\frac{\partial \bar{u}}{\partial y} = \rho\nu_T\frac{\partial \bar{u}}{\partial y}\,, \tag{4}$$

where now μ_T and ν_T are called turbulent, or eddy, viscosities. Note that

- μ (or ν) is a thermodynamic quantity depending on the properties of the fluid, mainly the temperature, but
- μ_T (or ν_T) depends on the properties of the turbulent flow (coming from the averaged quantity $\overline{u'v'}$) and, as we shall see, cannot be justified in the same manner as a molecular viscosity.

- We need a theory or a model for μ_T, or ν_T.

Unfortunately no deductive theory exists for μ_T or ν_T; however, some models have been developed which, when used with caution, can be very useful. To model ν_T, we follow Equation (2) and hypothesize that ν_T can be written in the form:

$$\nu_T = \mathcal{C}_T \ell_T \mathcal{U}_T \,, \tag{5}$$

where now

- $\mathcal{C}_T$ is a constant, not predicted by the modeling, but ultimately determined by matching the modeling results with data (like curve fitting);
- ℓ_T is a length scale of the turbulence, which is the source of the Reynolds stress,
- and $\mathcal{U}_T$ is a velocity, usually taken to be related to $\overline{u'^2}^{1/2}$, where $u' = u - \bar{u}$ (see the notes on the $\bar{k}$-ϵ modeling to follow).

Unfortunately, it is found experimentally that

$$\ell_{\bar{u}} \sim \ell_T \,,$$

i.e., the scale ℓ_T of the mechanism which is a source of the turbulent viscosity is of the same order as the scale of the turbulent average velocity itself, $\ell_{\bar{u}}$. This means that the inequality given by Equation (3), which is needed for Equation (2) to be valid, does not hold. This implies that assumptions given by Equations (4) and (5) are wrong in principle. However the models using them are often very useful in practice. But the user must be very aware of their limitations.

There are a number of models for ℓ_T and $\mathcal{U}_T$, some simpler and some more complicated. A principal model implemented in Fluent, STAR-CCM+, and other commercial codes, and employed by many users of turbulence models, is called the $\bar{k}$-ϵ model, and will be explained next.