There are some very good photographs of shock waves in the book *An Album of Fluid Motion* by Milton Van Dyke, which is on the course website. See especially Chapters 9 (subsonic flow), 10 (shock waves), and 11 (supersonic flow). It is interesting to follow the photographs of the 'projectile', starting with a Mach number, based upon the projectile speed, of 0.84 in Chapter 9, and ending with the Mach number of 1.015 in Chapter 11.

A shock wave is a region of a flow where the flow speed goes from supersonic to subsonic across a very thin, almost discontinuous, layer. To understand shock waves, begin by considering the control volume given in Figure 1. It is assumed that the shock is flat. Furthermore, the control volume is assumed to move with the shock so that, in this frame of reference, the flow is steady. Furthermore, it is assumed that the flow is normal to the shock (oblique shocks will be considered later), and that the area of the section of the control volume normal to the flow is $\mathcal{A}$.

Assume that the conditions upstream of the shock (to the left) are given, that is: V_1, the shock speed; p_1, the upstream, ambient pressure; ρ_1, the upstream, ambient density; and T_1, the upstream, ambient temperature. Given these upstream conditions, it is of interest, then, to determine what the downstream conditions are, i.e.: to determine V_2, p_2, ρ_2, and T_2. To do this, the conservations of mass and energy, the momentum balance, and the gas laws are considered.

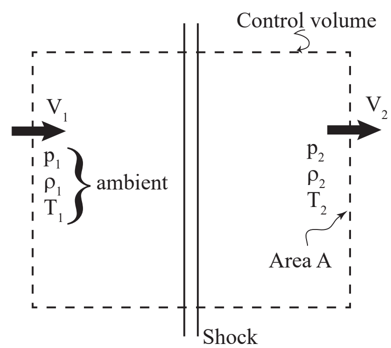

Figure 1: Sketch of shock wave geometry.

With the control volume moving with the shock as shown, the problem is steady-state and one-dimensional. The continuity equation is:

$$\underbrace{\frac{d}{dt}\int_{CV} \rho d\mathcal{V}}_{=0} = -\int_{CS} \rho(\mathbf{V}\cdot\mathbf{n})dA\,,\text{ or} \tag{1}$$

$$0 = -(-\rho_1 V_1 \mathcal{A} + \rho_2 V_2 \mathcal{A})\,,\text{ or, finally}$$

$$\rho_1 V_1 = \rho_2 V_2\,. \tag{2}$$

Assuming steady-state and one-dimensional, the momentum equation in the x (flow) direction becomes,

- neglecting gravity (unimportant here);
- neglecting viscous forces at the left and right boundaries, assuming that the flow is uniform there, so that
- there are only pressure forces at the right and left boundaries, then

$$\underbrace{\frac{d}{dt}\int_{CV} \rho u d\mathcal{V}}_{=0} = F_x - \int_{CS} \rho u (\mathbf{V}\cdot\mathbf{n}) dA \,, \text{or} \tag{3}$$

$$0 = (p_1 - p_2)\mathcal{A} - (-\rho_1 V_1 V_1 \mathcal{A} + \rho_2 V_2 V_2 \mathcal{A}) \,, \text{ or}$$

$$-\rho_1 V_1^2 + \rho_2 V_2^2 = p_1 - p_2 \,, \text{ i.e.,}$$

$$p_1 + \rho_1 V_1^2 = p_2 + \rho_2 V_2^2 \,. \tag{4}$$

Note that the latter equation, although similar, is not Bernoulli's equation.

Defining e as the total energy per unit mass, which in this case includes the kinetic $[(1/2)|\mathbf{V}|^2]$ and the internal ($\mathcal{U}$) energies per unit mass, so that $e = \mathcal{U} + (1/2)|\mathbf{V}|^2$, then the energy equation for this steady-state, one-dimensional problem is:

$$\underbrace{\frac{d}{dt}\int_{CV} \rho e d\mathcal{V}}_{=0} = \dot{Q} + \dot{W} - \int_{CS} \rho e (\mathbf{V}\cdot\mathbf{n}) dA \,. \tag{5}$$

Assuming

- no heat transfer to the control volume, i.e., $\dot{Q} = 0$,
- the only work is done by the pressure forces at the control surfaces, then the energy equation simplifies to

$$0 = -\int_{CS} \rho (e + \frac{p}{\rho})(\mathbf{V}\cdot\mathbf{n}) dA \,. \tag{6}$$

(See the handout entitled "Notes on the energy equation" for more explanation of this.)

Note that $e + \frac{p}{\rho} = \mathcal{U} + \frac{1}{2}|\mathbf{V}|^2 + \frac{p}{\rho} = h + \frac{1}{2}|\mathbf{V}|^2$, where the enthalpy h is given by $h = \mathcal{U} + \frac{p}{\rho}$, so that Equation (6) becomes, using Equation (2),

$$0 = -\rho_1 V_1 \mathcal{A}(h_1 + \frac{1}{2}V_1^2) + \rho_2 V_2 \mathcal{A}(h_2 + \frac{1}{2}V_2^2) \,, \text{ or}$$

$$h_1 + \frac{1}{2}V_1^2 = h_2 + \frac{1}{2}V_2^2 \,. \tag{7}$$

For an idea gas, the enthalpy is given by $h = c_p T$, where c_p is the specific heat at constant pressure, and T is the temperature. Introducing this into Equation (7), assuming that c_p is constant, and dividing by c_p gives:

$$T_1 + \frac{V_1^2}{2c_p} = T_2 + \frac{V_2^2}{2c_p} \,, \tag{8}$$

a reduced form of the conservation of energy. Note that with the total temperature defined by

$$T_t = T + \frac{V^2}{2c_p} \,,$$

the energy equation, Equation (8), may be written as

$$T_{t1} = T_{t2} \,,$$

i.e., the total temperature is conserved. This is analogous to Bernoulli's equation for incompressible flows, i.e.,

$$p + \frac{1}{2}\rho|\mathbf{V}|^2 = p_0 = \text{constant}\,,$$

where p_0 is the total pressure.

Up to this point there are three equations, Equations (2), (4), and (8), with four unknowns, T_2, V_2, ρ_2, and p_2. The final equation is the gas law, which for an idea gas is:

$$p_2 = \rho_2 R T_2\,, \tag{9}$$

where R is the gas constant.

To work with these equations, it is useful to introduce the local Mach number $M = V/c$, where the local speed of sound is given by $c = \sqrt{\gamma R T}$, where $\gamma = c_p/c_v$ is the ratio of the specific heats. In terms of the local Mach number, the momentum equation, Equation (4), becomes, using the gas law:

$$\begin{gathered}
p_1 + \rho_1 V_1^2 = p_2 + \rho_2 V_2^2\,, \text{ or} \\
p_1 + \frac{\gamma p_1}{\gamma R T_1} = p_2 + \frac{\gamma p_2}{\gamma R T_2} V_2^2\,, \text{ or} \\
p_1(1+\gamma M_1^2) = p_2(1+\gamma M_2^2)\,, \text{ or, finally} \\
\frac{p_1}{p_2} = \frac{1+\gamma M_1^2}{1+\gamma M_2^2}\,. \qquad (10)
\end{gathered}$$

Later we will find that, across a shock, $M_1 > 1$ and $M_2 < 1$, so that $p_2 > p_1$, i.e., the pressure increases across a shock.

From the energy equation, Equation (8),

$$T_1 + \frac{V_1^2}{2c_p} = T_2 + \frac{V_2^2}{2c_p}\,, \text{ or}$$

$$T_1 + \frac{T_1 V_1^2}{T_1 2c_p} = T_2 + \frac{T_2 V_2^2}{T_2 2c_p}\,.$$

But with $\gamma = c_p/c_v$ and $R = c_p - c_v$,

$$\begin{gathered}
\frac{V^2}{2c_p T} = \frac{\gamma R V^2}{2c_p(\gamma R T)} = \frac{(c_p/c_v)(c_p - c_v)}{2c_p} M^2 = \frac{\gamma - 1}{2} M^2\,, \text{ so} \\
T_1\left(1 + \frac{\gamma-1}{2} M_1^2\right) = T_2\left(1 + \frac{\gamma-1}{2} M_2^2\right), \text{ or, finally} \\
\frac{T_2}{T_1} = \frac{1 + \frac{\gamma-1}{2} M_1^2}{1 + \frac{\gamma-1}{2} M_2^2}\,. \qquad (11)
\end{gathered}$$

Again, since $M_1 > 1$ and $M_2 > 1$ across a shock, then $T_2 > T_1$, i.e., the temperature increases across a shock [noting that $(\gamma - 1) > 0$].

Finally, from the continuity equation, Equation (2), and using the ideal gas law and the definition of the sound speed,

$$\frac{p_1}{RT_1}\frac{V_1}{\sqrt{\gamma R T_1}}\sqrt{\gamma R T_1} = \frac{p_2}{RT_2}\frac{V_2}{\sqrt{\gamma R T_2}}\sqrt{\gamma R T_2}\,, \text{ or, finally}$$

$$\frac{p_1}{p_2}\frac{M_1}{M_2} = \sqrt{\frac{T_1}{T_2}}\,. \tag{12}$$

Equations (10) and (11) can be substituted into Equation (12) to give an equation for M_2 in terms of M_1. Once M_2 is determined, then Equation (10) can be used to determine p_2, and Equation (11) can be used to find T_2. Furthermore, given p_2 and T_2, the ideal gas law, Equation (9), can be used to solve for ρ_2, and the continuity equation, Equation (2), can be used to solve for V_2.

The resulting equation for M_2 in terms of M_1 is quadratic. One solution is the trivial one, $M_2 = M_1$, i.e., there is no shock. The other solution is:

$$M_2^2 = \frac{(\gamma-1)M_1^2 + 2}{2\gamma M_1^2 - (\gamma-1)}\,. \tag{13}$$

Note that this equation is symmetric in M_1 and M_2, i.e., if M_1 is solved for in terms of M_2, the exact same equation would be obtained, with M_1 and M_2 interchanged. Note also that this is true of all of the equations obtained up to this point; there is no distinction in the equations themselves between the quantities upstream and downstream of the shock. Finally, note that Equation (13) implies that if M_1 is greater than (less than) 1, then M_2 is less than (greater than) 1. Equation (13) is plotted in Figure 2.

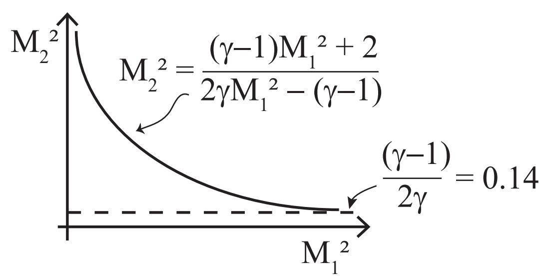

Figure 2: Plot of M_2^2 versus M_1^2 given by Equation (13).

As mentioned above, once the solution for M_2 in terms of M_1 is obtained, then the solution for any downstream quantity can be obtained. For example, substituting Equation (13) into Equation (10) gives, after some considerable algebra:

$$\frac{p_2}{p_1} = 1 + \frac{2\gamma}{(\gamma+1)}(M_1^2 - 1) \doteq 1 + 1.17(M_1^2 - 1)\,, \tag{14}$$

assuming that $\gamma = 7/5 = 1.4$. Note that, if $M_1 > 1$, i.e., so that oncoming flow is supersonic, then $p_2 > p_1$. Furthermore, if $M_1 = 1.5$, 2.0, and 2.5, then $p_2/p_1 = 2.46$, 4.51, and 7.14. Note, however, that for M_1 in the range of 2 or more, then the oncoming flow would be hypersonic, and more complicated thermodynamics would have to be considered. Also note that this equation is usually written in the form:

$$\frac{p_2 - p_1}{p_1} = \frac{\Delta p}{p_1} = \frac{2\gamma}{(\gamma+1)}(M_1^2 - 1)\,,$$

where $\Delta p/p_1 = (p_2 - p_1)/p_1$ is called the 'shock strength', which is often used to indicate the strength of a shock wave.

As an example of the change in pressure, consider the effect of a shock wave moving through a standard atmosphere at $M_1 = 1.5$. The standard atmospheric pressure is $p_a = p_1 = 101.3\,\text{kPa}$, and the change is pressure is given by $p_2/p_1 = 2.46$. Therefore $p_2 = 2.46 \cdot p_1 = 249.27\,\text{kPa}$, a very large pressure. This is large but not extreme. For example, the pressure on the ocean floor where the depth is about 1 mile (the depth of the Deepwater Horizon oil spill) is approximately $p_{of} = p_a \cdot 153$!

Using Equations (2) and (9), and Equation (13), after some algebra it can be found that:

$$\frac{\rho_2}{\rho_1} = \frac{V_1}{V_2} = \frac{(\gamma+1)M_1^2}{(\gamma-1)M_1^2+2}\,,$$

which gives $\rho_2 > \rho_1$ and $V_2 < V_1$ for $M_1 > 1$.

Up to this point, two possibilities exist, either $M_1 > 1$ and $M_2 < 1$, or vice-versa, $M_1 < 1$ and $M_2 > 1$, and the equations can be satisfied by either possibility. To determine which one of these is physically possible, the second law of thermodynamics and entropy must be brought in. For an idea gas with constant specific heats, i.e., constant c_p and c_v, the change in entropy per unit mass, s, when a system is taken from state 1 to state 2, is given by

$$s_2 - s_1 = c_p \ln(T_2/T_1) - R\ln(p_2/p_1)\,. \tag{15}$$

Some estimate of how the entropy will change near $M_1 = 1$ can be obtained in the following way.

- Using the expressions for p_2/p_1 and T_2/T_1 in terms of M_1^2, as in Equation (14);
- writing $M_1^2 - 1 = \epsilon$, where ϵ is assumed to be 'small',
- expanding the resulting logarithms as

$$\ln(1+\epsilon) = \epsilon - \frac{\epsilon^2}{2} + \cdots\,,$$

it is found that, after considerable algebra,

$$\frac{s_2 - s_1}{R} = \frac{2\gamma}{(\gamma+1)^2}\frac{(M_1^2-1)^3}{3} + \text{higher order terms in } (M_1^2-1)\,.$$

Since the entropy cannot decrease for this adiabatic flow, then $s_2 > s_1$, and $M_1^2 - 1 > 0$, or $M_1 > 1$ is the only solution which is possible, consistent with the second law of thermodynamics. The same result can be found for larger values of $M_1^2 - 1$ by numerically evaluating Equation (15). Furthermore, it can be shown that the increase in entropy as the flow goes through the shock can be related to the conversion of mechanical energy into internal energy. That is, mechanical energy is converted into random (disordered) molecular energy, resulting in large increases in the pressure, temperature, and entropy.

When the coordinate system is switched to moving with the upstream (ambient) fluid, the shock wave propagates to the left into the ambient fluid with a speed V_1 such that $V_1/c_1 = M_1 > 1$, i.e., the shock was is traveling faster than the speed of sound. For a very weak shock, however, $\Delta p/p \ll 1$. But with

$$\frac{\Delta p}{p} = \frac{2\gamma}{(\gamma+1)}(M_1^2 - 1) \ll 1\,,$$

then $M_1 \doteq 1$, i.e., the shock travels at the speed of sound (it is a sound wave).