Surface forces are due to direct contact of surfaces, and are proportional to the surface area. For example, consider one surface 'sliding' over another, which for a fluid would produce a shear force due to the viscosity of the fluid (see Figure 1). But it would also produce a pressure force applied normal to the same surface.

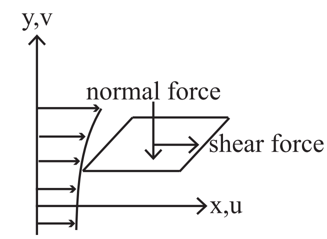

Figure 1: Forces of the fluid in the upper layer on the fluid in the lower layer.

To get the total contribution to the forces on a differential control volume we need to sum the forces on all six surfaces of the control volume. On each surface there can be 3 force components (see Figure 2). To mathematically describe these forces we need a 9 component stress matrix (tensor):

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_y x & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix} .$$

The components of this matrix are called (surface) stresses.

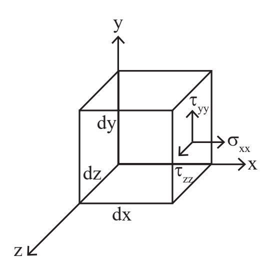

Figure 2: Example of surface stresses needed to define surface forces.

Important properties of these stresses are the following.

1. Subscript notation: $\tau_{xy} dydz$ is a force in the y-direction on a surface of area $dydz$ whose normal is in the x-direction. (Be careful in this notation when reading other books and papers; sometimes the indices are reversed.) Similar notation is used for the other components of the matrix $\boldsymbol{\sigma}$.

2. Definitions: σ_{xx}, σ_{yy}, and σ_{zz}, the diagonal components, are called the normal stresses. The off-diagonal components, τ_{xy}, etc., are called the shear stresses.

3. Sign convention: a components of $\boldsymbol{\sigma}$ (e.g., τ_{xy}, etc.) is positive if the force vector component and the area normal are either both positive or both negative. This is needed for consistency with Newton's third law (see Figure 3, which examines the normal force of volume *1* on volume *2*, and vice-versa).

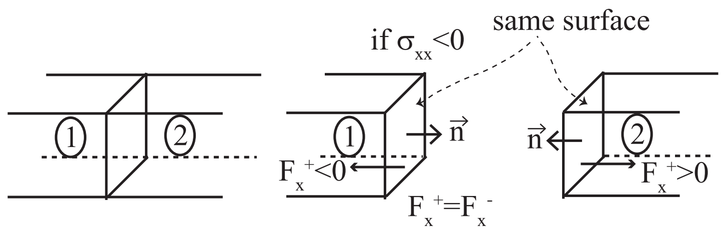

Figure 3: Example of the application of the sign convention.

Note that since pressure is always directed inwards (compression), opposite to the direction of the outward normal, then from the sign convention it is always a negative normal stress, and written as $-p$.

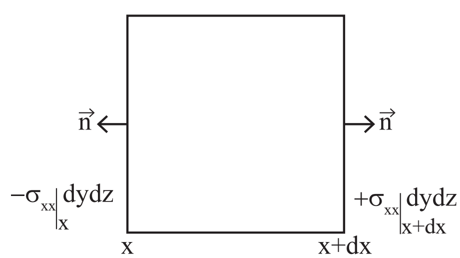

Figure 4: Normal stresses for forces in the s-direction on surfaces in the x-direction.

Consider the force on the cube in Figure 4 in the x-direction. Contributions with come from all 6 surfaces. The force on the left face is:

$$-\sigma_{xx}\Big|_x dydz\,.$$

Note that the (-) sign is needed since if σ_{xx} is positive, the force must be in the $-x$-direction (i.e., it must be negative) since the surface normal is in the $-x$-direction, consistent with the sign convention. The force on the right face is, using a Taylor series expansion:

$$\sigma_{xx}\Big|_{x+dx} dydz = \left(\sigma_{xx} + \frac{\partial \sigma_{xx}}{\partial x}\Big|_x dx\right) dydz\,.$$

Therefore the total contribution of the surface force in the x-direction, due to surfaces with their normals in the x (or -x) direction, is:

$$\frac{\partial \sigma_{xx}}{\partial x} dxdydz\,.$$

Therefore, for example, if $\dfrac{\partial \sigma_{xx}}{\partial x} = 0$, i.e., σ_{xx} is locally uniform, there is no net force on the cube in the x-direction due to the left and right faces, as the forces balance. On the other hand, if, for example, $\sigma_{xx} > 0$ and $\dfrac{\partial \sigma_{xx}}{\partial x} > 0$, there would be a net force in the $+x$ direction (see Figure 5).

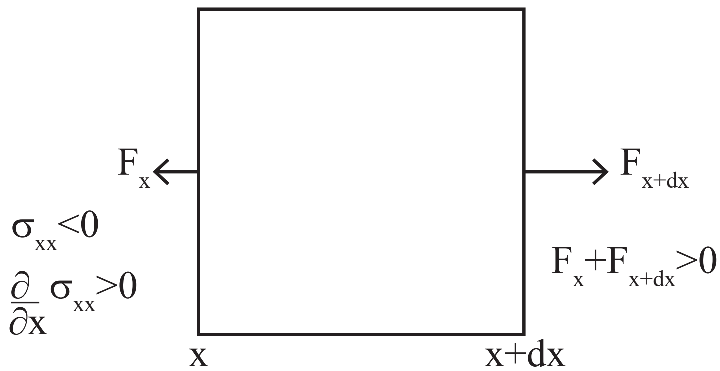

Figure 5: Net normal force is in the positive x-direction if $\sigma_{xx} > 0$ and $\partial\sigma_{xx}/\partial x > 0$.

In a similar manner, if we compute the contributions to the forces in the x-direction due to the surfaces with normals in the y-direction (see Figure 6), we would obtain:

$$\frac{\partial \tau_{yx}}{\partial y} dxdydz \, ,$$

and force surfaces wit normals in the z-direction:

$$\frac{\partial \tau_{zx}}{\partial z} dxdydz \, .$$

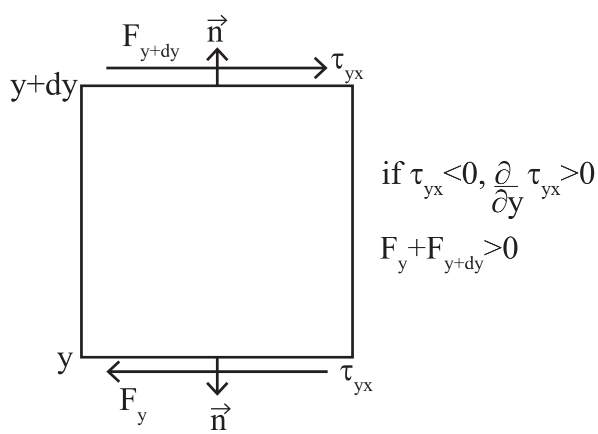

Figure 6: Net normal force is in the positive x-direction if $\tau_{yx} > 0$ and $\partial\tau_{yx}/\partial x > 0$.

Therefore, for the x-component of the momentum equation we obtain, dividing out $dxdydz$:

$$\underbrace{\rho\frac{Du}{Dt}}_{\substack{\text{rate-of-change}\\ \text{following fluid}}} = \underbrace{\rho g_x}_{\text{body force}} + \underbrace{\left(\frac{\partial}{\partial x}\sigma_{xx} + \frac{\partial}{\partial y}\tau_{yx} + \frac{\partial}{\partial z}\tau_{zx}\right)}_{\text{surface forces}} \quad \text{(per unit volume)}\,,$$

where $\dfrac{D}{Dt} = \dfrac{\partial}{\partial t} + u\dfrac{\partial}{\partial x} + v\dfrac{\partial}{\partial y} + w\dfrac{\partial}{\partial z}$ is the substantial, or material, derivative. Similar arguments for the y- and z-components of the momentum equation lead to:

$$\rho\frac{Dv}{Dt} = \rho g_y + \left(\frac{\partial}{\partial x}\tau_{xy} + \frac{\partial}{\partial y}\sigma_{yy} + \frac{\partial}{\partial z}\tau_{zy}\right), \text{ and}$$

$$\rho\frac{Dw}{Dt} = \rho g_z + \left(\frac{\partial}{\partial x}\tau_{xz} + \frac{\partial}{\partial y}\tau_{yz} + \frac{\partial}{\partial z}\sigma_{zz}\right).$$

So far we have the following equations:

- conservation of mass: 1 equation
- momentum balance: 3 equations
- conservation of energy: 1 equation (discussed later)

We can also add the angular momentum balance, which can be shown to lead to the fact that $\boldsymbol{\sigma}$ is a symmetric matrix, reducing the number of its components from 9 to 6; this will be considered later. So we have at most 8 equations. But we have at least 13 unknowns:

$$\rho\,,\ u\,,\ v\,,\ w\,,\ \sigma_{xx}\,,\ \tau_{xy}\,,\ldots$$

Therefore we have more unknowns than equations so that we don't have a well-posed mathematical problem yet; we still need more information. The equations at this point are valid for any continuum, e.g., a solid, a fluid, a visco-elastic material, etc. We next have to bring in the properties of the medium that are particular to a fluid, i.e.,

1. Constitutive equations, giving the mechanical properties of the medium. For example, for a linearly elastic solid, this is Hooke's law that the stress is a linear function of the strain.

2. Gas laws, i.e. the thermodynamic properties of the medium, e.g., $p = \rho RT$. Note that these are not needed for an incompressible fluid.

The subject of the constitutive equations for a fluid is complex, and can be made the topic of a graduate course. Here the results will be given, along with some motivation, but without any derivation. More complex discussions are given in ME503, the graduate course on continuum mechanics, and ME507, the graduate course on fluid mechanics.

Recall that, by definition, a fluid is a material which cannot support a shear stress without continuously deforming. So if a shear stress is applied to it, it will continue to deform. If it is not deforming, there are no shear stresses. Therefore, when a fluid is static, it can only support normal stresses. When the fluid is static, this normal stress is: (i) compressive, i.e., the force is opposite to the outward normal of the surface (the stress is negative); (ii) at a given point, the same force is applied over the same area in any given direction; and (iii) this normal stress is the usual static pressure. (It is the thermodynamic pressure for a compressible fluid.) Therefore, when the fluid is static,

$$\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = -p \, , \; \tau_{xy} = \tau_{xz} = \ldots = 0 \, ,$$

that is, all of the shear stresses are 0.

This same pressure is assumed to exist for moving fluids, and is separated from the rest of the stresses, i.e.,

$$\boldsymbol{\sigma} = \begin{bmatrix} -p + \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_y x & -p + \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & -p + \tau_{zz} \end{bmatrix}$$

Here the matrix $\boldsymbol{\tau}$, composed of τ_{xx}, τ_{xy}, etc., represents the remainder of the surface stresses after the pressure has been removed, and is due to the viscosity of the fluid.

For a solid, the stresses are functions of the strain, i.e., of the relative displacements, e.g., in Hooke's law where stress $\propto$ strain. For a fluid, to be consistent with the definition of a fluid discussed above, the stresses are functions of the rate-of-strain, i.e., the rate-of-change of the relative displacements. Relative motion is needed to have viscous stresses. For example, in a simple shear flow (see Figure 7), the rate-of-strain is related to $\dfrac{\partial u}{\partial y}$, the spatial derivative of u in the y direction.

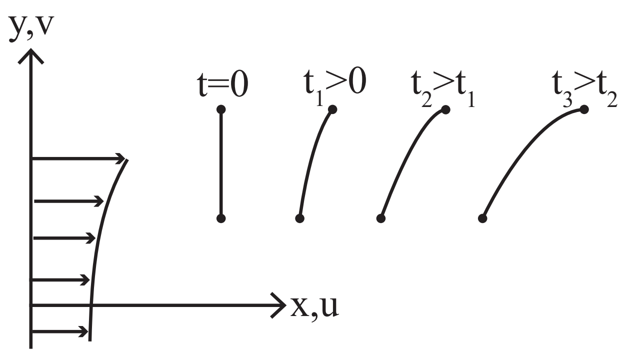

Figure 7: Strain-rate due to a shearing flow.

For most fluids of interest, e.g., air, water, etc., the stresses are linear functions of the rate-of-strains, which is the basic assumption for a Newtonian fluid. For an incompressible flow, this leads to the following relationships between $\boldsymbol{\tau}$ and $\mathbf{V} = (u, v, w)$:

$$\tau_{xx} = 2\mu \frac{\partial u}{\partial x} \, , \; \tau_{yy} = 2\mu \frac{\partial v}{\partial y} \, , \; \tau_{zz} = 2\mu \frac{\partial w}{\partial z} \, ,$$

$$\tau_{xy} = \tau_{yx} = \mu \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) , \; \tau_{xz} = \tau_{zx} = \mu \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) , \; \tau_{yz} = \tau_{zy} = \mu \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) .$$

The constant of proportionality, μ, is the fluid viscosity.

Note the following facts about the viscous stresses $\boldsymbol{\tau}$:

- The shear stress matrix $\boldsymbol{\tau}$ is a symmetric matrix, e.g., $\tau_{xy} = \tau_{yx}$. This is due to the angular momentum balance, assuming that there are no internal moments in the fluid, e.g., due to magnetic effects.
- The equations can be derived from the kinetic theory of gases for simple fluids, e.g., hydrogen.
- For more general fluids, including liquids, these relations are postulated, based upon physical reasoning and mathematical constraints.
- Comparisons between predictions using these relationships and experiments are generally very good, lending confidence in them.

When these equations are plugged into the momentum equation there results, for an incompressible fluid (check this):

$$\rho \frac{Du}{Dt} = \rho g_x - \frac{\partial p}{\partial x} + \mu \nabla^2 u \, ,$$

where $\nabla \cdot (\nabla u) = \nabla^2 u = \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2}$, and $\nabla^2 u$ is called the LaPlacian of u. Similarly, the y and z components of the momentum equation are:

$$\rho \frac{Dv}{Dt} = \rho g_y - \frac{\partial p}{\partial y} + \mu \nabla^2 v \, ,$$

$$\rho \frac{Dw}{Dt} = \rho g_z - \frac{\partial p}{\partial z} + \mu \nabla^2 w \, .$$

Together with the continuity equation,

$$\nabla \cdot \mathbf{V} = 0 \, ,$$

with ρ and μ known, this gives 4 equations for the 4 unknowns u, v, w, and p. These 4 equations taken together are called the incompressible form of the Navier-Stokes equations, and hold at each point in the flow. For incompressible flows, it is these equations that are solved numerically by codes such as Fluent, STAR CCM+, and COMSOL.

What remains is to discuss initial conditions and boundary conditions.