ME431A/538A/538B Taylor Series[1] 30 September 2019

In several problems that we will be addressing in this course, we will be approximating the value of a function at a point, or the value of its derivative at that point, by information about the function at neighboring points. This issue will usually arise because we work with a function over some interval in y, say $0 \leq y \leq L$, by its values at discrete points on the interval, say at $y_i = (\Delta y)i\,,\, i = 0, 1, 2, \ldots, n$ (where $n = L/\Delta y$ is an integer; see Figure 1). Considering specific discrete points is sometimes referred to as discretizing the interval or, when using numerical methods, in creating a computational mesh.

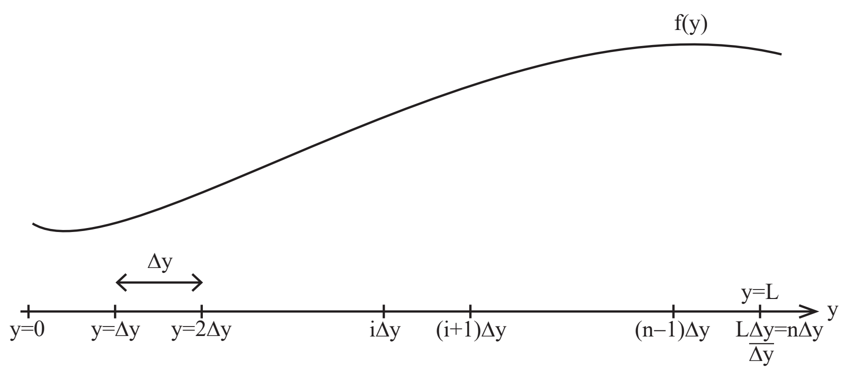

Figure 1: Discretization of the interval $0 \leq y \leq L$.

Suppose that we would like to approximate the value of a function $f(y)$ at a point y, given the value of f and its derivatives at a neighboring point $y - \Delta y$, where it will be assume that, in some sense, Δy is 'small' (see Figure 2). A first approximation might be

$$f(y) \approx f(y - \Delta y)\,.$$

If y and $y - \Delta y$ are close enough, then this might be a good enough approximation. To improve on this, a better approximation might be to also use the slope of f at $y - \Delta y$ to estimate $f(y)$ as

$$f(y) \approx f(y - \Delta y) + f'(y - \Delta y) \cdot (\Delta y)\,, \tag{1}$$

where $f'(y - \Delta y)$ denotes the derivative of f at the point $y - \Delta y$. As seen from Figure 2, this should give a better approximation to the value of $f(y)$. If $f(y)$ is a straight line over the interval $y - \Delta y$ to y, then Equation 1 gives the exact value for $f(y)$, given $f(y - \Delta y)$ and $f'(y - \Delta t)$. However, if $f(y)$ has curvature in the interval, then a better approximation is:

$$f(y) \approx f(y - \Delta y) + f'(y - \Delta y) \cdot (\Delta y) + f''(y - \Delta y) \cdot \frac{(\Delta y)^2}{2!}\,.$$

This process can be improved indefinitely by continually adding new terms based upon the next higher derivative and higher powers of Δy, i.e.,

$$\begin{aligned} f(y) \approx f(y - \Delta y) \quad &+ \quad f'(y - \Delta y) \cdot (\Delta y) + f''(y - \Delta y) \cdot \frac{(\Delta y)^2}{2!} + f'''(y - \Delta y) \cdot \frac{(\Delta y)^3}{3!} \\ &+ \quad \cdots + f^{(n)}(y - \Delta y) \cdot \frac{(\Delta y)^n}{n!} + \cdots \end{aligned}$$

[1]See also Appendix A.10 of Part I of the text.

This series on the right-hand side is called a Taylor series expansion of the function f about the point $(y - \Delta y)$, and is discussed in more detail in most books on calculus. It can be shown that this series converges to the actual value of $f(y)$ as n becomes large, and that the error in truncating the series at the n^{th} term can be estimated as:

$$E_n = f^{(n+1)}(y - \Delta y)\frac{(\Delta y)^{n+1}}{n + 1!}\,.$$

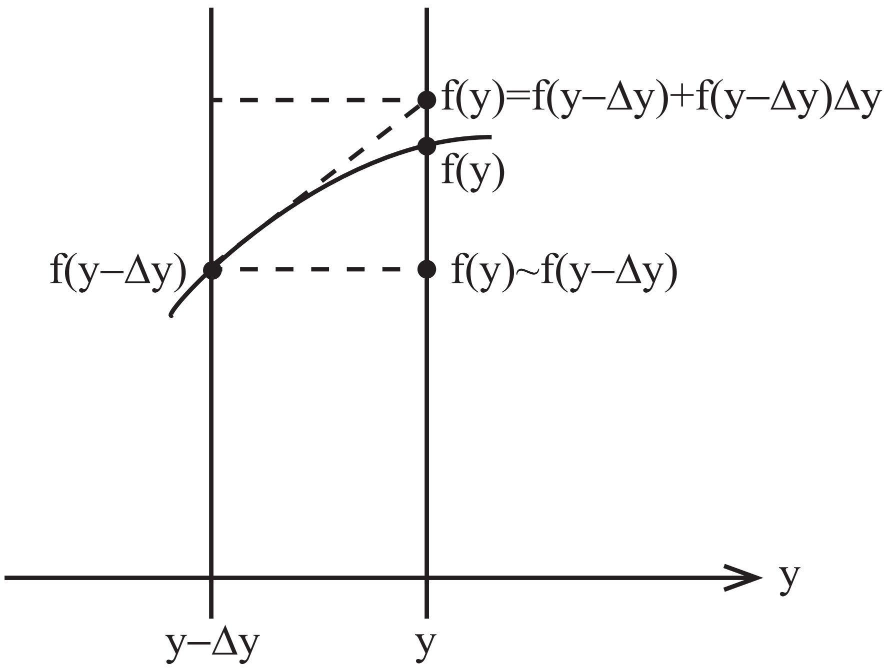

Figure 2: Approximation of $f(y)$ by a Taylor series.

Taylor series expansions will be used in both deriving the differential forms of the control volume equations for the conservation of mass, the momentum balance, and the conservation of energy from their control volume forms, and the expansions will also be used to obtain the finite-volume forms of the equations used in many of the computational fluid mechanics (CFD) codes.