ME 431A/538A/538B Handout 4 November 2019

Introduction to Turbulence

1. Background

Many flows in nature and technology are turbulent, having the feature of being very random, or chaotic looking, as opposed to smooth laminar flows. Some examples are: the power plant plume (a buoyant, turbulent jet, see the video on the course web site); the motion of the clouds in the sky; the boundary layers over cars and airplanes; the flow in internal combustion engines, in gas turbine engines, and in jet engine exhaust; other atmospheric flows, especially near the ground. Some flows, however, are clearly not turbulent, for example, flow in micro-electro-mechanical (MEMS) devices, cardiovascular flows (unless the veins or arteries are diseased), and other low Reynolds number flows.

Turbulent flows cannot be defined by a one- or two-sentence definition, but are defined by their characteristics, some of which follow.

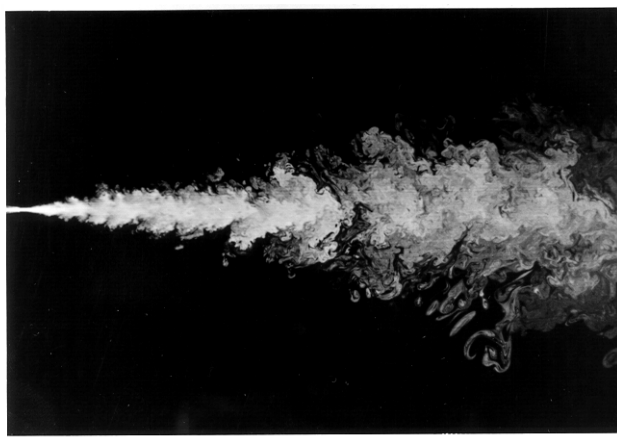

Figure 1: Photograph of a turbulent jet.

- Turbulent flows are highly random, irregular, chaotic, with a broad range of space and time scales (see Figure 1). This suggests introducing averaging, usually time averaging. But note that some flows which are random do not have all of the properties of turbulence. For example, the flow directly under ocean waves is usually random, because the surface waves themselves are random; but the flow is not necessarily turbulent.

- Turbulent flows are unstable flows, so that even small perturbations can significantly affect the flows; however small perturbations do not affect their (time) averages. In fact the reason that a particular flow is turbulent is because a laminar flow under the same conditions would be unstable.

- Turbulent flows are three dimensional. This is related to the fact that in two dimensions, vorticity is conserved, but in three dimensions it is not, especially due to vortex stretching, which does not exist in two dimensions. To see that it is not consider a vortex in the x-direction, as shown in Figure 2. Assume that, locally, the velocity u increases in the x-direction, i.e., that $\partial u/\partial x > 0$. Due to the conservation of angular momentum, as the flow evolves, the area of the vortex, $\mathcal{A}$, times the rotation rate, ζ_x, is conserved, i.e., $\mathcal{A}\zeta_x =$

constant. As the vortex is stretched because $\partial u/\partial x > 0$, then the area $\mathcal{A}$ must decrease to conserve its volume. Therefore ζ_x must increase. That is, the vorticity is increased as the fluid is stretched. It is found that this is a very important mechanism in turbulence, especially in making length scales small and vorticity higher. This implies that you cannot numerically simulate turbulence in two dimensions. We will find, however, that some flows are two-dimensional after averaging (e.g., circular pipe flows, where the averages depend only on r and x).

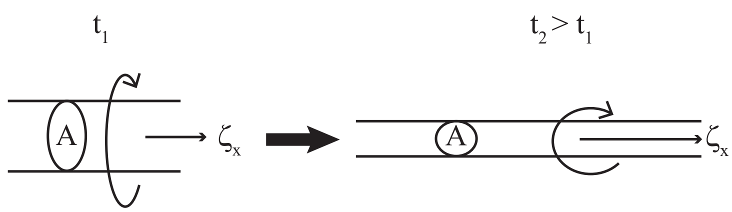

Figure 2: Sketch of vortex stretching.

- Turbulent flows are characterized by high rates of diffusion and mixing of mass, heat, chemical species, etc. This is a very important feature of turbulence in some technological processes. For example, in combustion, turbulence can be extremely helpful in mixing the fuel and air to maintain the combustion process.

- Turbulent flows are highly dissipative, meaning that in a turbulent flow there is a high rate of loss of mechanical energy into heat (internal energy). This implies that there are larger power requirements to maintain a turbulent flow, e.g., in a pipe flow. There are higher drag forces on an object with a turbulent boundary layer, so that the forces to move the object must be stronger. So, for example, there is a strong motivation for reducing the turbulence in the boundary layer over an airplane to reduce the drag on the airplane, and hence decrease the power requirements.

- Turbulent flows satisfy the Navier-Stokes equations, so that the underlying assumptions in their derivation are still valid, i.e.,

  - the continuum approximation,
  - conservation of mass and the momentum balance still hold, and
  - a Newtonian fluid.

Turbulent flows are so complex that mathematical solutions are out of the question. Numerical solutions, however, are becoming more and more useful. There are three general types of numerical approaches to simulate turbulent flows.

- Direct numerical simulation (DNS). In this approach, the three-dimensional, time-dependent Navier-Stokes equations are solved numerically, with all the relevant length and time scales of the flow resolved. This usually requires high-performance computing (HPC) using high-order of accuracy numerical methods. Due to computing limitations, it is difficult to use DNS in complex geometries, and it is mainly used for research in studying problems in simply geometries.

- Large-eddy simulation (LES). In this case, instead of as in DNS, the smaller-scale motions are filtered out and modeled, so that only the larger-scale motions are computed directly. This approach, while often requiring high-performance computing, still has significantly less computing requirements than DNS, and so is beginning to be used in applications. Most commercial codes include modeling for LES, although the numerical methods employed are often not accurate enough for the problems addressed.

- Reynolds-averaged Navier-Stokes (RANS). In this approach the (time) averaged Navier-Stokes equations are solved, instead of the instantaneous equations. This is a much easier and less expensive (in computer resources) computational problem than LES or DNS. An number of theoretically unjustifiable assumptions, however, must be introduced. This is by far the main approach used today in applications, and will be discussed in this course.

2. Averaging

Because of its complexity, averaging, usually time averaging, is almost always introduced in simulating turbulent flows. For example, consider the velocity profiles of turbulent flow in a pipe, as shown in Figure 3. If one uses a modern measurement method, such as PIV (particle image velocimetry) to measure the instantaneous velocity profile in this flow, a profile of the axial velocity $u(r)$, sketched on the left of the figure, might be obtained. This profile is is very complex, characterized by many length scales. If one averages the measurement over time, however, a very smooth profile is obtained, as sketched on the right. This is the profile of the average velocity, which we will denote by $\bar{u}$. The average (mean) velocity $\bar{u}$ is defined as follows.

$$\overline{u(x,y,z,t)} = \frac{1}{T}\int_t^{t+T} u(x,y,z,t')dt' = \bar{u}(x,y,z)\,. \tag{1}$$

Here T, the averaging time, is a time very large compared to the time scales of the flow. Note that $\bar{u}$ is independent of T if T is large enough.

In defining this time average, we are assuming that the flow is 'statistically steady', or 'statistically stationary'. For example, consider the flow in a pipe driven by a pump, as shown in Figure 4. The pump is assumed to be operating at constant conditions, i.e., constant impeller rotation rate. If the Reynolds number is high enough, however, i.e., larger than about 2,300, this pipe flow will be turbulent, and the a probe placed in it will measure an unsteady signal. But one would expect to be able to define a meaningful time average. This is an example of a statistically steady flow, and its average would not depend on time.

Suppose, however, that the impeller rotation rate was slowly decreasing. Then one would expect that the 'average' velocity would keep decreasing with time. In fact the average velocity would strongly depend on the averaging time T. This would be an example of a statistically unsteady flow, and it is not clear what a time average would mean. In this case sometimes other means of averaging are used, if possible, such as spatial averaging, or possibly ensemble averaging (over many events).

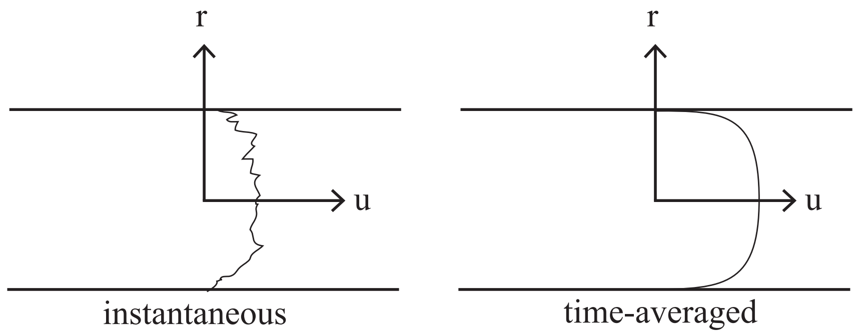

Figure 3: Axial velocity profiles for the instantaneous, u and average, $\bar{u}$ velocities.

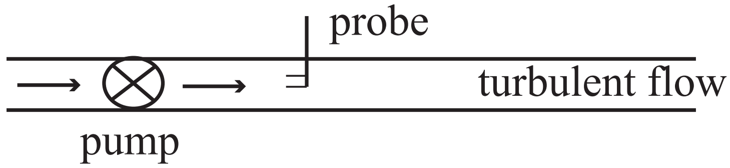

Figure 4: Sketch of a turbulent pipe flow driven by a pump.

Here we will only consider statistically steady flows. There are several properties of time averages that we will be using, which can be proven easily from the definition of a time average (Equation (1)).

1. If c is a constant (in time), then:

$$\bar{c} = c\,. \tag{2}$$

2. If u is a random function, such as a component of the velocity, then

$$\overline{cu} = c\bar{u}\,. \tag{3}$$

3. Since $\overline{u(x, y, z, t)}$ is a constant in time, then

$$\overline{\overline{u(x, y, z, t}} = \overline{u(x, y, z, t)}\,, \tag{4}$$

that is, the average of an average is the average itself.

4. When the spatial derivative of a function is averaged, the following is obtained:

$$\overline{\frac{\partial u}{\partial x}} = \frac{\partial \bar{u}}{\partial x}\,. \tag{5}$$

This is due to the fact that the time integral in the averaging operation and the spatial derivative commute.

5. Finally when the sum of the random variables a and b are averaged, the following result is obtained:

$$\overline{(a+b)} = \bar{a} + \bar{b}\,. \tag{6}$$

This is the distributive property of the averaging operator, and can be obtained from the distributive property of an integral.

Self-test:
Show that the average of the velocity $u(x,y,t) = \alpha x \sin^2(2\pi t/T) + \beta y \sin(2\pi t/T)$ over the period T is $\bar{u}(x) = \alpha x/2$.
Show that the average of the velocity $u(x,y,t) = \alpha x y t e^{-(t/\tau)^2}$ from 0 to τ is $\bar{u}(x,y) = \alpha x y(1-e)$.

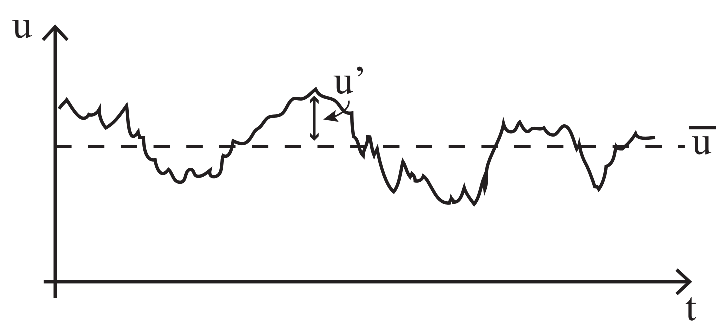

Figure 5: Defining the Reynolds decomposition, $u = \bar{u} + u'$.

In order to predict the behavior of a turbulent flow, we would like to develop equations for the time averages, in the incompressible flow case for $\bar{u}$, $\bar{v}$, $\bar{w}$, and $\bar{p}$. We will do this by time-averaging the Navier-Stokes equations. But before doing this it is useful to define the fluctuation about the mean (or average). For u it is written at u', and defined by (see Figure 5):

$$u = \bar{u} + u'\,, \text{ or } u' = u - \bar{u}\,. \tag{7}$$

Equation (7) is referred to as the Reynolds decomposition of the velocity u'. Note that averaging the velocity, with this decomposition, gives:

$$\bar{u} = \overline{\bar{u} + u'} = \bar{\bar{u}} + \overline{u'} = \bar{u} + \overline{u'}\,,$$

using averaging properties 5 and 3. Therefore $\overline{u'} = 0$, i.e., the average of the fluctuation about the average is 0. The result will be used often in deriving the equations for the time-averaged quantities.