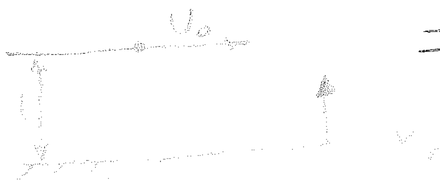
Homework 3 Solutions

Problem 1 Transient Couette flow.



Infinite plates, no pressure gradient. $\frac{dP}{dx}=0$

Fluid at rest for $t=0$.

One dimensional flow $v=0$

Incompressible flow $\rho =$ constant

Equations

Continuity : $\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\vec{V})=0 \Rightarrow \frac{\partial u}{\partial x}=0 \Rightarrow u=u(y,t)$

Momentum x-direction: $\frac{\partial u}{\partial t} + \underbrace{u\cdot\frac{\partial u}{\partial x}}_{0} + \underbrace{v\frac{\partial u}{\partial y}}_{0} = -\frac{1}{\rho}\underbrace{\frac{\partial P}{\partial x}}_{0} + \frac{\mu}{\rho}\left[\frac{\partial}{\partial x}\underbrace{\left(2\frac{\partial u}{\partial x}\right)}_{0} + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y} + \underbrace{\frac{\partial v}{\partial x}}_{0}\right)\right]$

$$\boxed{\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}}$$ 2

Boundary Conditions

2

$t=0 \Rightarrow u=0$ fluid at rest

$y=0 \Rightarrow u=0$ no-slip condition for the lower wall

$y=H \Rightarrow u=U_0$ no-slip condition for the moving upper wall.

Non dimensionalization

$u^* = \frac{u}{U_0}$; $y^* = \frac{y}{H}$; $t^* = \frac{t}{T}$

$\frac{U_0}{T}\frac{\partial u^*}{\partial t^*} = \nu \frac{U_0}{H^2}\frac{\partial^2 u^*}{\partial y^{*2}}$ Let $T = \frac{H^2}{\nu}$ and we drop the stars:

$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial y^2}$ with $u=0$ $(t=0)$, $u=0$ $(y=0)$, $u=1$ $(y=1)$ 1

To solve the equation we try separation of variables:

First we remove the stationary state:

$$\frac{\partial^2 u_s}{\partial y^2} = 0 \qquad u_s = Ay + B \qquad \begin{array}{ll} u_s = 0 & y = 0 \\ u_s = 1 & y = 1 \end{array}$$

1

$$u_s(y) = y$$

$\hat{u}(y,t) = u(y,t) - u_s(y)$. The problem for the transient is

$$\frac{\partial \hat{u}}{\partial t} = \frac{\partial^2 \hat{u}}{\partial y^2} \qquad \begin{array}{l} \hat{u}(t=0) = -y \\ \hat{u}(y=0) = 0 \\ \hat{u}(y=1) = 0 \end{array}$$

2

Now that we have homogeneous boundary conditions for y, we do

Separation of variables: $\hat{u}(y,t) = Y(y) \cdot T(t)$.

$Y\dot{T} = TY''$ Dividing by $\hat{u}$ we have

$\frac{\dot{T}}{T} = \frac{Y''}{Y} = -\lambda^2$ The only way a function of t can be equal to a function of y is that both are equal to constant.

$$\dot{T} = -\lambda^2 T \Rightarrow T(t) = c\, e^{-\lambda^2 t}$$

2

$Y'' + \lambda^2 Y = 0$; We try the solution $Y(y) = D\sin(\lambda y) + E\cos(\lambda y)$

$$Y'(y) = +\lambda D\cos(\lambda y) - \lambda E\sin(\lambda y); \quad Y'' = -\lambda^2 D\sin(\lambda y) - \lambda^2 E\cos(\lambda y)$$

$$-\lambda^2 D\sin(\lambda y) - \lambda^2 E\cos(\lambda y) + \lambda^2[D\sin(\lambda y) + E\cos(\lambda y)] \equiv 0$$

Applying the boundary conditions we have:

$Y(0) = \boxed{E = 0}$ $\quad Y(1) = D\sin\lambda = 0$. If $D = 0$ we

+1

have $Y \equiv 0$ which is the trivial solution. We need $\sin\lambda = 0$.

That gives us the eigenvalues of the problem: $\lambda_n = n\pi$

Thus the solution is of the form:

$$\hat{u}(y,t) = T(t)\cdot Y(y) = \sum_{n=1}^{\infty} a_n e^{-n^2\pi^2 t} \cdot \sin(n\pi y)$$

But we still have to satisfy the initial condition. That way we obtain the a_n.

$$\overset{n}{u}(y, t=0) = \sum_{n=1}^{\infty} a_n \cdot 1 \cdot \sin(n\pi y) = -y$$ +1

Multiplying both members by $\sin m\pi y$ and integrating between 0 and 1.

$$\int_0^1 \sum a_n \sin(n\pi y)\sin(m\pi y)\,dy = \int_0^1 -y \sin(m\pi y)\,dy.$$

Using the property that the eigenfunctions are orthogonal, that is that $2 \cdot \int_0^1 \sin(n\pi y)\sin(m\pi y)\,dy = \delta_{nm} \begin{cases} 0 & n \neq m \\ 1 & n = m \end{cases}$ we can +1

calculate the values of a_n:

$$\frac{1}{2} a_m = \int_0^1 -y \sin(m\pi y)\,dy.$$

Integrating by parts $\int_0^1 y \sin(m\pi y)\,dy = -y \frac{\cos m\pi y}{m\pi}\Big|_0^1 -$

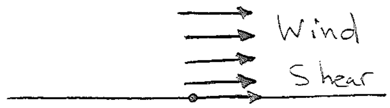
$$-\int_0^1 -\frac{\cos m\pi y}{m\pi}\,dy = -1\,\frac{\cos m\pi}{m\pi} + \cancel{\frac{\sin m\pi y}{m^2\pi^2}\Big|_0^1} = (-1)^{m+1}\frac{1}{m\pi}$$

We have that $\boxed{a_m = (-1)^{m+1}\frac{2}{m\pi}}$ 0 +1

The final solution for the transient Couette flow is

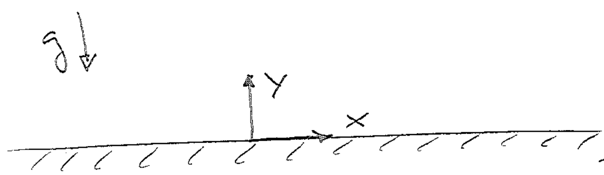
+1 $$\boxed{\overset{n}{u}(y,t) = U_0 \cdot \sum (-1)^{n+1} \frac{2}{n\pi} e^{-\frac{n^2\pi^2\nu}{H^2}t} \cdot \sin\left(\frac{n\pi y}{H}\right)}$$

Problem 2



$S = \mu\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right)$ at the free surface

Assume incompressible flow



The lake can be assumed to be infinitely wide

Continuity: $\cancel{\frac{\partial v_x}{\partial x}} + \frac{\partial v_y}{\partial y} + \cancel{\frac{\partial v_z}{\partial z}} = 0$

2

$\frac{\partial v_y}{\partial y} = 0$ and $v_y = 0$ @ $y = 0 \Rightarrow \boxed{v_y = 0}$

Conservation of momentum:

x-direction: $\rho\left(\frac{\partial v_x}{\partial t} + v_x \cancel{\frac{\partial v_x}{\partial x}} + \cancel{v_y} \frac{\partial v_x}{\partial y} + \cancel{v_z} \cancel{\frac{\partial v_x}{\partial z}}\right) = -\frac{\partial P}{\partial x} + \mu\left(\cancel{\frac{\partial^2 v_x}{\partial x^2}} + \frac{\partial^2 v_x}{\partial y^2} + \cancel{\frac{\partial^2 v_x}{\partial z^2}}\right)$

at the free surface $P = P_{atm}$ along all of x:

From y-direction we get $\frac{\partial P}{\partial y} = -\rho g \Rightarrow P = P_o(x) - \rho g y$

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and at $y = d$ we have $P_o(x) - \rho g d = P_{atm} \Rightarrow P_o(x) = P_{atm} + \rho g d$

and the pressure distribution is $P = P_{atm} + \rho g (d - y) \Rightarrow \underline{\underline{\frac{\partial P}{\partial x} = 0}}$

$$\boxed{\rho \frac{\partial v_x}{\partial t} = \mu \frac{\partial^2 v_x}{\partial y^2}}$$

2

With boundary conditions

$v_x(y, t \le 0) = 0$

$v_x(y = 0, t) = 0$

$\mu \frac{\partial v_x}{\partial y}(y = d, t) = S$

2

For short times ($\nu t \ll d^2$) the water near the surface moves without the effect of the bottom, as if the lake was infinite deep: $\eta = \frac{d-y}{\delta(t)}$ such that at $y=d$ $\mu \frac{d v_x}{d\eta}(\eta=0) = \mu \delta \frac{\partial v_x}{\partial y}\Big|_{y=d} = \delta(-$

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \eta}\cdot\underbrace{\frac{\partial \eta}{\partial t}}_{-\frac{d-y}{\delta^2}\delta'} + \frac{\partial}{\partial t}\cdot\underbrace{\frac{\partial t}{\partial t}}_{=1} \; ; \quad \frac{\partial}{\partial y} = \frac{\partial}{\partial \eta}\cdot\underbrace{\frac{\partial \eta}{\partial y}}_{=-1/\delta} + \frac{\partial}{\partial t}\underbrace{\frac{\partial t}{\partial y}}_{=0}$$

$$-\frac{d-y}{\delta}\cdot\frac{\delta'}{\delta}\frac{\partial v_x}{\partial \eta} + \cancel{\frac{\partial v_x}{\partial t}} = \nu\left(-\frac{1}{\delta}\right)^2\frac{\partial^2 v_x}{\partial \eta^2}$$

$$\frac{\delta\delta'}{\nu}\eta\frac{dv_x}{d\eta} + \frac{d^2 v_x}{d\eta^2} = 0$$

+1 $\frac{\delta\delta'}{\nu} = 2 \Rightarrow \frac{d\delta^2}{dt} = 2\nu \Rightarrow \underline{\delta(t) = \sqrt{4\nu t}}, \cancel{+C} \to 0 \quad \delta(t=0)=0$

$$\frac{d}{d\eta}\frac{dv_x}{d\eta} + 2\eta\frac{dv_x}{d\eta} = 0$$

$$\int \frac{d\left(\frac{dv_x}{d\eta}\right)}{\frac{dv_x}{d\eta}} = \int -2\eta\, d\eta$$

$$\ln\left(\frac{dv_x}{d\eta}\right) = -\eta^2 + \eta_0^2$$

$$\frac{dv_x}{d\eta} = e^{-\eta^2+\eta_0^2}$$ 1

$\frac{dv_x}{d\eta}\Big| = e^{\eta_0^2} = -\frac{\delta S}{}$

1 $$v_x = -\frac{S\cdot\delta}{\mu}\int_0^{\eta} e^{-\eta'^2}\,d\eta' + C$$

$$v_x(\eta\to\infty)\to 0 \qquad -\frac{S\cdot\delta}{\mu}\underbrace{\int_0^{\infty} e^{-\eta'^2}\,d\eta'}_{\sqrt{\pi}/2} + C' = 0$$

1

$$C' = +\frac{\sqrt{\pi}}{2}\,\frac{S\delta}{\mu}$$

$$v_x(y,t) = \frac{\sqrt{\pi}}{2}\,\frac{S\delta}{\mu}\left[1 - \frac{2}{\sqrt{\pi}}\int_0^{\eta} e^{-\eta'^2}\,d\eta'\right]$$ 1

$$v_x(y,t) = \frac{\sqrt{\pi}}{2}\,\frac{S\delta}{\mu}\left[1 - \mathrm{erf}\left(\frac{y}{\sqrt{4\nu t}}\right)\right]$$ for short times

When the wind has been blowing for a long time and a steady state has been set up, the surface rises at a small uniform elevation rate α and there is a pressure gradient that induces a recirculation cell in the lake. This ensures that the mass flow rate through any cross section is zero $\Rightarrow$ mass is conserved and cannot accumulate anywhere in steady-state.

CONSERVATION OF MOMENTUM (N-S EQ.)

x-direction

$$\rho\left(\underbrace{\cancel{\frac{\partial v_x}{\partial t}}}_{\text{STEADY}} + v_x \underbrace{\cancel{\frac{\partial v_x}{\partial x}}}_{\text{FULLY DEVELOPED}} + \overbrace{\cancel{v_y}}^{\text{negligible}} \frac{\partial v_x}{\partial y} + v_z \cancel{\frac{\partial v_x}{\partial z}}\right) = -\frac{\partial P}{\partial x} + \mu\left(\overbrace{\cancel{\frac{\partial^2 v_x}{\partial x^2}}}^{\text{FULLY DEVELOPED}} + \frac{\partial^2 v_x}{\partial y^2} + \cancel{\frac{\partial^2 v_x}{\partial z^2}}\right)$$

y-direction

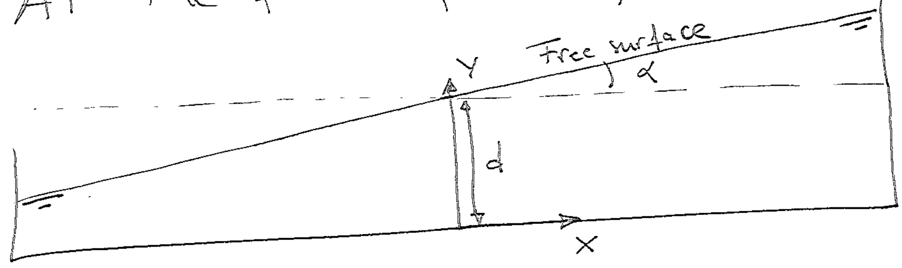
$$\rho\left(\cancel{\frac{\partial v_y}{\partial t}} + v_x \cancel{\frac{\partial v_y}{\partial x}} + v_y \cancel{\frac{\partial v_y}{\partial y}} + v_z \cancel{\frac{\partial v_y}{\partial z}}\right) = -\frac{\partial P}{\partial y} + \mu\overbrace{\cancel{\left(\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2}\right)}}^{\text{negligible}} - \rho g$$

negligible v_y

$$\frac{\partial P}{\partial y} = -\rho g \Rightarrow P(x,y) = P_0(x) - \rho g y$$

At the free surface: $y = d + \alpha x \Rightarrow P_{atm} = P_0(x) - \rho g(d + \alpha x)$

$$P_0(x) = P_{atm} + \rho g (d + \alpha x)$$



$$\frac{\partial P}{\partial x} = \frac{\partial P_0}{\partial x} = \rho g \alpha$$

From the x-direction equation: $\quad 0 = -\rho g \alpha + \mu \dfrac{d^2 v_x}{dy^2}$

$$\frac{dv_x}{dy} = \frac{g\alpha}{\nu} y + C_1 \Rightarrow \mu \left.\frac{dv_x}{dy}\right|_{y=d} = S' = \rho g \alpha d + C_1 \mu$$

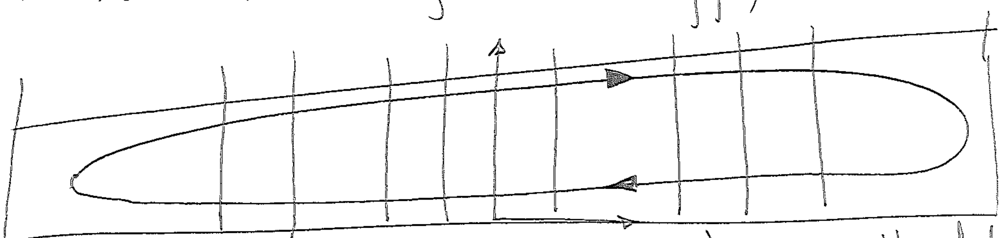
$$C_1 = S'/\mu - \frac{g\alpha}{\nu} d$$

$$v_x = \frac{g\alpha}{\nu} \frac{y^2}{2} + \left(\frac{S}{\mu} - \frac{g\alpha}{\nu} d\right) y + C_2$$

$$v_x(y=0) = 0 = C_2$$

$$\boxed{v_x(y) = \frac{g\alpha}{\nu} \frac{y^2}{2} + \left(\frac{S}{\mu} - \frac{g\alpha}{\nu} d\right) y}$$

But α and S are not independent, the elevation angle of the free surface is a function of the shear rate imposed by the wind. To link the together we apply conservation of mass:



Along all these cross sections of the lake, away from the ends, the flow is quasi-unidirectional but mass has to be conserved and that means the net flux across a section has to be zero:

$$\int_{\text{cross-section}} \rho\, v_x(y)\, dA = \rho \int_0^d v_x(y) \cdot \cancel{b}\, dy = 0$$

(width of the lake perpendicular to the paper)

$$\int_0^d \left[\frac{g\alpha}{\nu} \frac{y^2}{2} + \left(\frac{S}{\mu} - \frac{g\alpha}{\nu} d\right) y\right] dy = \frac{g\alpha}{\nu}\frac{y^3}{6} + \left(\frac{S}{\mu} - \frac{g\alpha}{\nu} d\right)\frac{y^2}{2}\Bigg|_0^d =$$

$$\frac{g\alpha}{\nu}\frac{d^3}{6} + \frac{S d^2}{\mu 2} - \frac{g\alpha}{\nu}\frac{d^3}{2} = 0$$

$$\frac{S d^2}{2\mu} = \frac{g\alpha}{\nu}\frac{d^3}{3} \Rightarrow \boxed{\alpha = \frac{3}{2}\frac{S}{\rho g d}}$$