ME 507 FLUID MECHANICS

HOMEWORK #1 SOLUTIONS

Problem 1

$$V_x = \frac{x}{1+t} \qquad V_y = \frac{y}{1+2t}$$

- Streamlines: $\frac{dy}{dx} = \frac{V_y}{V_x} \Rightarrow \frac{dy}{V_y} = \frac{dx}{V_x}$

$$\frac{dy}{y/1+2t} = \frac{dx}{x/1+t} \Rightarrow \int \frac{dy}{y} = \frac{1+t}{1+2t}\int\frac{dx}{x} \Rightarrow \ln y = \frac{1+t}{1+2t}\ln x + C$$

For the streamline that goes through (x_0, y_0):

$$\ln y_0 = \frac{1+t}{1+2t}\ln x_0 + C' \Rightarrow C' = \ln y_0 - \frac{1+t}{1+2t}\ln x_0 \Rightarrow$$

$\Rightarrow$ Equation for the streamline is $\ln(y/y_0) = \frac{1+t}{1+2t}\ln(x/x_0) \Rightarrow$

$$\Rightarrow \boxed{y = y_0 \cdot \left(\frac{x}{x_0}\right)^{\frac{1+t}{1+2t}}}$$

Pathlines: $\frac{dx}{dt} = V_x$; $\frac{dy}{dt} = V_y$

$$\frac{dx}{dt} = \frac{x}{1+t} \Rightarrow \int\frac{dx}{x} = \int\frac{dt}{1+t} \Rightarrow \ln x = \ln(1+t) + C_1$$

$$\frac{dy}{dt} = \frac{y}{1+2t} \Rightarrow \int\frac{dy}{y} = \int\frac{dt}{1+2t} \Rightarrow \ln y = \frac{1}{2}\ln(1+2t) + C_2$$

For the pathline that goes through (x_0, y_0) at $t = t_0$

$$\left.\begin{array}{l} \ln x_0 = \ln(1+t_0) + C_1 \Rightarrow \ln(x/x_0) = \ln\left(\frac{1+t}{1+t_0}\right) \\ \ln y_0 = \ln(1+2t_0) + C_2 \Rightarrow \ln(y/y_0) = \ln\left(\frac{1+2t}{1+2t_0}\right)^{1/2} \end{array}\right\} \Rightarrow$$

$$\Rightarrow \begin{cases} x = x_0\left(\frac{1+t}{1+t_0}\right) \\ y = y_0\left(\frac{1+2t}{1+2t_0}\right)^{1/2} \end{cases}$$

Eliminating t : $t = (1+t_0)\frac{x}{x_0} - 1 \Rightarrow$

$$\Rightarrow y = y_0 \sqrt{\frac{1+2\left[(1+t_0)\frac{x}{x_0} - 1\right]}{1+2t_0}} \Rightarrow \boxed{y = y_0 \sqrt{\frac{2(1+t_0)\frac{x}{x_0} - 1}{1+2t_0}}}$$

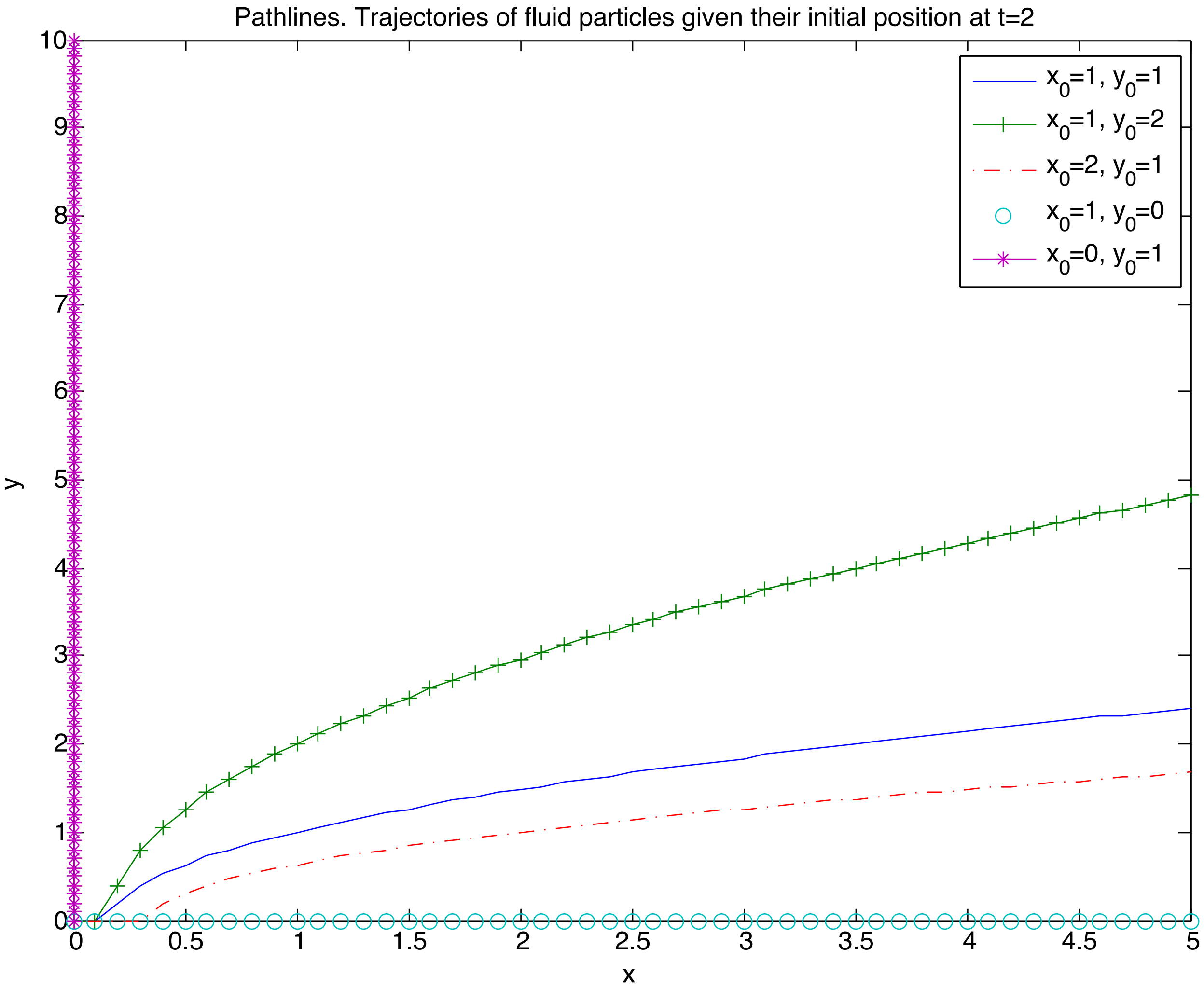
Compare with streamlines $y = y_0 \left(\frac{x}{x_0}\right)^{\frac{1+t}{1+2t}}$

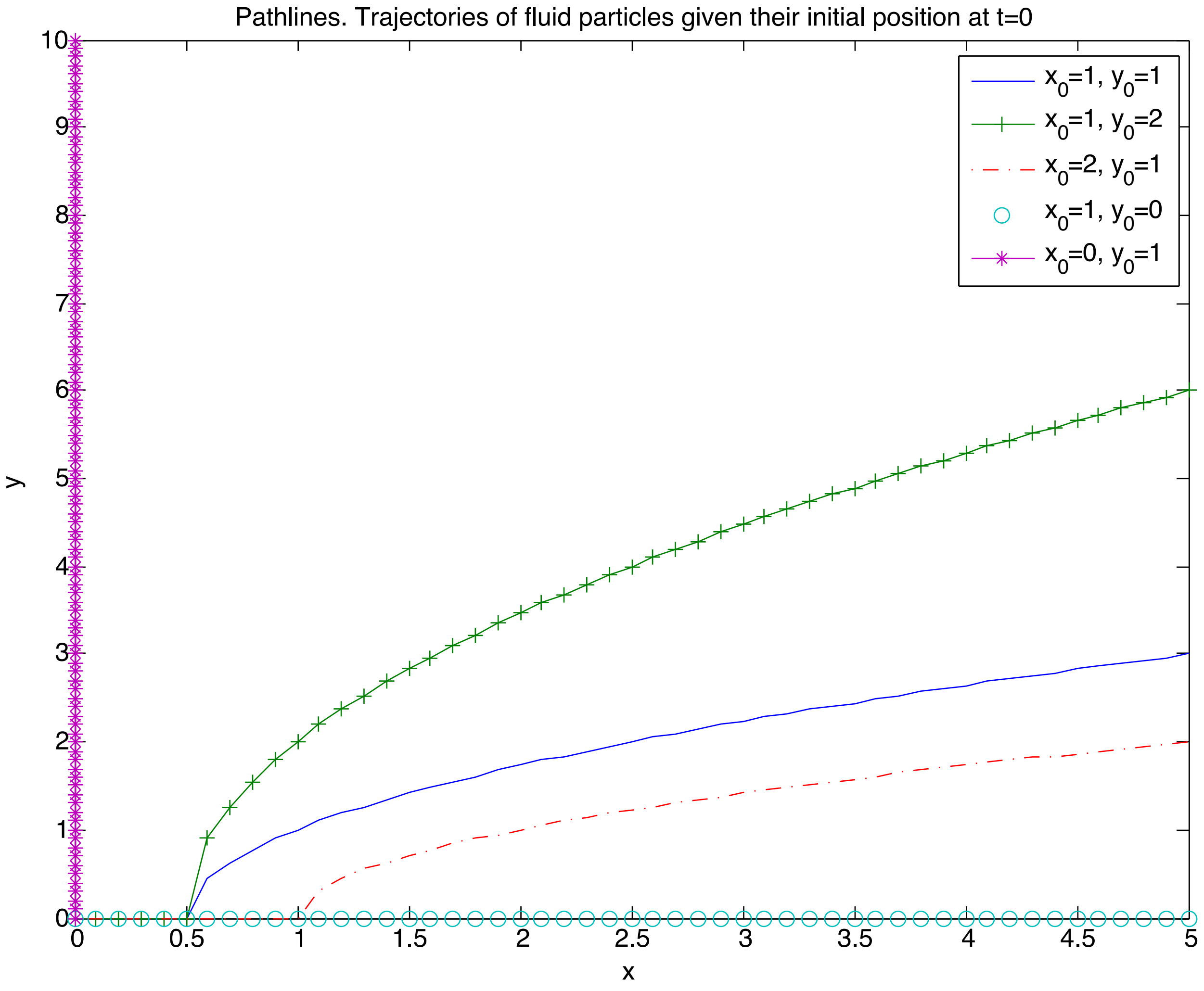
At $t=0 \Rightarrow$ Streamlines: $y = y_0\left(\frac{x}{x_0}\right)$

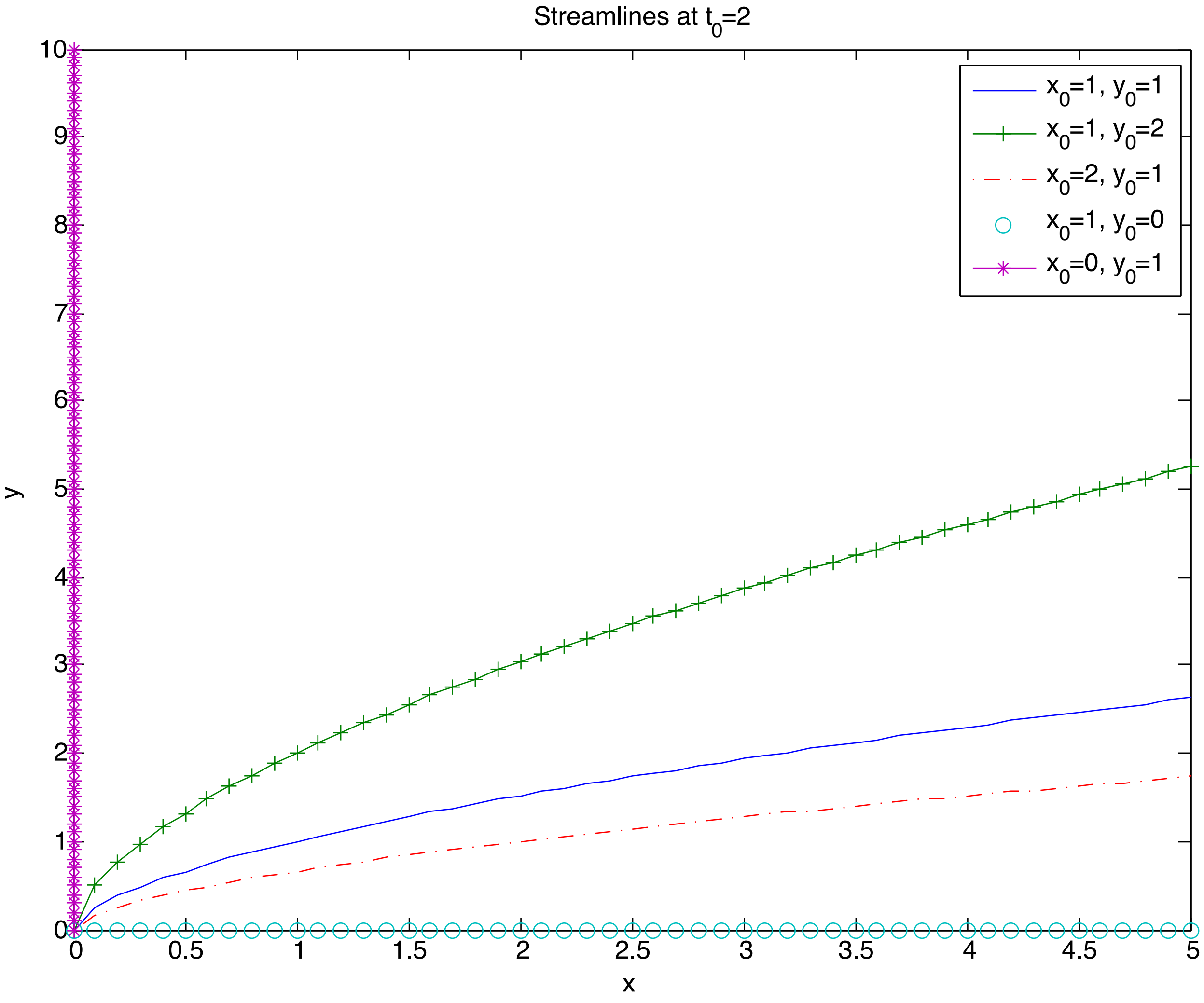
Pathline $(t_0=0)$: $y = y_0\sqrt{\left(2\frac{x}{x_0} - 1\right)}$: Straight line

At $t=2 \Rightarrow$ Streamlines: $y = y_0\left(\frac{x}{x_0}\right)^{3/5}$

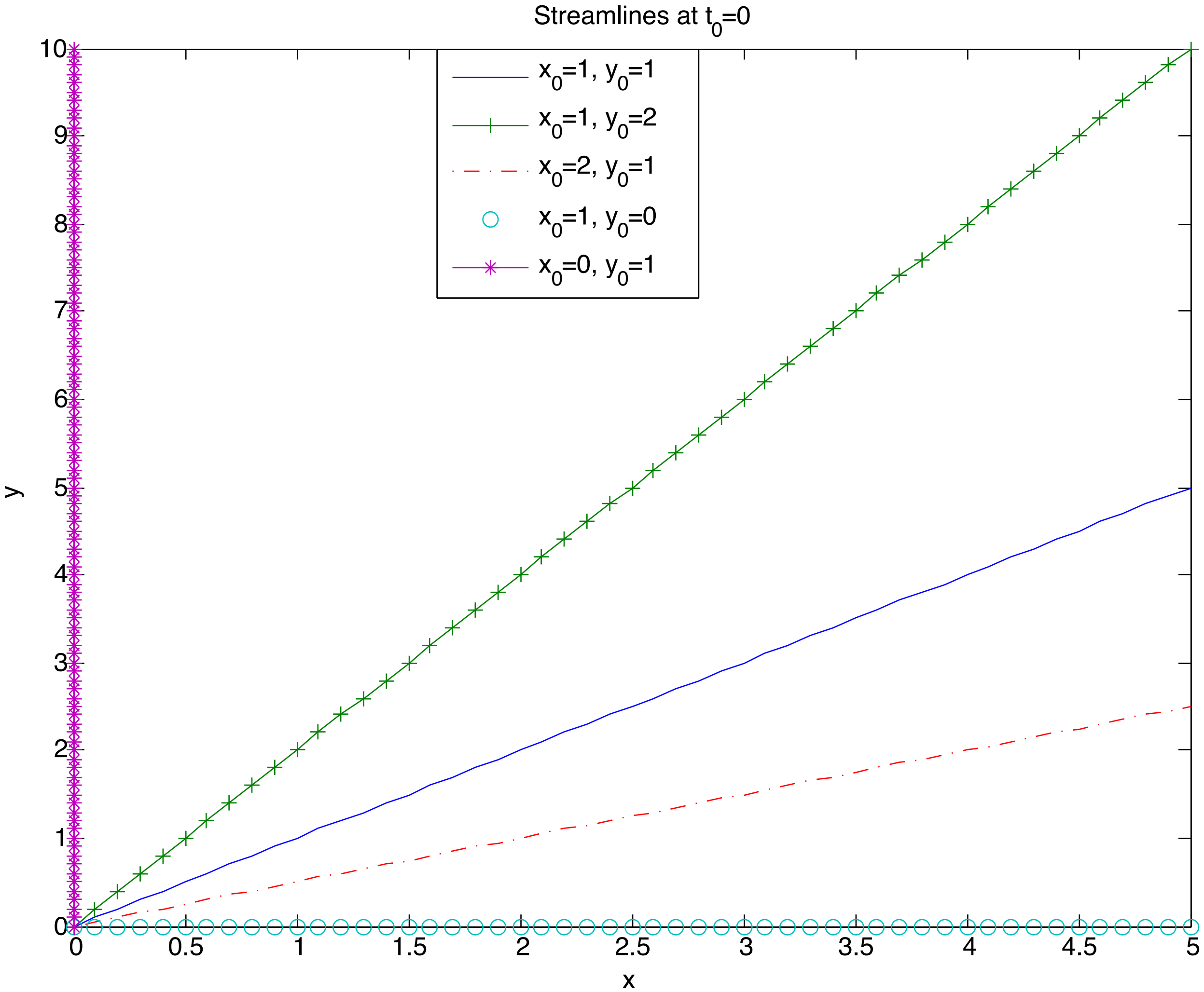
Pathlines $(t_0=2)$: $y = y_0\sqrt{\left(\frac{6}{5}\frac{x}{x_0} - \frac{1}{5}\right)}$ Straight line


Pathlines. Trajectories of fluid particles given their initial position at t=2
y
x
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5
0
1
2
3
4
5
6
7
8
9
10
x_0=1, y_0=1
x_0=1, y_0=2
x_0=2, y_0=1
x_0=1, y_0=0
x_0=0, y_0=1

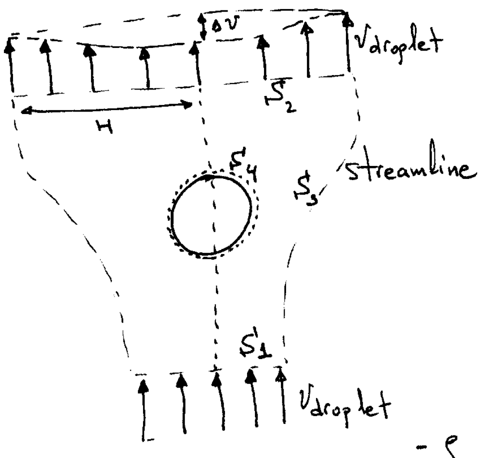

Pathlines. Trajectories of fluid particles given their initial position at t=0
y
x
0
1
2
3
4
5
6
7
8
9
10
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5
x_0=1, y_0=1
x_0=1, y_0=2
x_0=2, y_0=1
x_0=1, y_0=0
x_0=0, y_0=1



Streamlines at t_0=2
y
x
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5
0
1
2
3
4
5
6
7
8
9
10
x_0=1, y_0=1
x_0=1, y_0=2
x_0=2, y_0=1
x_0=1, y_0=0
x_0=0, y_0=1


Streamlines at t_0=0
x_0=1, y_0=1
x_0=1, y_0=2
x_0=2, y_0=1
x_0=1, y_0=0
x_0=0, y_0=1
y
x
0
1
2
3
4
5
6
7
8
9
10
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5

Problem 2



Conservation of mass:

$$\cancel{\frac{\partial}{\partial t}\int_{c.v.}\rho\, d\mathcal{V}} + \int_{c.s.}\rho(\vec{v}-\cancel{\vec{v}_c})\cdot\vec{n}\, dA = 0$$

STEADY

$$\int_{S_1}\rho\vec{v}\cdot\vec{n}\,dA + \int_{S_2}\rho\vec{v}\cdot\vec{n}\,dA + \cancel{\int_{S_3}\rho\vec{v}\cdot\vec{n}\,dA} + \cancel{\int_{S_4}\rho\vec{v}\cdot\vec{n}\,dA} = 0$$

Streamline $\vec{v}\perp\vec{n}$

no flux across the water droplet surface (no evaporation/condensation)

$$-\rho_{air}V_{droplet}A_1 + \rho_{air}\int_0^H\left\{V_{droplet}-\Delta V\left[1-\left(\frac{x}{H}\right)^2\right]^2\right\}2\pi x\,dx = 0$$

$$\cancel{\rho_{air}}\cancel{V_{droplet}}A_1 = \cancel{\rho_{air}}\cancel{V_{droplet}}\left[\left(1-\cancel{\frac{\Delta V}{V_{droplet}}}\right)\pi H^2 \cancel{+\frac{\Delta V}{2V_{droplet}}\pi H^2} - \frac{\Delta V}{V_{droplet}}\frac{\pi H^2}{3}\right]$$

$$\boxed{A_1 = \pi H^2\left(1-\frac{1}{3}\frac{\Delta V}{V_{droplet}}\right)}$$

Conservation of momentum

$$\cancel{\frac{\partial}{\partial t}\int_{c.v.}\rho\vec{v}\,d\mathcal{V}} + \int_{c.s.}\rho\vec{v}(\vec{v}-\cancel{\vec{v}_c})\cdot\vec{n}\,dA = -\int_{c.s.}p\vec{n}\,dA + \int_{c.s.}\bar{\bar{\tau}}\cdot\vec{n}\,dA + \cancel{\int_{c.v.}\rho\vec{g}\,d\mathcal{V}}$$

STEADY

negligible $\rho_{air} \ll \rho_{liq}$

$$\int_{S_1}\rho_{air}\vec{v}(\vec{v}\cdot\vec{n})dA + \int_{S_2}\rho_{air}\vec{v}(\vec{v}\cdot\vec{n})dA + \int_{S_3}\rho_{air}\vec{v}\cancel{(\vec{v}\cdot\vec{n})}dA + \int_{S_4}\rho\vec{v}\cancel{(\vec{v}\cdot\vec{n})}dA = \cancel{-\int_{S_1}P_{atm}\vec{n}\,dA - \int_{S_2}P_{atm}\vec{n}\,dA - \int_{S_3}P_{atm}\vec{n}}$$

0 streamlines

no flux across droplet surface

0 uniform pressure along a closed surface

$$-\int_{S_4}P\vec{n}\,dA + \int_{S_4}\bar{\bar{\tau}}\cdot\vec{n}\,dA + \cancel{\int_{S_1}\bar{\bar{\tau}}\cdot\vec{n}\,dA} + \cancel{\int_{S_2}\bar{\bar{\tau}}\cdot\vec{n}\,dA} + \cancel{\int_{S_3}\bar{\bar{\tau}}\cdot\vec{n}\,dA}$$

negligible inlet

negligible outlet

negligible along streamlines

$$\rho_{air}V_{droplet}\vec{k}(-V_{droplet})A_1 + \rho_{air}\int_0^H V_{droplet}^2\left\{1-\frac{\Delta V}{V_{droplet}}\left[1-\left(\frac{x}{H}\right)^2\right]^2\right\}^2(+\vec{k})2\pi x\,dx =$$

$$= \underbrace{-\int_{S_4} p\vec{n}\,dA + \int_{S_4} \bar{\bar{\tau}}\cdot\vec{n}\,dA}_{\vec{F}_{droplet\to fluid} \;=\; -\overrightarrow{D_{rag}}}$$

Applying Newton's 2nd Law to the droplet:

$$\cancel{\frac{d\,m\vec{v}}{dt}} = \Sigma \vec{F}_{ext} \begin{cases} \text{Aerodynamic drag} \\ \text{Weight} \end{cases}$$

0 by definition of the terminal velocity

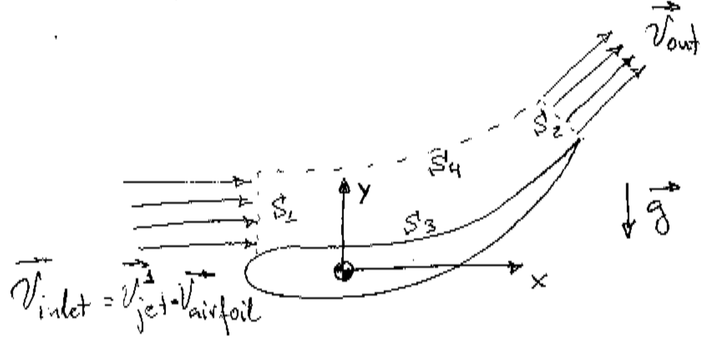
$$0 = -m_{drop}g\,\vec{k} + \overrightarrow{D_{rag}} \Rightarrow \boxed{\overrightarrow{D_{rag}} = \frac{\pi d^3}{6}\rho_{droplet}\,g\,\vec{k}}$$

$$\rho_{air}\,\underline{\underline{v_{droplet}^2}}\left\{\int_0^H \left\{1 - \frac{2\Delta v}{\underline{\underline{v_{droplet}}}}\left[1-\left(\frac{x}{H}\right)^2\right]^2 + \left(\frac{\Delta v}{\underline{\underline{v_{droplet}}}}\right)^2\left[1-\left(\frac{x}{H}\right)^2\right]^4\right\}2\pi x\,dx - A_1\right\}\vec{k} =$$

$$= -\frac{\pi d^3}{6}\rho_{droplet}\,g\,\vec{k} \Rightarrow$$

This is an algebraic 4th order equation that can be easily solved numerically with any iterative method.

Problem 3



Choosing a frame of reference that moves with the airfoil (inertial because the velocity is rectilinear and constant), the control volume **does not** move. The velocity of the liquid however is $\vec{V}_{rel} = \vec{V}_{abs} - \vec{V}_{airfo}$

Conservation of mass

$$\frac{D\, M_{sys}}{Dt} = 0 \Rightarrow \cancel{\frac{\partial}{\partial t}\int_{C.V.} \rho\, dV} + \int_{C.S.} \rho\,(\vec{v} - \cancel{\vec{v}_c}^{\,0}) \cdot \vec{n}\, dA = 0$$

STEADY

$$\int_{S_1} \rho\, \underbrace{\vec{v}_1 \cdot \vec{n}_1}_{-V_{inlet}}\, dA + \int_{S_2} \rho\, \underbrace{\vec{V}_2 \cdot \vec{n}_2}_{+V_{outlet}}\, dA + \int_{S_3} \rho\, \cancel{\vec{V}_3 \cdot \vec{n}_3} + \int_{S_4} \rho\, \cancel{\vec{V}_4 \cdot \vec{n}_4}\, dA = 0$$

SOLID WALL FREE SURFACE

$$-\cancel{\rho_{liq}}\, V_{inlet} A_1 + \cancel{\rho_{liq}}\, V_{outlet} A_2 = 0 \Rightarrow \boxed{V_{outlet} = \frac{A_1}{A_2} V_{inlet} = (V_{jet} - V_{airfoil}) \frac{A_1}{A_2}}$$

Conservation of momentum

$$\frac{D\,(M_{sys}\, \vec{V}_{CG})}{Dt} = \sum \vec{F}_{ext}$$

$$\cancel{\frac{\partial}{\partial t}\int_{C.V.} \rho\, \vec{v}\, dV} + \int_{C.S.} \rho\, \vec{v}\, (\vec{v} - \cancel{\vec{V}_c}^{\,0}) \cdot \vec{n}\, dA = -\int_{C.S.} p\, \vec{n}\, dA + \int_{C.S.} \bar{\bar{\tau}} \cdot \vec{n}\, dA + \int_{C.V.} \rho\, \vec{g}\, dV$$

STEADY

$$\int_{S_1}\rho\vec{V}_1(\vec{V}_1\cdot\vec{n}_1)\,dA+\int_{S_2}\rho\vec{V}_2(\vec{V}_2\cdot\vec{n}_2)\,dA+\underbrace{\int_{S_3}\rho\vec{V}(\cancel{\vec{V}\cdot\vec{n}})\,dA}_{0\ \text{SOLID WALL}}+\underbrace{\int_{S_4}\rho\vec{V}(\cancel{\vec{V}\cdot\vec{n}})\,dA}_{0\ \text{FREE SURFACE}}=-\int_{S_1+S_2+S_4}P_{atm}\vec{n}\,dA-\int_{S_3}P_{atm}\vec{n}\,dA$$

$$-\int_{S_3}P\vec{n}\,dA+\int_{S_3}\bar{\bar{\tau}}'\cdot\vec{n}\,dA+\underbrace{\cancel{\int_{S_1+S_2+S_4}\bar{\bar{\tau}}'\cdot\vec{n}\,dA}}_{\text{negligible}}+\int_{c.V.}\rho\vec{g}\,dV$$

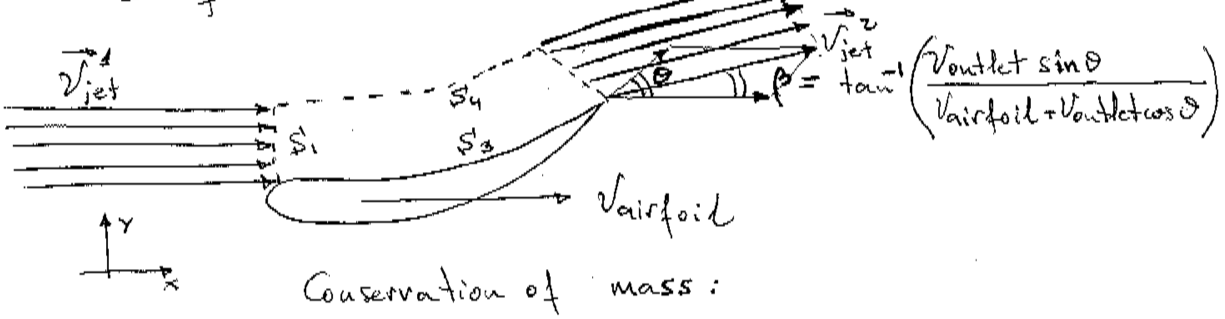
$$+\int_{S_3}P_{atm}\vec{n}\,dA$$

$$-\rho_{liq}V_{inlet}\left(V_{inlet}\vec{i}\right)A_1+\rho_{liq}V_{outlet}\left(V_{outlet}\cos\theta\,\vec{i}+V_{outlet}\sin\theta\,\vec{j}\right)A_2=$$

$$-\underbrace{\cancel{\int_{S_1+S_2+S_3+S_4}P_{atm}\vec{n}\,dA}}_{\substack{0\ \text{uniform}\\ \text{pressure}\\ \text{on a close surface}}}\quad\underbrace{-\int_{S_3}(P-P_{atm})\vec{n}+\int_{S_3}\bar{\bar{\tau}}'\cdot\vec{n}\,dA}_{-\vec{F}_{airfoil}}\quad-\rho gV\vec{j}$$

$$\vec{F}_{airfoil}=\rho_{liq}V_{inlet}A_1\left[\left(V_{inlet}-V_{outlet}\cos\theta\right)\vec{i}-V_{outlet}\sin\theta\,\vec{j}\right]-\rho gV\vec{j}$$

If we choose to use a fixed reference frame:



Conservation of mass:

$$\underbrace{\cancel{\frac{\partial}{\partial t}\int_{c.V}\rho\,dV}}_{\text{STEADY}}+\int_{c.S}\rho(\vec{V}-\vec{V}_c)\cdot\vec{n}\,dA=0$$

$$\int_{S_1}\rho\underbrace{\left(\vec{V}_{jet}^{1}-\vec{V}_{airfoil}\right)\cdot\vec{n}}_{-V_{inlet}}\,dA+\int_{S_2}\rho\underbrace{\left(\vec{V}_{jet}^{2}-\vec{V}_{airfoil}\right)\cdot\vec{n}}_{+V_{outlet}}\,dA+\underbrace{\int_{S_3}\rho\cancel{(\vec{V}-\vec{V}_{airfoil})\cdot\vec{n}}\,dA}_{\text{SOLID WALL}}+\underbrace{\int_{S_4}\rho\cancel{(\vec{V}-\vec{V}_{airfoil})}\cdot\vec{}}_{\text{FREE SURFA}}$$

$$\boxed{-\rho V_{inlet}A_1+\rho V_{outlet}A_2=0}$$

Conservation of momentum

$$\underbrace{\cancel{\frac{\partial}{\partial t}\int \rho \vec{v}\, dV}}_{\text{STEADY}} + \int_{c.s.} \rho \vec{v}\,(\vec{v}-\vec{v}_c)\cdot\vec{n}\, dA = -\int_{c.s} p\vec{n} + \int_{c.s.} \bar{\bar{\tau}}'\cdot\vec{n}\, dA + \int \rho\vec{g}\, dV$$

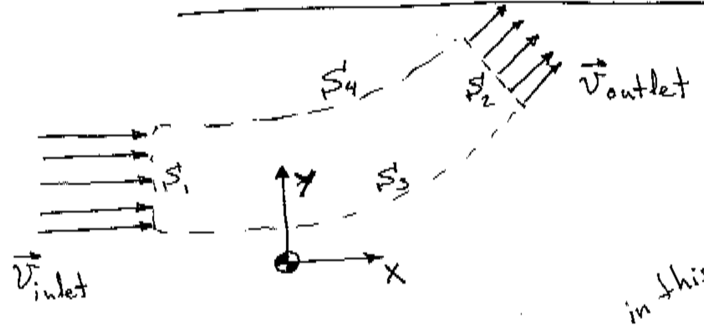
$$\int_{S_1} \rho \vec{v}^{\,1}_{jet} \underbrace{(\vec{v}-\vec{v}_c)\cdot\vec{n}}_{-V_{inlet}}\, dA + \int_{S_2} \rho \vec{v}^{\,2}_{jet} \underbrace{(\vec{v}-\vec{v}_c)\cdot\vec{n}}_{+V_{outlet}}\, dA = \overbrace{-\int_{S_1+S_2+S_4} P_{atm}\,\vec{n}\, dA - \int_{S_3} P_{atm}\,\vec{n}\, dA}^{0 \text{ UNIFORM PRESSURE ON A CLOSED SURFACE}} + \underbrace{\cancel{\int_{S_1+S_2+S_4} \bar{\bar{\tau}}'\cdot\vec{n}\, dA}}_{\text{negligible}} + \int_{c.v.} \rho\vec{g}\, dV + \underbrace{\left(-\int_{S_3} (P-P_{atm})\vec{n} + \int_{S_3} (\bar{\bar{\tau}}'\cdot\vec{n})\, d\right.}_{\shortparallel \;\; -\vec{F}_{airfoil}}$$

$$\vec{F}_{airfoil} = -\rho g V \vec{j} + \rho_{liq} V_{inlet} A_1 \Big(\underbrace{\vec{V}^{\,1}_{jet} - \vec{V}^{\,2}_{jet}}_{\shortparallel \;\; (\cancel{V_{airfoil}} + V_{inlet})\vec{i} - (\cancel{V_{airfoil}} + V_{outlet}\cos\theta\, \vec{i} + V_{outlet}\sin\theta\, \vec{j})}\Big)$$

$$\boxed{\vec{F}_{airfoil} = \rho_{liq} V_{inlet} A_1 \left[(V_{inlet} - V_{outlet}\cos\theta)\,\vec{i} - V_{outlet}\sin\theta\, \vec{j}\right] - \rho g V \vec{j}}$$

The subtlety is in the energy equation.

Conservation of Energy



On a reference frame moving with the airfoil.

in this reference frame

negligible

$$\cancelto{0}{\frac{\partial}{\partial t}\int_{c.v}\rho e\,dV} + \int_{c.s}\rho e\,(\vec{v}-\cancelto{0}{\vec{v}_o})\cdot\vec{n}\,dA = -\int_{c.s}p\,\vec{n}\cdot\vec{v}\,dA + \int_{c.s.}\vec{v}\cdot\bar{\bar{\tau}}'\cdot\vec{n}\,dA + \cancel{\int_{c.v.}\rho\,\vec{g}\cdot\vec{v}\,d}$$

STEADY

$$+\cancel{\int_{c.v}\dot{Q}_{rad/che}\,dV} - \cancel{\int_{c.s.}\vec{q}\cdot\vec{n}\,dA}$$

ADIABATIC

$$\int_{S_1}\rho e_{in}\,\vec{v}_{in}\cdot\vec{n}\,dA + \int_{S_2}\rho e_{out}\,\vec{v}_{out}\cdot\vec{n}\,dA + \cancelto{0}{\int_{S_3+S_4}\rho e\,\vec{v}\cdot\vec{n}\,dA} = -\int_{S_1}P_{in}\,\vec{n}\cdot\vec{v}_{in}\,dA - \int_{S_2}P_{out}\,\vec{n}\cdot\vec{v}_{ou}$$

SOLID WALL
FREE SURFACE

$$-\int_{S_3}p\,\vec{n}\cdot\cancelto{0}{\vec{v}}\,dA - \int_{S_4}\cancelto{0}{p\,\vec{n}\cdot\vec{v}}\,dA + \cancel{\int_{S_1+S_2+S_4}\vec{v}\cdot\bar{\bar{\tau}}'\cdot\vec{n}\,dA} + \cancelto{0}{\int_{S_3}\vec{v}\cdot\bar{\bar{\tau}}'\cdot\vec{n}\,dA}$$

STATIONARY SOLID WALL

FREE SURFACE

negligible viscous stresses

STATIONARY SOLID WALL

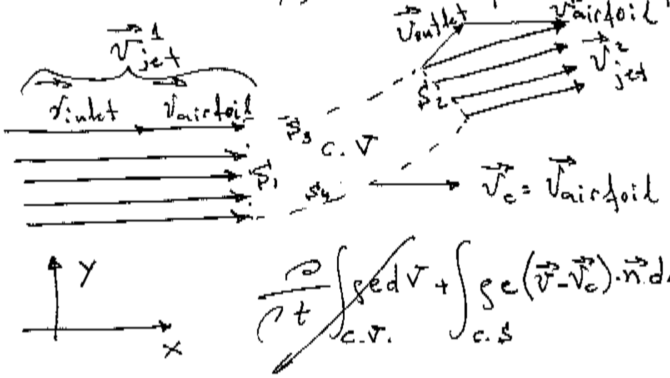
$$\rho_{liq}\left(u_{in}+\frac{1}{2}v_{in}^2\right)\left(-v_{in}\right)A_{in} + \rho_{liq}\left(u_{out}+\frac{1}{2}v_{out}^2\right)v_{out}A_{out} = \cancel{-P_{in}(-v_{in})}A_{in} - \cancel{P_{out}v_{out}}A_{out}$$

Since $\rho = \rho_{liq}$ and $P_{in} = P_{out} = P_{atm}$

$$P_{in}/\rho\;\rho v_{in}A_{in} = P_{out}/\rho\;\rho v_{out}A_{out} = \dot{m}\,P_{atm}/\rho$$

$$\dot{m}\left(u_{in}+\frac{1}{2}\rho v_{in}^2\right) = \dot{m}\left(u_{out}+\frac{1}{2}\rho v_{out}^2\right)$$

Alternatively, on a fixed reference frame:



$$\cancel{\frac{\partial}{\partial t}\int_{c.\forall}\rho e\,d\forall}+\int_{c.s}\rho e(\vec v-\vec v_c)\cdot\vec n\,dA=-\int_{c.s}p\,\vec v\cdot\vec n\,dA+\int_{c.s}\vec v\cdot\bar{\bar{\tau}}'\cdot\vec n\,dA+\cancel{\int_{c.\forall}\rho\vec g\cdot\vec v}$$

STEADY

negligible

$$+\cancel{\int_{c.\forall}\dot Q_{rad,che}\,d\forall}-\cancel{\int_{c.s}\vec q\cdot\vec n\,dA}$$

ADIABATIC

$$\int_{S_1}\underset{\substack{\parallel\\ \left(u_{in}+\frac{1}{2}V^{1\,2}_{jet}\right)}}{\rho e_{in}}\underbrace{\left(\vec V^1_{jet}-\vec V_{airfoil}\right)\cdot\vec n}_{-V_{in}}dA+\int_{S_2}\underset{\substack{\parallel\\ \left(u_{out}+\frac{1}{2}V^{2\,2}_{jet}\right)}}{\rho e_{out}}\underbrace{\left(\vec V^2_{jet}-\vec V_{airfoil}\right)\cdot\vec n}_{V_{out}}dA+\int_{S_3+S_4}\rho e\,\underbrace{\cancel{(\vec v-\vec v_c)\cdot\vec n}}_{0}\,dA=$$

NO FLUX THROUGH SOLID WALL OR FREE SURFACE

$$=-\int_{S_1}p_{in}\left(\vec V^1_{jet}\cdot\vec n\right)dA-\int_{S_2}p_{out}\vec V^2_{jet}\cdot\vec n\,dA-\int_{S_3}p_3\,\vec v\cdot\vec n\,dA-\int_{S_4}p_4\vec v\cdot\vec n\,dA+\cancel{\int_{S_1+S_2+\dots}\vec v\cdot\bar{\bar{\tau}}'\cdot\vec n}+\int_{S_4}\vec v\cdot\bar{\bar{\tau}}'\cdot\vec n\,dA$$

negligible

$$+\int_{S_1}p_{atm}\vec V_{airfoil}\cdot\vec n\,dA+\int_{S_2}p_{atm}\vec V_{airfoil}\cdot\vec n+\int_{S_3}p_{atm}\vec V_{airfoil}\cdot\vec n\,dA+\int_{S_4}p_{atm}\vec V_{airfoil}\cdot\vec n\,dA$$

$$\left(\int_{S_1+S_2+S_3+S_4}p_{atm}\vec n\,dA\right)\cdot\vec V_{airfoil}=0$$

$$\underbrace{\left[-\int_{S_4}(p_4-p_{atm})\vec n\,dA+\int_{S_4}\bar{\bar{\tau}}'\cdot\vec n\,dA\right]}_{-\vec F_{airfoil}}\cdot\vec V_{airfoil}$$

$$-\rho V_{in}A_{in}\left(u_{in}+\frac{1}{2}V^2_{jet_1}\right)+\rho V_{out}A_{out}\left(u_{out}+\frac{1}{2}V^2_{jet_2}\right)=-\int_{S_1}p_{atm}\left(\vec V_{jet_1}-\vec V_{airfoil}\right)$$

$$-\int_{S_2}p_{atm}\left(\vec V_{jet_2}-\vec V_{airfoil}\right)\cdot\vec n\,dA-\int_{S_3}p_{atm}\underbrace{\cancel{\left(\vec V-\vec V_{airfoil}\right)}}_{0}\cdot\vec n\,dA-\vec F_{airfoil}\cdot\vec V_{airfoil}$$

NO FLUX THROUGH FREE SURFACE

$$\dot{m}\left\{h_{out} + \frac{1}{2}\left[\left(V_{airfoil} + V_{out}\cos\theta\right)^2 + \left(V_{out}\sin\theta\right)^2\right] - h_{in} - \frac{1}{2}\left(V_{in} + V_{airfoil}\right)^2\right\} =$$

$$- \cancel{P_{atm}\left(V_{in}\right)} A_{in} - \cancel{P_{atm} V_{out} A_{out}} - \vec{F}_{airfoil} \cdot \vec{V}_{airfoil}$$

$$\dot{m}\left(h_{out} + \frac{1}{2} V_{out}^2\right) - \dot{m}\left(h_{in} + \frac{1}{2} V_{in}^2\right)$$

$$+ \dot{m}\,\frac{1}{\cancel{2}}\left(\cancel{V_{airfoil}^2} + \cancel{2} V_{airfoil} V_{out}\cos\theta\right) - \dot{m}\,\cancel{\frac{1}{2}}\left(\cancel{V_{airfoil}^2} + \cancel{2} V_{in} V_{airfoil}\right) =$$

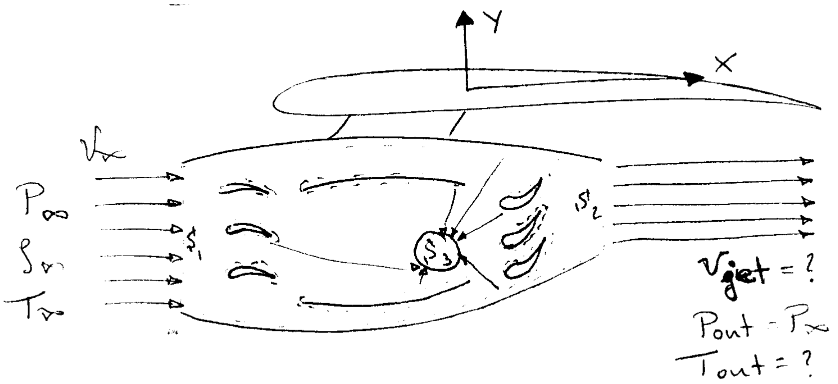
$$= -\left\{\dot{m}\left[\left(V_{in} - V_{out}\cos\theta\right)\vec{i} - V_{out}\sin\theta\,\vec{j}\right] - \rho g \forall \vec{j}\right\} \cdot V_{airfoil}\,\vec{i}$$

$$\dot{m}\left(h_{out} + \frac{1}{2} V_{out}^2\right) - \dot{m}\left(h_{in} + \frac{1}{2} V_{in}^2\right) + \dot{m}\left(\cancel{V_{airfoil} V_{out}\cos\theta} - V_{in}\cancel{V_{airfoi}}\right.$$

$$- \dot{m}\,V_{in}\cancel{V_{airfoil}} + \dot{m}\,V_{out}\cancel{\cos\theta\,V_{airfoil}}$$

$$\boxed{\dot{m}\left(h_{out} + \frac{1}{2} V_{out}^2\right) - \dot{m}\left(h_{in} + \frac{1}{2} V_{in}^2\right) = 0}$$

Problem 4



Conservation of mass

$$\cancel{\frac{\partial}{\partial t}\int_{C.V.}\rho\, dV} + \int_{C.S.}\rho(\vec{v}-\cancelto{0}{\vec{v}_c})\cdot\vec{n}\, dA = 0$$

STEADY

$$\int_{S_1}\rho\,\vec{v}_1\cdot\vec{n}_1\, dA + \int_{S_2}\rho\,\vec{v}_2\cdot\vec{n}_2\, dA + \int_{S_3}\rho\,\cancel{\vec{v}\cdot\vec{n}}\, dA = 0$$

SOLID WALLS

$$\rho_\infty(-V_\infty)A_{in} + \rho_{out}V_{jet}A_{out} = 0$$

$$\dot{m} = \rho_\infty V_\infty A_{in} = \rho_{out}V_{jet}A_{out}$$

Conservation of momentum

$$\cancel{\frac{\partial}{\partial t}\int_{C.V.}\rho\vec{v}\, dV} + \int_{C.S.}\rho\vec{v}(\vec{v}-\cancelto{0}{\vec{v}_c})\cdot\vec{n}\, dA = -\int_{C.S.}p\,\vec{n}\, dA + \int_{C.S.}\overline{\overline{\tau}}'\cdot\vec{n}\, dA + \int_{C.V.}\rho\vec{g}\, dV$$

STEADY

$$\int_{S_1}\rho V_\infty\vec{i}(-V_\infty)\, dA + \int_{S_2}\rho V_{jet}\vec{i}\,V_{jet}\, dA + \int_{S_3}\rho\vec{v}(\cancelto{0}{\vec{v}\cdot\vec{n}})\, dA = \cancelto{0}{\left(-\int P_\infty\vec{n}\, dA - \int_{S_3}P_\infty\vec{n}\, dA\right)} +$$

UNIFORM PRESSURE

SOLID WALL

$$+\cancel{\int_{S_1+S_2}\overline{\overline{\tau}}'\cdot\vec{n}\, dA} + \underbrace{\int_{S_3}\overline{\overline{\tau}}'\cdot\vec{n}\, dA + \cancel{\int_{C.V}\rho\vec{g}\, dV} - \int_{S_3}p\,\vec{n}\, dA + \int_{S_3}P_\infty\vec{n}\, dA}_{-\vec{Thrust}}$$

negligible at inlets and outlets

negligible for gases except in atmospheric scales

$$\vec{T}\text{hrust} = -\underbrace{\rho\, v_{jet}\, v_{jet}\, A_{out}}_{\dot m}\,\vec{i} + \underbrace{\rho\, V_\infty\, V_\infty\, A_{in}}_{\dot m}\,\vec{i}$$

$$\vec{T}\text{hrust} = -\,\dot m\left(v_{jet} - V_\infty\right)\vec{i}$$

So far we have Thrust in terms of Vjet and Vjet in terms of ρ_{out}.

Energy equation

$$\underbrace{\cancel{\frac{\partial}{\partial t}\int_{C.V.}\rho e\, dV}}_{\text{STEADY}} + \int_{C.S.}\rho e\,(\vec{v} - \cancelto{0}{\vec{V}_c})\cdot\vec{n}\, dA = -\int_{C.S.} p\,\vec{v}\cdot\vec{n}\, dA + \int_{C.S.}\vec{v}\cdot\overline{\overline{\tau}}'\cdot\vec{n}\, dA +$$

$$+ \underbrace{\cancel{\int_{C.V.}\rho\,\vec{g}\cdot\vec{v}\, dV}}_{\text{negligible}} + \underbrace{\int_{C.V}\dot Q_{\substack{rad\\ chc}}\, dV - \int_{C.S.}\vec{q}\cdot\vec{n}\, dA}_{\text{The net effect of these two terms is the heat addition given in the statement of the problem } \dot Q_{comb}}$$

$$\int_{S_1}\rho e\,\vec{v}_1\cdot\vec{n}_1\, dA + \int_{S_2}\rho e\,(\vec{v}_2\cdot\vec{n}_2)\, dA + \int_{S_3}\rho e\,(\underbrace{\cancel{\vec{v}}}_{\text{SOLID WALL}}\cdot\vec{n})\, dA = -\int_{S_1} p\,\vec{v}_1\cdot\vec{n}_1\, dA - \int_{S_2} p\,\vec{v}_2\cdot\vec{n}_2\, dA$$

$$\underbrace{-\int_{S_3} p\,(\vec{v}\cdot\vec{n})\, dA + \int_{S_3}\vec{v}\cdot\overline{\overline{\tau}}'\cdot\vec{n}\, dA}_{\text{These terms are zero on the stationary solid surfaces but non-zero on the rotating parts of the compressor and turbine.}} + \underbrace{\cancel{\int_{S_1+S_2}\vec{v}\cdot\overline{\overline{\tau}}'\cdot\vec{n}\, dA}}_{\text{negligible}} + \dot Q_{comb}$$

In this example however the engine is not designed to extract power, so the power of the turbine and compressor are equal but with opposite sign and therefore cancel out.

The energy equation becomes:

$$\int_{S_1} \rho \left(u+\tfrac{1}{2}v^2\right) \underbrace{(\vec{v}_1\cdot\vec{n}_1)}_{-v_\infty} dA + \int_{S_1} p \underbrace{(\vec{v}_1\cdot\vec{n}_1)}_{-v_\infty} dA + \int_{S_2} \rho\left(u+\tfrac{1}{2}v^2\right)\underbrace{(\vec{v}_2\cdot\vec{n}_2)}_{v_{jet}} dA +$$

$$+ \int_{S'_2} p \underbrace{(\vec{v}_2\cdot\vec{n}_2)}_{v_{jet}} dA = \dot{Q}_{comb}$$

$$\dot{Q}_{comb} = -\rho_\infty v_\infty A_{in} \left(u + P/\rho + \tfrac{1}{2}v^2\right)_\infty + \rho_{out} v_{jet} A_{out} \left(u + P/\rho + \tfrac{1}{2}v^2\right)_{out}$$

$$\dot{m}\left[\left(h_{out} + \tfrac{1}{2}v_{jet}^2\right) - \left(h_\infty + \tfrac{1}{2}v_\infty^2\right)\right] = \dot{Q}_{comb}$$

We have h_{out} in terms of v_{jet}

We only a thermodynamic relationship between h_{out} and ρ_{out} to close the problem

$$\left.\begin{aligned} h_{out} &= c_p \cdot T_{out} \\ T_{out} &= \frac{P_{out}}{R_g \rho_{out}} = \frac{P_\infty}{R_g \rho_{out}} \end{aligned}\right\} h_{out} = \frac{c_p}{R_g}\frac{P_\infty}{\rho_{out}} = \frac{c_p}{c_p - c_v}\frac{P_\infty}{\rho_{out}}$$

$$\underline{R_g = c_p - c_v}$$

$$\boxed{h_{out} = \frac{1}{1-\frac{c_v}{c_p}} P_\infty/\rho_{out} = \frac{\gamma}{\gamma - 1}\frac{P_\infty}{\rho_{out}}}$$

using $\underline{\gamma = c_p/c_v}$