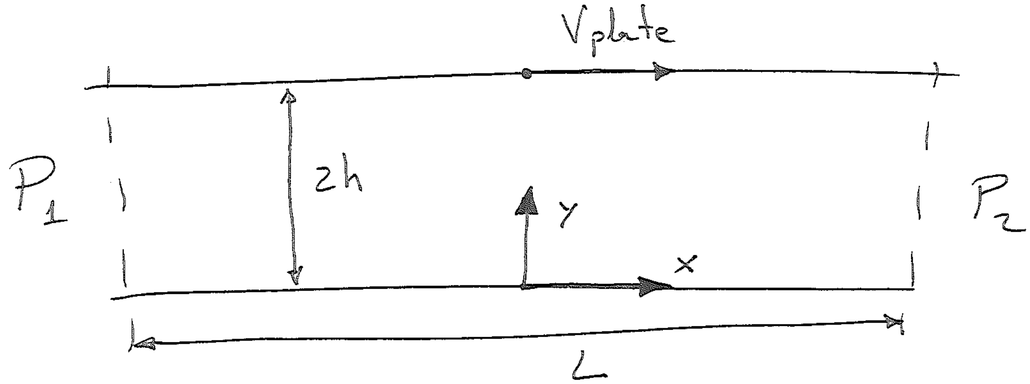


Continuity

$$\cancel{\frac{\partial \rho}{\partial t}} + \cancel{\frac{\partial}{\partial x}}(\rho V_x) + \frac{\partial}{\partial y}(\rho V_y) + \cancel{\frac{\partial}{\partial z}}(\rho V_z) = 0$$

$$\rho \frac{\partial V_y}{\partial y} = 0 \Rightarrow \frac{\partial V_y}{\partial y} = 0 \longrightarrow V_y(y) = 0$$

$V_y = 0$ @ $y = 0$ and $y = 2h$

Conservation of momentum

$$\rho\left(\cancel{\frac{\partial V_x}{\partial t}} + V_x \cancel{\frac{\partial V_x}{\partial x}} + \cancel{V_y}\frac{\partial V_x}{\partial y} + \cancel{V_z}\cancel{\frac{\partial V_x}{\partial z}}\right) = -\frac{\partial P}{\partial x} + \mu\left(\cancel{\frac{\partial^2 V_x}{\partial x^2}} + \frac{\partial^2 V_x}{\partial y^2} + \cancel{\frac{\partial^2 V_x}{\partial z^2}}\right)$$

$$0 = -\frac{\partial P}{\partial x} + \mu \frac{\partial^2 V_x}{\partial y^2}$$

Take the $\frac{\partial}{\partial x}$ derivative:

$$0 = -\frac{\partial}{\partial x}\left(\frac{\partial P}{\partial x}\right) + \mu \frac{\partial^2}{\partial y^2}\left(\cancel{\frac{\partial V_x}{\partial x}}\right) \to 0$$

$$\frac{\partial}{\partial x}\left(\frac{\partial P}{\partial x}\right) = 0 \Rightarrow \frac{\partial P}{\partial x} = \text{constant} = \frac{P_2 - P_1}{L}$$

$$\mu \frac{d^2 V_x}{dy^2} = \frac{P_2 - P_1}{L}$$

$$\frac{dV_x}{dy} = -\frac{P_1 - P_2}{\mu L} y + C_1$$

$$V_x(y) = -\frac{P_1 - P_2}{2\mu L} y^2 + C_1 y + C_2$$

Boundary Conditions

$$V_x(y=0) = \boxed{C_2 = 0}$$

$$V_x(y=2h) = -\frac{P_1 - P_2}{2\mu L} 4h^2 + C_1 2h = V_{plate}$$

$$C_1 = \frac{V_{plate}}{2h} + \frac{P_1 - P_2}{\mu L} h$$

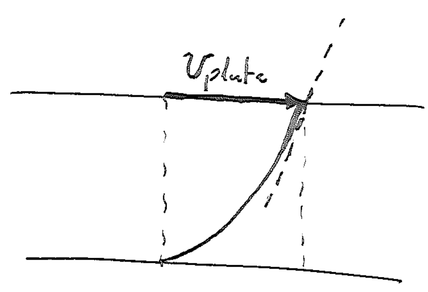
$$\boxed{V_x(y) = \frac{P_1 - P_2}{\mu L} y\left(h - y/2\right) + V_{plate} \frac{y}{2h}}$$

Slope of the velocity profile at the upper plate ($y=2h$)

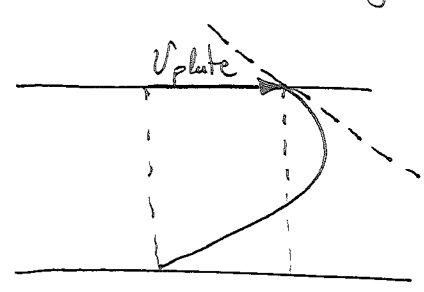
$$\left.\frac{dV_x}{dy}\right|_{y=2h} = -\frac{P_1 - P_2}{\mu L} \cancel{2}h + \frac{V_{plate}}{2h} + \cancel{\frac{P_1 - P_2}{\mu L}} h$$

$$\frac{V_{plate}}{2h} \quad \begin{matrix} > \\ = \\ < \end{matrix} \quad \frac{P_1 - P_2}{\mu L} h$$

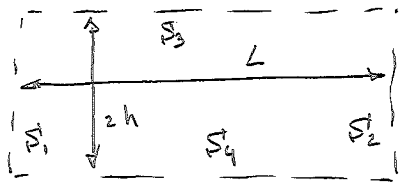
positive slope
zero
negative slope



$$V_{plate} > \frac{P_1 - P_2}{\mu L} 2h^2$$



$$V_{plate} < \frac{P_1 - P_2}{\mu L} 2h^2$$



$$\underbrace{\cancel{\frac{\partial}{\partial t}\int_{c.v.}\rho\vec{v}\,dV}}_{\text{Steady}} + \int_{c.s.}\rho\vec{v}(\vec{v}\cdot\vec{n})\,dA = -\int_{c.s.}p\vec{n}\,dA + \int_{c.s.}\bar{\bar{\tau}}\cdot\vec{n}\,dA + \int_{c.v.}\rho\vec{g}\,dV$$

$$-\rho\int_0^{2h}v^2(y)\,b\,dy\,\Big|_{S_1} + \rho\int_0^{2h}v^2(y)\,b\,dy\,\Big|_{S_2} + \int_{S_3+S_4}\rho\vec{v}(\underbrace{\cancel{\vec{v}\cdot\vec{n}}}_{0})\,dA =$$

$$= -P_1A_1(-\vec{i}) - P_2A_2\vec{i}\;\underbrace{-\int_{S_3}P_3\vec{n}\,dA - \int_{S_4}P_4\vec{n}\,dA + \int_{c.v}\rho\vec{g}\,dV}_{\text{Y-axis}}$$

$$+\underbrace{\cancel{\int_{S_1+S_2}\bar{\bar{\tau}}\cdot\vec{n}\,dA}}_{\text{negligible inlet and outlet}} + \int_{S_3+S_4}\bar{\bar{\tau}}\cdot\vec{n}\,dA$$

<u>X-axis</u>

$$0 = (P_1-P_2)\,b\cdot 2h - \int_0^L\mu\frac{dv_x}{dy}\Big|_{y=0}b\,dx + \int_0^L\mu\frac{dv_x}{dy}\Big|_{y=2h}b\,d$$

<u>Y-axis</u>

$$0 = -\int_{S_3}P_3\vec{n}\,dA - \int_{S_4}P_4\vec{n}\,dA + \int\rho\vec{g}\,dV \rightarrow$$ This step can be skipp

$$0 = (P_1-P_2)\,b\cdot 2h - \mu\left(\frac{U_{plate}}{2h}+\frac{P_1-P_2}{\mu L}h\right)bL +$$

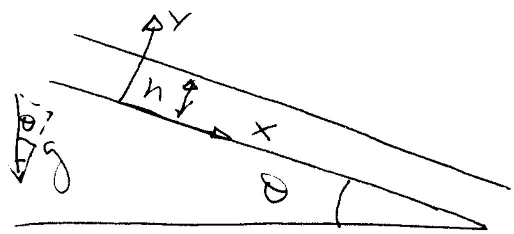
$$+\mu\left(\frac{U_{plate}}{2h}-\frac{P_1-P_2}{\mu L}h\right)b\cdot L$$

$$0 = \left(P_1 - P_2\right) b \cdot 2h - \cancel{\mu \frac{V_{plate}}{2h} bL} - \cancel{\mu} \frac{P_1 - P_2}{\cancel{\mu}\cancel{L}} h b \cancel{L}$$

$$+ \cancel{\mu \frac{V_{plate}}{2h} bL} - \cancel{\mu} \frac{P_1 - P_2}{\cancel{\mu}\cancel{L}} h b \cancel{L}$$

$$\underline{\underline{0 = \left(P_1 - P_2\right) b \cdot 2h - 2 \, \underline{P_1 - P_2} \, b \cdot h}}$$

PROBLEM 2



Continuity (Conservation of mass)

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \vec{v} = 0 \Rightarrow \nabla \cdot \vec{v} = 0 \Rightarrow \cancel{\frac{\partial V_x}{\partial x}} + \frac{\partial V_y}{\partial y} + \cancel{\frac{\partial V_z}{\partial z}} = 0$$

incompressible   fully developed   no z-dependency

$$\frac{\partial V_y}{\partial y} = 0 \Rightarrow V_y = 0 \text{ at } y=0 \Rightarrow V_y = 0 \text{ everywhere.}$$

Conservation of momentum

$$\rho\left[\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v}\right] = -\nabla p + \mu \nabla^2 \vec{v} + \rho \vec{g}$$

$$\rho\left(\cancel{\frac{\partial V_x}{\partial t}} + V_x \cancel{\frac{\partial V_x}{\partial x}} + \cancel{V_y}\frac{\partial V_x}{\partial y} + V_z \cancel{\frac{\partial V_x}{\partial z}}\right) = -\frac{\partial P}{\partial x} + \mu\left(\cancel{\frac{\partial^2 V_x}{\partial x^2}} + \frac{\partial^2 V_x}{\partial y^2} + \cancel{\frac{\partial^2 V_x}{\partial z^2}}\right) + \rho g \sin\theta$$

steady   fully developed   0   no z-dependency   fully developed   no z-dependency

Since $P(y=h) = P_{atm}$ for all values of x: $\frac{\partial P}{\partial x} = 0$

$$0 = \mu \frac{d^2 V_x}{dy^2} + \rho g \sin\theta \Rightarrow \frac{dV_x}{dy} = -\frac{\rho g \sin\theta}{\mu} y + C_1 \Rightarrow$$

$$\Rightarrow V_x(y) = -\frac{\rho g \sin\theta}{\mu} \cdot \frac{y^2}{2} + C_1 y + C_2$$

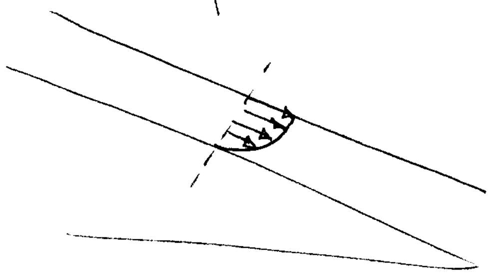
At the wall: $y=0 \Rightarrow V_x = 0 \Rightarrow C_2 = 0$.

At the free surface, because $\mu_{air} \ll \mu_{water} \Rightarrow \left.\frac{dV_x}{dy}\right|_{y=h} = 0$

$$0 = -\frac{\rho g \sin\theta}{\mu} h + C_1 \Rightarrow C_1 = \frac{\rho g \sin\theta}{\mu} h$$

$$\boxed{V_x(y) = \frac{\rho g \sin\theta}{\mu} h y \left(1 - \frac{y}{2h}\right)}$$

The flow rate through any cross-section is



$$\dot{m} = \int_S \vec{v}\cdot\vec{n}\, dA$$

or for incompressible flow (ρ constant)

$$Q = \int_S \vec{v}\cdot\vec{n}\, dA = \int_0^h V_x(y)\cdot b\, dy =$$

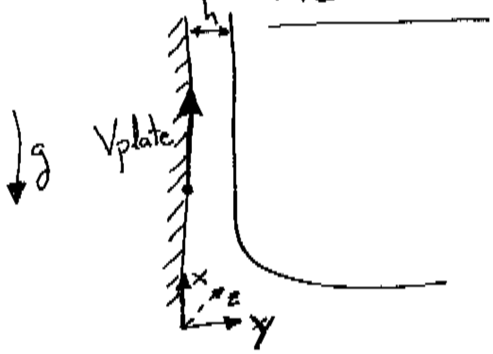
$$= \int_0^h b \frac{\rho g \sin\theta}{\mu} h y \left(1 - \frac{y}{2h}\right) dy = \frac{\rho g \sin\theta}{\mu} b h \left[\frac{y^2}{2} - \frac{y^3}{6h}\right]_0^h$$

$$= \frac{\rho g \sin\theta}{\mu} b \cdot h^3 \left(\frac{1}{2} - \frac{1}{6}\right) = \frac{1}{3}\underbrace{\frac{\rho g \sin\theta}{\mu} h^2}_{2 V_{max}} b\cdot h = \frac{2}{3} V_{max}\cdot b\cdot h$$

The friction coefficient can be defined such that $\tau_w = f \frac{1}{2}\rho V_{average}^2$ but in this problem we have an exact solution for $V_x(y)$ and so we can calculate $\tau_w = \mu \left.\frac{dV_x}{dy}\right|_{y=0}$ exactly

$$\cancel{\mu} \frac{\rho g \sin\theta}{\cancel{\mu}} h \left(1 - \frac{y}{h}\right)_{y=0} = \cancel{\rho} g \sin\theta\, h = f \frac{1}{2} \cancel{\rho} \left(\frac{\rho g \sin\theta}{\mu} \frac{h^2}{3}\right)^2$$

$$\boxed{f = \frac{18 \left(\mu/\rho\right)^2}{g \sin\theta\, h^3}}$$



- Steady
- Fully developed
- Incompressible
- No z-dependency

<u>Continuity</u> :

$$\underbrace{\cancel{\frac{\partial \rho}{\partial t}}}_{\text{incompressible}} + \nabla\cdot(\rho \vec{v}) = 0 \Rightarrow \underbrace{\cancel{\frac{\partial v_x}{\partial x}}}_{\substack{\text{fully}\\ \text{developed}}} + \frac{\partial v_y}{\partial y} + \underbrace{\cancel{\frac{\partial v_z}{\partial z}}}_{\text{no } z \text{ dependency}} = 0$$

$$\left.\begin{aligned} \frac{\partial v_y}{\partial y} = 0 \Rightarrow \; & v_y = \text{constant} \\ & \frac{\partial}{\partial t} = \frac{\partial}{\partial x} = \frac{\partial}{\partial z} = 0 \\ & v_y = 0 \; @ \; y = 0 \end{aligned}\right\} \quad v_y \equiv 0$$

<u>Conservation of momentum</u> : <u>x-direction</u>

$$\rho\left(\underbrace{\cancel{\frac{\partial v_x}{\partial t}}}_{\text{Steady}} + v_x \underbrace{\cancel{\frac{\partial v_x}{\partial x}}}_{\substack{\text{fully}\\ \text{developed}}} + \underbrace{\cancel{v_y}}_{\substack{\text{unidirectional}\\ \text{flow}}} \frac{\partial v_x}{\partial y} + v_z \underbrace{\cancel{\frac{\partial v_x}{\partial z}}}_{\text{no } z\text{-dependency}}\right) = -\frac{\partial P}{\partial x} + \mu\left(\underbrace{\cancel{\frac{\partial^2 v_x}{\partial x^2}}}_{\substack{\text{fully}\\ \text{developed}}} + \frac{\partial^2 v_x}{\partial y^2} + \underbrace{\cancel{\frac{\partial^2 v_x}{\partial z^2}}}_{\text{no } z\text{-dependen}}\right) + \rho$$

$\frac{\partial P}{\partial x} = 0$ since at $y = h$ (free surface) $P = P_{atm}$ so P is constant along x (direction parallel to the free surface) and from conservation of momen in the y-direction we get $\frac{\partial P}{\partial y} = 0$

$$0 = \mu \frac{d^2 v_x}{dy^2} - \rho g$$

<u>b.c.</u> $y = 0 \rightarrow v_x = V_{plate}$ (no slip condition: velocity of th fluid matches the velocity of the wall with which it is in contact

$y = h \rightarrow \frac{\partial v_x}{\partial y} = 0$ (free slip condition: zero shear on th free surface due to the mismatch in t values of viscosity between liquid and

$$\frac{dV_x}{dy} = \frac{\rho g}{\mu} y + C_1$$

$$@\, y=h \Rightarrow 0 = \frac{\rho g}{\mu} h + C_1 \rightarrow C_1 = -\frac{\rho g}{\mu} h$$

$$V_x = \frac{\rho g}{\mu}\left(\frac{y^2}{2} - hy\right) + C_2$$

$$@\, y=0 \rightarrow V_{plate} = C_2 \rightarrow \boxed{V_x(y) = V_{plate} - \frac{\rho g}{\mu} y\left(h - \frac{y}{2}\right)}$$

<u>Energy equation</u> $\quad e = u + \frac{1}{2} V^2$

steady, fully developed, unidirectional, no z dependency, incompressible, no pressure gradient, unidirectional, no z dependency

$$\rho\left(\cancel{\frac{\partial e}{\partial t}} + \cancel{V_x \frac{\partial e}{\partial x}} + \cancel{V_y \frac{\partial e}{\partial y}} + \cancel{V_z \frac{\partial e}{\partial z}}\right) = -p\,\cancel{\nabla\cdot\vec{v}} - \left(\cancel{V_x \frac{\partial P}{\partial x}} + \cancel{V_y \frac{\partial P}{\partial y}} + \cancel{V_z \frac{\partial P}{\partial z}}\right) +$$

$$+ \left[2\mu\left(\frac{1}{2}(\nabla\vec{v} + \nabla\vec{v}^T)\right) + \lambda\,\underbrace{\cancel{\nabla\cdot\vec{v}}}_{\text{incompressible}}\,\overline{\overline{I}}\right] : (\nabla\vec{v}) + \nabla\cdot\left[2\mu\left(\frac{1}{2}(\nabla\vec{v} + \nabla\vec{v}^T)\right) + \lambda\,\underbrace{\cancel{\nabla\cdot\vec{v}}}_{\text{incompressible}}\,\overline{\overline{I}}\right]\cdot\vec{v}$$

$$- K\left(\underbrace{\cancel{\frac{\partial^2 T}{\partial x^2}}}_{\text{fully developed}} + \frac{\partial^2 T}{\partial y^2} + \underbrace{\cancel{\frac{\partial^2 T}{\partial z^2}}}_{\text{no z dependency}}\right) + \rho\vec{g}\cdot\vec{v}$$

$$0 = 2\mu\,\frac{1}{2}\left(\frac{\partial V_x}{\partial y}\right)^2 + \left[2\mu\,\frac{1}{2}\,\frac{\partial}{\partial y}\left(\frac{\partial V_x}{\partial y}\right)\right]\cdot V_x - K\frac{\partial^2 T}{\partial y^2} - \rho$$

$$0 = \mu\left[\frac{\rho g}{\mu}(y-h)\right]^2 + \cancel{\mu\,\frac{\rho g}{\mu}\cdot\left[V_{plate} - \frac{\rho g}{\mu} y\left(h - \frac{y}{2}\right)\right]} - K\frac{d^2T}{dy^2} -$$

$$\cancel{-\rho g\left[V_{plate} - \frac{\rho g}{\mu} y\left(h - \frac{y}{2}\right)\right]}$$

$$K\frac{d^2T}{dy^2} = \frac{(\rho g)^2}{\mu}(y-h)^2$$

$$@\, y=h \rightarrow -K\frac{\partial T}{\partial y} = -q_s\,(\vec{q}\cdot\vec{n}) \qquad @\, y=0 \rightarrow T(0) = T_{plate}$$

$$\frac{dT}{dy} = \frac{(\rho g)^2}{\mu K}\,\frac{(y-h)^3}{3} + c_1$$

$$-K\left.\frac{dT}{dy}\right|_{y=h} = -\frac{(\rho g)^2}{\mu}\cdot 0 - K c_1 = -q_s$$

$$c_1 = \frac{q_s}{K}$$

$$T(y) = \frac{q_s}{K}\,y + \frac{(\rho g)^2}{\mu K}\,\frac{(y-h)^4}{12} + c_2$$

$$T(0) = T_{plate} = \frac{(\rho g)^2}{12\mu K}\,h^4 + c_2$$

$$T(y) = T_{plate} + q_s/K\,y + \frac{(\rho g)^2}{12\mu K}\left[(y-h)^4 - h^4\right]$$

Because the temperature (internal energy) and velocity
(kinetic energy) become fully developed, the left hand sid
of the energy equation becomes exactly zero. Note that th
is only true when we DON'T take into account potenti
energy. If we account for potential energy, there is a
flux as the fluid flows upwards $v_x \frac{\partial e}{\partial x} = v_x\left(\cancel{\frac{\partial u}{\partial x}} + \cancel{\frac{\partial \frac{1}{2}v^2}{\partial x}} + \frac{\partial \rho}{\partial}\right.$

$$= v_x \cdot \rho g$$

We account for this term on the right hand side a
work done by gravity forces.

On the left hand side we see that pressure does not do any work because the flow is incompressible and there is no pressure gradient on the fluid. The first viscous term is viscous dissipation $\Phi = \bar{\bar{\tau}} : \nabla \vec{v}$ that results in $\mu \left(\frac{\partial V_x}{\partial Y}\right)^2 = \frac{(\rho g)^2}{\mu}(Y-h)^2$. This term increases the temperature gradient on the wall (heat flux) and the temperature profile throughout the fluid.

The second viscous term represents work done by viscous stresses on the fluid. This term depends on both ρg and Vplate, as both determine the velocity profile and the shear in the fluid. $(\nabla \cdot \bar{\bar{\tau}}) \cdot \vec{v}$. This term cancels exactly with the work done by gravity forces $\rho \vec{g} \cdot \vec{v}$. This may seem surprising as they are apparently unrelated: Because the flow is bounded betw the moving wall and the free surface with the gas, in the absence of gravity, the liquid would move uniformly with velocity Vplate. So the shear (velocity difference) in the fluid is due to gravity only and not on Vplate. This the reason why the viscous dissipation Φ does not depend on Vplate (but it does on ρg). Part of the tota work done by the plate goes to increasing the potential energy of the fluid (or to balancing the negative work do by the effect of gravity). Another fraction (the rest which is solely dependent on gravity) goes to dissipation and increasing the internal energy of the fluid.

Because we assume the problem is fully developed in both the velocity and temperature fields, the solution does not depend on the initial temperature that the fluid has in the pool. The fully developed temperature profile is fully determined by :

- the temperature at the wall
- the slope of the temperature at the free surface (heat flux). There is another characteristic quan[tity] that determines the temperature field which is the viscous dissipation. This also determines what is the heat flux to the wall.

The effect of pool temperature is on the develop ment length for the temperature profile. The colder th[e] fluid is initially, the longer (larger development length along th[e] plate) it will take to reach the equilibrium temperature profi[le] (assuming $T_{plate} > T_0$ and q_s is positive). Similarly, changing the fluid viscosity and conductivity would alter the length of development for velocity and temper[ature] respectively.