When the wind has been blowing for a long time and a steady state has been set up, the surface rises at a small uniform elevation rate α and there is a pressure gradient that induces a recirculation cell in the lake. This ensures that the mass flow rate through any cross section is zero $\Rightarrow$ mass is conserved and cannot accumulate anywhere in steady-state.

CONSERVATION OF MOMENTUM (N-S EQ.)

x-direction

negligible — FULLY DEVELOPED

$$\rho\left(\cancel{\frac{\partial v_x}{\partial t}} + \cancel{v_x \frac{\partial v_x}{\partial x}} + \cancel{v_y}\,\frac{\partial v_x}{\partial y} + \cancel{v_z \frac{\partial v_x}{\partial z}}\right) = -\frac{\partial P}{\partial x} + \mu\left(\cancel{\frac{\partial^2 v_x}{\partial x^2}} + \frac{\partial^2 v_x}{\partial y^2} + \cancel{\frac{\partial^2 v_x}{\partial z^2}}\right)$$

STEADY — FULLY DEVELOPED

y-direction

negligible

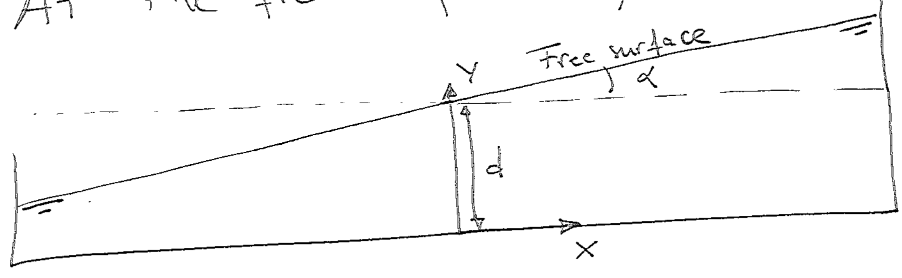
$$\rho\left(\cancel{\frac{\partial v_y}{\partial t}} + \cancel{v_x \frac{\partial v_y}{\partial x}} + \cancel{v_y \frac{\partial v_y}{\partial y}} + \cancel{v_z \frac{\partial v_y}{\partial z}}\right) = -\frac{\partial P}{\partial y} + \mu\cancel{\left(\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2}\right)} - \rho g$$

negligible v_y

$$\frac{\partial P}{\partial y} = -\rho g \Rightarrow P(x,y) = P_0(x) - \rho g y$$

At the free surface: $y = d + \alpha x \Rightarrow P_{atm} = P_0(x) - \rho g(d + \alpha x)$

$$P_0(x) = P_{atm} + \rho g(d + \alpha x)$$



$$\frac{\partial P}{\partial x} = \frac{\partial P_0}{\partial x} = \rho g \alpha$$

From the x-direction equation: $0 = -\rho g \alpha + \mu \dfrac{d^2 v_x}{dy^2}$

$$\frac{dv_x}{dy} = \frac{g\alpha}{\nu} y + C_1 \Rightarrow \mu \left.\frac{dv_x}{dy}\right|_{y=d} = S' = \rho g \alpha d + C_1 \mu$$

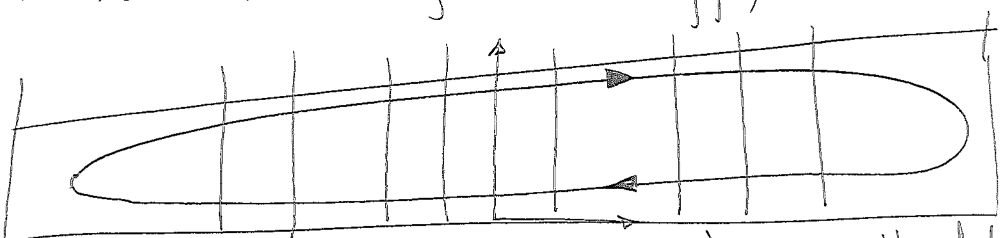
$$C_1 = S'/\mu - \frac{g\alpha}{\nu} d$$

$$v_x = \frac{g\alpha}{\nu}\, y^2/2 + \left(\frac{S}{\mu} - \frac{g\alpha}{\nu} d\right) y + C_2$$

$$v_x(y=0) = 0 = C_2$$

$$\boxed{v_x(y) = \frac{g\alpha}{\nu}\, y^2/2 + \left(\frac{S}{\mu} - \frac{g\alpha}{\nu} d\right) y}$$

But α and S are not independent, the elevation angle of the free surface is a function of the shear rate imposed by the wind. To link the together we apply conservation of mass:



Along all these cross sections of the lake, away from the ends, the flow is quasi-unidirectional but mass has to be conserved and that means the net flux across a section has to be zero:

width of the lake perpendicular to the paper

$$\int_{\text{cross-section}} \rho\, v_x(y)\, dA = \rho \int_0^d v_x(y) \cdot \cancel{b}\, dy = 0$$

$$\int_0^d \left[\frac{g\alpha}{\nu} y^2/2 + \left(S/\mu - \frac{g\alpha}{\nu} d\right) y\right] dy = \frac{g\alpha}{\nu}\frac{y^3}{6} + \left(\frac{S}{\mu} - \frac{g\alpha}{\nu} d\right) y^2/2 \Big|_0^d =$$

$$\frac{g\alpha}{\nu}\frac{d^3}{6} + \frac{S d^2}{\mu 2} - \frac{g\alpha}{\nu}\frac{d^3}{2} = 0$$

$$\frac{S d^2}{2\mu} = \frac{g\alpha}{\nu} d^3/3 \Rightarrow \boxed{\alpha = \frac{3}{2}\frac{S}{\rho g d}}$$