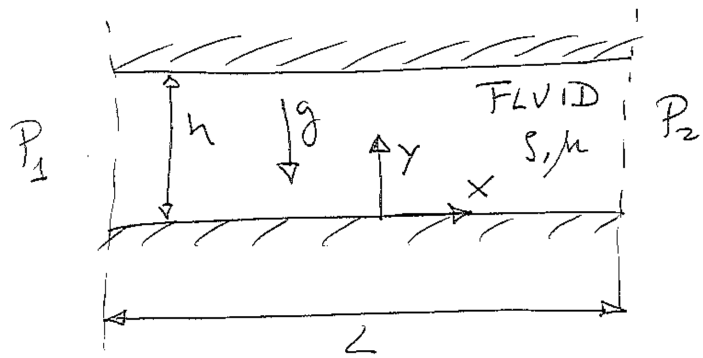
UNSTEADY POISEUILLE FLOW



- Incompressible
- Fully developed $\dfrac{\partial \vec{v}}{\partial x} = 0$
- Infinite plate: $\dfrac{\partial}{\partial z} = 0$

Continuity

$$\cancel{\frac{\partial \rho}{\partial t}} + \frac{\partial \rho v_x}{\partial x} + \frac{\partial \rho v_y}{\partial y} + \frac{\partial \rho v_z}{\partial z} = 0$$

$$\rho \left(\cancel{\frac{\partial v_x}{\partial x}} + \frac{\partial v_y}{\partial y} + \cancel{\frac{\partial v_z}{\partial z}} \right) = 0 \Rightarrow \frac{\partial v_y}{\partial y} = 0 ;$$

B.C.s
$v_y = 0$ @ $y = 0$
$v_y = 0$ @ $y = h$

$$v_y \equiv 0 \quad \forall y$$

Navier - Stokes

$$\rho \left(\frac{\partial v_x}{\partial t} + v_x \cancel{\frac{\partial v_x}{\partial x}} + \cancel{v_y} \frac{\partial v_x}{\partial y} + v_z \cancel{\frac{\partial v_x}{\partial z}} \right) = - \frac{\partial P}{\partial x} + \mu \left(\cancel{\frac{\partial^2 v_x}{\partial x^2}} + \frac{\partial^2 v_x}{\partial y^2} + \cancel{\frac{\partial^2 v_x}{\partial z^2}} \right)$$

$$\frac{\partial v_x}{\partial t} = - \frac{1}{\rho} \frac{\partial P}{\partial x} + \nu \frac{\partial^2 v_x}{\partial x^2} ;$$

B.C.s
$v_x = 0$ @ $y = 0, h$
$v_x = 0$ @ $t = 0$

Non-dimensionalizing the equation, and boundary conditions

with: $\dfrac{dP}{dx}(t) = \dfrac{P_2 - P_1}{L} = -G \quad ; \quad t^* = \dfrac{t}{T} ; \; v_x^* = \dfrac{v_x}{U_0}$

$$\frac{T}{U_0}\frac{\partial V_x^*}{\partial t^*} = \frac{G}{\rho} + \frac{\nu U_0}{L^2}\frac{\partial^2 V_x^*}{\partial y^{*2}}$$

$$\frac{L^2}{\nu T}\frac{\partial V_x^*}{\partial t^*} = \frac{G L^2}{\mu U_0} + \frac{\partial^2 V_x^*}{\partial y^{*2}}$$

so making $T = \frac{L^2}{\nu}$ and $U_0 = \frac{G L^2}{\mu}$ we simplify

the equation to: $\frac{\partial V_x}{\partial t} = 1 + \frac{\partial^2 V_x}{\partial y^2}$ (dropping the *)

To solve this we need to first find a particular solution (steady state in this case) so the resulting PDE is homogeneous is both eq. and b.c.s

$$\frac{d^2\overline{V}_x}{dy^2} = -1 \Rightarrow \left.\begin{array}{l}\overline{V}_x(y=0)=0\\ \overline{V}_x(y=1)=0\end{array}\right\} \overline{V}_x(y) = \frac{1}{2}y(1-y)$$

$$V_x = \overline{V}_x(y) + \tilde{V}_x(y,t)$$

$$\frac{\partial \tilde{V}_x}{\partial t} = \cancel{1} + \Bigg(\underbrace{\frac{d^2\overline{V}_x}{dy^2}}_{= \cancel{-1}} + \frac{\partial^2 \tilde{V}_x}{\partial y^2}\Bigg);$$

B.C.s

$\tilde{V}_x(t=0) = -\frac{1}{2}y(1-y)$

$\tilde{V}_x(y=0,t) = 0$

$\tilde{V}_x(y=1,t) = 0$

Now, we can separate variables:

$$\overset{n}{v}_x(y,t) = Y(y)\cdot T(t)$$

$$\dot{T}\,Y = T\cdot Y''$$ (where $\dot{}$ denotes a temporal derivative and $'$ denotes a spatial derivative)

$$\underbrace{\frac{\dot{T}}{T}}_{\text{function of } t} = \underbrace{\frac{Y''}{Y}}_{\text{function of } y} = \underbrace{-\lambda^2}_{\text{constant}} \Rightarrow$$

$$\Rightarrow \quad \dot{T}(t) = -\lambda^2\, T(t) \Rightarrow \underline{\underline{T(t) = A e^{-\lambda^2 t}}}$$

$$Y''(y) = -\lambda^2\, Y(y) \Rightarrow \underline{\underline{Y(y) = B\sin(\lambda y) + C\cos(\lambda y)}}$$

Boundary Conditions:

for any value of t: $\quad Y(y=0) = 0 \Rightarrow C = 0$

$$Y(y=1) = 0 \Rightarrow B\sin\lambda = 0$$

Either $B=0 \Rightarrow \overset{n}{v}_x \equiv 0$

or $\sin\lambda = 0 \Rightarrow \lambda = n\pi$

for $\underline{n = 0, 1, \ldots}$

Initial condition:

$$T(t=0)\cdot \sum_{n=0}^{\infty} B_n \sin(n\pi y) = -\frac{1}{2}\, y(1-y)$$

To find the values of the constants, we multiply both sides of the equation by the term $\sin(m\pi y)$ and integrate from 0 to π (a period of the function $\sin(m\pi y)$).

$$\int_0^1 \Big[A \underbrace{e^{-n^2\pi^2 t(=0)}}_{1} \cdot \sum_{n=0}^{\infty} B_n \underbrace{\sin(n\pi y)\Big] \sin(m\pi y)\, dy}_{0 \text{ if } m \neq n} = \int_0^1 \Big[\frac{1}{2} y(1-y) \cdot \sin(m\pi y) \Big] dy$$

$$A B_m \underbrace{\int_0^1 \sin^2(m\pi y)\, dy}_{1/2} = -\frac{1}{2} \underbrace{\int_0^1 y \sin(m\pi y)\, dy}_{\text{integration by parts} \Rightarrow \frac{-1}{2m\pi}\left[(-1)^m - 1\right]} + \frac{1}{2} \underbrace{\int_0^1 y^2 \sin(m\pi y)\, dy}_{\substack{\frac{-1}{2m\pi}\left[(-1)^m-1\right] \\ \frac{-2}{(m\pi)^3}\left[(-1)^m-1\right]}}$$

$$\frac{1}{2} A B_m = -\frac{1}{2} \left\{ \cancel{\frac{-1}{m\pi}\left[(-1)^m - 1\right]} \right\} + \frac{1}{2} \left\{ \cancel{\frac{-1}{m\pi}\left[(-1)^m - 1\right]} - \frac{2}{(m\pi)^3}\left[(-1)^m - 1\right] \right\}$$

$$A B_m = \frac{-2}{(m\pi)^3} \underbrace{\left[(-1)^m - 1\right]}_{\substack{-2 \text{ if } m \text{ is odd} \\ 0 \text{ if } m \text{ is even}}}$$

$$\boxed{v_x(y,t) = \underbrace{\frac{1}{2} y(1-y)}_{\bar{v}_x(y)} + \underbrace{\sum_{m=1}^{\infty} \frac{-4}{(2m-1)^3 \pi^3} \sin\left[(2m-1)\pi y\right] e^{-(2m-1)^2 \pi^2 t}}_{\tilde{v}_x(y,t)}}$$