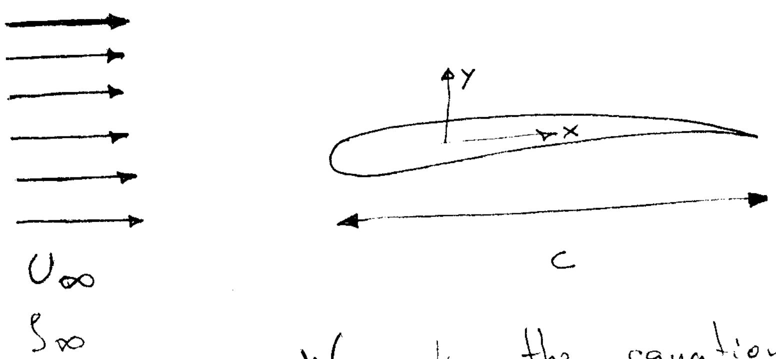
BOUNDARY LAYER THEORY



We make the equations non dimensional:

$$\frac{\partial \rho}{\partial t} + \nabla . \rho \vec{v} = 0$$

$$\rho \frac{\partial \vec{v}}{\partial t} + \rho \vec{v} . \nabla \vec{v} = -\nabla p + \nabla . \bar{\bar{\tau}} + \rho \vec{g}$$

For incompressible, steady flow with negligible gravity term:

$$\boxed{\begin{aligned} &\nabla . \vec{v} = 0 \\ &\vec{v} . \nabla \vec{v} = -\frac{1}{\rho_\infty} \nabla p + \nu \nabla^2 \vec{v} \end{aligned}}$$

$$\nabla^* = \frac{1}{c} \nabla$$

$$\vec{v}^* = \frac{\vec{v}}{U_\infty}$$

$$p^* = \frac{p - p_\infty}{\rho_\infty U_\infty^2}$$

$$\cancel{\left(\frac{U_\infty}{c}\right)} \nabla^* . \vec{v}^* = 0$$

$$\cancel{\left(\frac{U_\infty^2}{c}\right)} \vec{v}^* . \nabla^* \vec{v}^* = - \frac{\rho_\infty \cancel{U_\infty^2}}{\rho_\infty c} \nabla^* p^* + \nu \frac{U_\infty}{c^2} \nabla^{*2} \vec{v}^*$$

$$\vec{v}^* . \nabla^* \vec{v}^* = - \nabla^* p^* + \underbrace{\frac{\nu}{c . U_\infty}}_{= 1/Re} \nabla^{*2} \vec{v}^*$$

If $Re >> 1 \Rightarrow \frac{1}{Re} = \epsilon << 1 \Rightarrow$ Singular perturbation problem

Regular Perturbation Theory:

$$\frac{d^2 f}{dx^2} + \epsilon \frac{df}{dx} + f = \epsilon g(x)$$

$\epsilon \ll 1$ is the perturbation parameter.

Boundary conditions: $f(L) = 0$

$f(0) = 1 + \epsilon$

Singular Perturbation Theory:

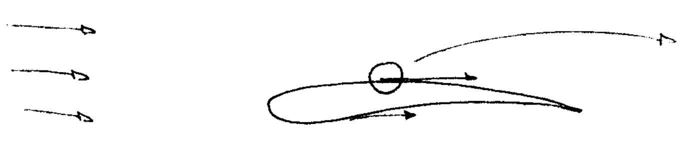
$$\epsilon \frac{d^2 f}{dx^2} + \frac{df}{dx} + f = g(x)$$

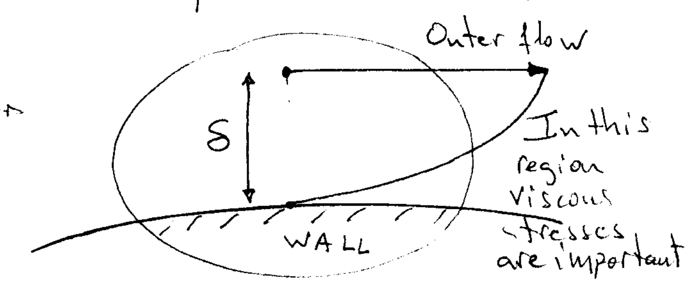
B.c's: $f(0) = 0$

$f(L) = 0$

If we neglect ϵ, the equation changes character, we have a 1^{st} order ODE and we can only impose 1 boundary condition

In boundary layer problems, we neglect viscous terms over inertial terms $\rho \vec{v} \cdot \nabla \vec{v} \gg \mu \nabla^2 \vec{v}$ but the solution produces velocity discontinuities at boundaries. Viscous stresses are very high at those discontinuities, in fact they are high enough to be comparable with the inertial terms.





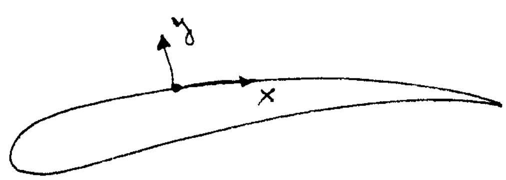
δ is the thickness of the fluid layer where viscosity is not negligible. It is an structure that develops within the flow, it is imposed by a length scale given by the geometry of the problem.

$$\underbrace{\mu \nabla^2 \vec{v} \sim \mu \frac{U_\infty}{\delta^2} \sim \frac{\rho_\infty U_\infty^2}{\ }}\sim \rho \vec{v}\cdot\nabla\vec{v}$$

$$\frac{\delta^2}{L^2} \sim \frac{\mu/\rho_\infty}{U_\infty} \cdot \frac{L}{L^2} \Rightarrow \boxed{\frac{\delta}{L} \sim \sqrt{\frac{1}{Re}}}$$

The thickness of the boundary layer is determined by the Reynolds number of the flow.

Boundary layer equations:



x: direction along the wall

y: direction normal to the wall

Cartesian formulation is valid as long as the radius of curvature is large ($R \gg \delta$)

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0$$

$$v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2}\right)$$

$$v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2}\right)$$

Unlike in the previous analysis, we now know that

$$\frac{\partial}{\partial x} \sim \frac{1}{L}\frac{\partial}{\partial x^*} \quad \text{but} \quad \frac{\partial}{\partial y} \sim \frac{1}{\delta}\frac{\partial}{\partial y^*}$$

Continuity:

$$\frac{1}{L}\frac{\partial v_x}{\partial x^*} + \frac{1}{\delta}\frac{\partial v_y}{\partial y^*} = 0$$

v_x and v_y have different velocity scales

$v_x \approx U_\infty$ but $v_y \approx ???$

$$\frac{U_\infty}{L}\frac{\partial v_x^*}{\partial x^*} + \frac{\bar{V}_y}{\delta}\frac{\partial v_y^*}{\partial y^*} = 0 \Rightarrow \boxed{\bar{V}_y^* = \frac{\delta}{L}U_\infty \ll U_\infty}$$

This is imposed by continuity

Conservation of momentum

x-axis

$$\frac{\cancel{U_\infty^2}}{\cancel{L}} v_x^*\frac{\partial v_x^*}{\partial x^*} + \frac{\cancel{U_\infty \frac{\delta}{L} U_\infty}}{\cancel{\delta}} v_y^*\frac{\partial v_x^*}{\partial y^*} = -\frac{\cancel{\rho_\infty U_\infty^2}}{\cancel{\rho_\infty L}}\frac{\partial p^*}{\partial x^*} + \nu\left(\frac{U_\infty}{L^2}\frac{\partial^2 v_x^*}{\partial x^{*2}} + \frac{U_\infty L}{L\,\delta^2}\frac{\partial^2 v_x^*}{\partial y^{*2}}\right)$$

$$v_x^*\frac{\partial v_x^*}{\partial x^*} + v_y^*\frac{\partial v_x^*}{\partial y^*} = -\frac{\partial p^*}{\partial x^*} + \cancel{\frac{1}{Re}}\frac{\partial^2 v_x^*}{\partial x^{*2}} + \underbrace{\frac{1}{Re}\left(\frac{L}{\delta}\right)^2}_{O(1)}\frac{\partial^2 v_x^*}{\partial y^{*2}}$$

Y-axis

$$\frac{U_\infty \cancel{\frac{\delta}{L}U_\infty}}{L} v_x^*\frac{\partial v_y^*}{\partial x^*} + \frac{\left(\frac{\delta}{L}U_\infty\right)^2}{\cancel{\delta}} v_y^*\frac{\partial v_y^*}{\partial y^*} = -\frac{\cancel{\rho_\infty} U_\infty^2 L^2}{\cancel{\rho_\infty}\,\delta\,\delta}\frac{\partial p^*}{\partial y^*} + \nu\left(\frac{\delta U_\infty}{L}\right)\frac{L}{U_\infty}\left(\frac{1}{L^2}\frac{\partial^2 v_y^*}{\partial x^{*2}} + \frac{1}{\delta^2}\frac{\partial^2 v_y^*}{\partial y^{*2}}\right)$$

$$v_x^*\frac{\partial v_y^*}{\partial x^*} + v_y^*\frac{\partial v_y^*}{\partial y^*} = -\left(\frac{L}{\delta}\right)^2\frac{\partial p^*}{\partial y^*} + \frac{\nu}{V_\infty L}\frac{\partial^2 v_y^*}{\partial x^{*2}} + \frac{\nu}{V_\infty L}\left(\frac{L}{\delta}\right)^2\frac{\partial^2 v_y^*}{\partial y^{*2}}$$

$1 \qquad 1 \qquad Re^2 \qquad 1/Re^2 \qquad 1/Re^2 \cdot Re$

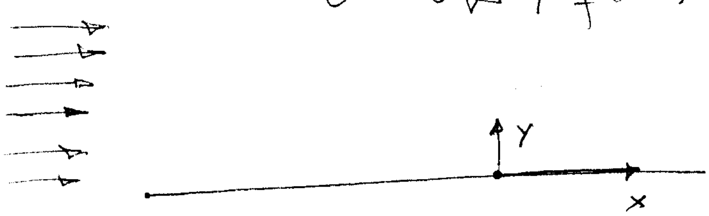
The leading order term is $\boxed{\dfrac{\partial P^*}{\partial y^*} = 0}$

The pressure across the boundary layer is approximately constant (and that is why the outer solution, often potential where we can use Bernoulli, is useful to estimate the lift but not the drag).

Example: Blasius solution

Flat plate, zero pressure gradient.

$$U = U_\infty \neq f(x) \Longleftrightarrow \frac{dP}{dx} = 0$$



Continuity

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0$$

Conservation of momentum

$$v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} = \nu \frac{\partial^2 v_x}{\partial y^2}$$

Because the flat plate is infinitely long, there is no characteristic length scale to this problem

$\Rightarrow$ Similarity solution

Using the stream function $\Psi(x,y)$ so that continuity is satisfied identically (by definition of Ψ)

$$\frac{\partial \Psi}{\partial y} \cdot \frac{\partial^2 \Psi}{\partial y\, \partial x} + \left(-\frac{\partial \Psi}{\partial x}\right) \cdot \frac{\partial^2 \Psi}{\partial y^2} = \nu \frac{\partial^3 \Psi}{\partial y^3}$$

Making the equations non-dimensional:

$$x^* = \frac{x}{L???} \; ; \quad y^* = \frac{y}{\delta(x)} \; ; \quad \psi^* = \frac{\psi}{\delta V_\infty}$$

$$v_x^* = \frac{v_x}{U_\infty} \; ; \quad v_y^* = \frac{v_y}{V_y}$$

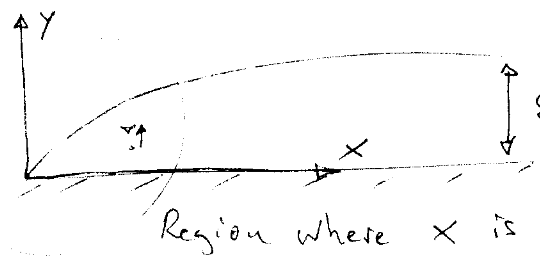
We know $\left(\frac{\delta}{L}\right)^2 \sim Re^{-1} = \left(\frac{V_\infty L}{\nu}\right)^{-1}$

$$\| \qquad\qquad \|$$

$$\frac{\partial \psi^*}{\partial y^*} \sim \frac{\partial \psi / \partial y}{U_\infty} \qquad \frac{\partial \psi^*}{\partial x^*} \sim \frac{\partial \psi / \partial x}{V_y}$$

$$\delta \sim \sqrt{\nu L / U_\infty}$$

But, what is L??? The only length scale in the problem, that determines the thickness of the boundary layer is the distance to the origin



$\delta(x) \to \delta \sim \sqrt{\frac{\nu x}{U_\infty}}$

Region where x is small $Re_x = \frac{U_\infty \cdot x}{\nu}$ is not large enough and the velocity is not unidirectional.

$$\psi^*(\eta) = \frac{\psi}{\sqrt{\nu U_\infty x}} \; ; \qquad x^* = \frac{x}{x} \text{ ?? not a variable}$$

$$y^* = \frac{y}{\sqrt{\nu x / U_\infty}} = \eta \quad \text{self similarity variable}$$

$$(x, y) \to (x, \eta)$$

$$\psi(x,y) \to \psi^*(\eta)$$

The non-dimensionalization <u>is</u> the change of variables

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$$\frac{\partial \Psi}{\partial x} = \sqrt{\nu U_\infty}\frac{1}{2\sqrt{x}} f(\eta) + \sqrt{\nu x U_\infty} f'(\eta)\cdot\frac{\partial \eta}{\partial x}$$

$$= \frac{1}{2}\sqrt{\frac{\nu U_\infty}{x}} f(\eta) - \frac{1}{2}\frac{y}{x} U_\infty f'$$

$$\frac{\partial \Psi}{\partial y} = \sqrt{\nu U_\infty}\, x^{1/2}\cdot f'(\eta)\cdot\frac{\partial \eta}{\partial y} = U_\infty f'$$

$$\frac{\partial^2 \Psi}{\partial x \partial y} = \frac{1}{2}\sqrt{\frac{\nu U_\infty}{x}} f'(\eta)\cdot\frac{\partial \eta}{\partial y} - \frac{1}{2}\frac{U_\infty}{x} f' - \frac{1}{2}\frac{y}{x} U_\infty f''\frac{\partial \eta}{\partial y}$$

$$\frac{\partial^2 \Psi}{\partial y^2} = U_\infty f''\cdot\frac{\partial \eta}{\partial y} = \frac{U_\infty f''}{\sqrt{\nu x/U_\infty}}$$

$$\frac{\partial^3 \Psi}{\partial y^3} = \frac{U_\infty f'''}{\sqrt{\nu x/U_\infty}}\cdot\frac{1}{\sqrt{\nu x/U_\infty}} = \frac{U_\infty^2}{\nu x} f'''$$

$$\nu\frac{\partial^3 \Psi}{\partial y^3} = \frac{\partial \Psi}{\partial y}\frac{\partial^2 \Psi}{\partial x \partial y} - \frac{\partial \Psi}{\partial x}\frac{\partial^2 \Psi}{\partial y^2}$$

$$\cancel{\nu}\frac{U_\infty^2}{\cancel{\nu} x} f''' = U_\infty f'\left[\frac{1}{2}\sqrt{\frac{\cancel{\nu} U_\infty}{x}} f'\frac{1}{\sqrt{\frac{\cancel{\nu} x}{U_\infty}}} - \cancel{\frac{1}{2}\frac{U_\infty}{x} f'} - \frac{1}{2}\frac{y}{x} U_\infty f''\frac{1}{\sqrt{\frac{\nu x}{U_\infty}}}\right]$$

$$-\left(+\frac{1}{2}\sqrt{\frac{\nu U_\infty}{x}} f - \frac{1}{2}\frac{y}{x} U_\infty f'\right)\frac{U_\infty}{\sqrt{\nu x/U_\infty}} f''$$

$$\frac{U_\infty^2}{x} f''' = \cancel{-\frac{1}{2}\frac{U_\infty^2}{x}\eta f''} - \frac{1}{2}\frac{U_\infty^2}{x} f f'' + \cancel{\frac{1}{2}\frac{U_\infty^2}{x}\eta f''}$$

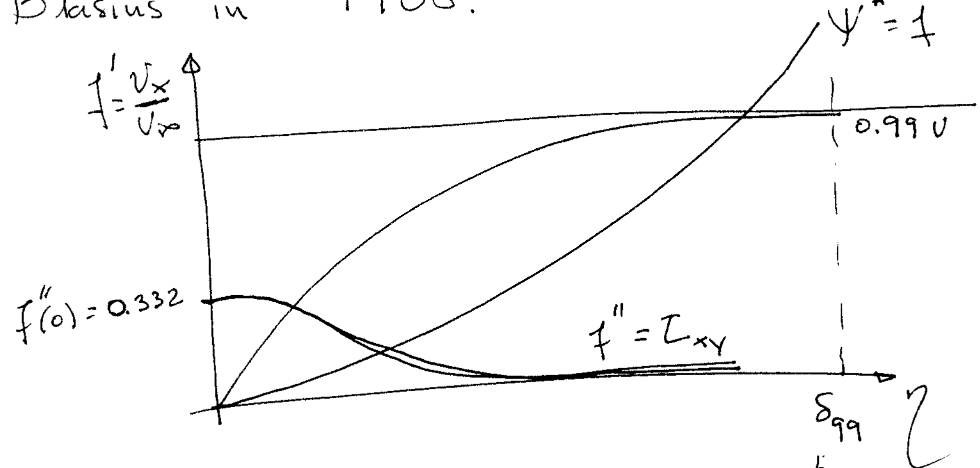
$$f''' + \frac{1}{2} f f'' = 0$$

Boundary conditions: $y = 0 \Rightarrow V_x = V_y = 0 \Rightarrow \eta = 0 \Rightarrow f'(0) = 0$

$y \to \infty \Rightarrow V_x = U_\infty \Rightarrow \eta \to \infty \Rightarrow f'(\eta \to \infty) = 1$

The wall is a streamline: $y = 0 \Rightarrow f(0) = 0$.

This problem accepts a series solution published by Blasius in 1908.



$$\eta_{99} = 4.9 \Rightarrow \delta_{99}/x = \frac{4.9}{\sqrt{Re_x}}$$

In water $U_\infty = 1\,m/s$, $x = 1\,m \Rightarrow Re_x \approx 10^6 \Rightarrow \delta_{99} = 0.5\,cm$

In air $U_\infty = 1\,m/s$, $x = 1\,m \Rightarrow Re_x \approx 10^7 \Rightarrow \delta_{99} = 0.2\,cm$