VON KARMAN MOMENTUM INTEGRAL

Exact solutions to the boundary layer equations are only possible in a few simple cases. Most of the time approximate solutions (analytic or numerical) need to be used. Von Karman - Pohlhausen uses integral formulation to provide simplified analysis to answer the important questions in the boundary layer, namely what is the wall shear stress? what is the boundary layer thickness? and where will it separate first?

$$V_x \frac{\partial V_x}{\partial x} + V_y \frac{\partial V_x}{\partial y} = V_{outer} \frac{dV_{outer}}{dx} + \nu \frac{\partial^2 V_x}{\partial y^2}$$

Assuming V_{outer} can be expressed as $U(x)$, velocity along the streamlines of the outer flow which is only a function of distance along the streamline (x):

We can ~~add~~ substract $V_x \frac{dU}{dx}$ from both sides

$$V_x \frac{\partial (V_x - U)}{\partial x} + V_y \frac{\partial V_x}{\partial y} = (U - V_x) \frac{dU}{dx} + \nu \frac{\partial^2 V_x}{\partial y^2}$$

Regrouping the terms that contain U to the LHS and using the fact that $\frac{\partial U}{\partial y} = 0$, we get

$$(U - V_x)\frac{dU}{dx} + V_x \frac{\partial (U - V_x)}{\partial x} + V_y \frac{\partial (U - V_x)}{\partial y} = -\nu \frac{\partial^2 V_x}{\partial y^2}$$

Integrating between $y=0$ and $y=h$ $(h > \delta)$

1st term LHS: $$\int_0^h (U - V_x)\frac{dU}{dx}\,dy = U\frac{dU}{dx}\underbrace{\int_0^h \left(1 - \frac{V_x}{U}\right)dy}_{\text{displacement thickness } \delta^*}$$

3rd term LHS: $$\int_0^h V_y \frac{\partial (U - V_x)}{\partial y}\,dy = \Big[\cancel{V_y}\,\cancel{(U - V_x)}\Big]_0^h - \int_0^h (U - V_x)\frac{\partial V_y}{\partial y}\,dy =$$

0 at $y=0$ — 0 at $y=h$

$$= -\int_0^h (U - V_x)\frac{\partial V_y}{\partial y}\,dy = +\int_0^h (U - V_x)\frac{\partial V_x}{\partial x}\,dy$$

continuity

RHS: $$-\nu\int_0^h \frac{\partial^2 V_x}{\partial y^2}\,dy = -\nu\left[\cancel{\frac{\partial V_x}{\partial y}}\Big|_{y=h} - \frac{\partial V_x}{\partial y}\Big|_{y=0}\right] = +\nu\frac{\partial V_x}{\partial y}\Big|_{y=0} = \frac{\tau_w}{\rho}$$

0

$$U\frac{dU}{dx}\delta^* + \int_0^h \underbrace{\left[V_x \frac{\partial (U - V_x)}{\partial x} + (U - V_x)\frac{\partial V_x}{\partial x}\right]}_{\frac{\partial}{\partial x}\left[V_x (U - V_x)\right]} dy = \frac{\tau_w}{\rho}$$

$$U \cdot \frac{dU}{dx}\delta^* + \frac{\partial}{\partial x}\int_0^h V_x (U - V_x)\,dy = \frac{\tau_w}{\rho}$$

$$\boxed{\delta^* U \frac{dU}{dx} + \frac{d}{dx}\left(U^2\theta\right) = \frac{\tau_w}{\rho}}$$

Where $\theta = \int_0^{h \text{ (or } \infty)} \frac{V_x}{U}\left(\frac{U-V_x}{U}\right) dy = \int_0^{\infty} \frac{V_x}{U}\left(1-\frac{V_x}{U}\right) dy$

The simplest velocity profile that can satisfy the boundary conditions is a parabola:

$$V_x(x,y) = U\left(\frac{2y}{\delta} - \frac{y^2}{\delta^2}\right)$$

at $y=0$ $\quad V_x(y=0)=0$

at $y=\delta$ $\quad \left.\frac{\partial V_x}{\partial y}\right|_{y=\delta} = 0$

With this we can calculate

- the displacement thickness:

$$\delta^* = \delta\int_0^1 \left(1 - \frac{2y}{\delta} + \frac{y^2}{\delta^2}\right) d\,y/\delta = \delta\left[\frac{y}{\delta} - \left(\frac{y}{\delta}\right)^2 + \frac{1}{3}\left(\frac{y}{\delta}\right)^3\right]_0^1$$

$$\boxed{\delta^* = \frac{\delta}{3}}$$

- the momentum thickness:

$$\theta = \delta\int_0^1 \left(2\,y/\delta \quad y^2/\delta^2\right)\left(1 - \frac{2y}{\delta} + y^2/\delta^2\right) d(y/\delta)$$

$$\theta = \delta\left[\left(\frac{y}{\delta}\right)^2 - \frac{5}{3}\left(y/\delta\right)^3 + \left(y/\delta\right)^4 - \frac{1}{5}\left(y/\delta\right)^5\right]_0^1 = \left(1 - 5/3 + 1 - \frac{1}{5}\right)\delta$$

$$= \left(\frac{1}{3} - \frac{1}{5}\right)\delta = \frac{2}{15}\,\delta$$

- the shear stress is

$$\delta^* U \frac{dU}{dx} + \frac{d}{dx}\left(U^2\theta\right) = \frac{\tau_w}{\rho}$$

but we don't know what $\frac{d\theta}{dx}$ because we don't really know what δ looks like.

Alternatively: $\tau_w = \mu \left.\frac{\partial V_x}{\partial y}\right|_{y=0} = \mu U \left.\left(\frac{2}{\delta} - \frac{2y}{\delta^2}\right)\right|_{y=0}$

$$\tau_w = 2\mu \frac{U}{\delta}$$

and therefore we have a differential equation for $\delta(x)$

$$\delta^* U \underbrace{\cancel{\frac{dU}{dx}}}_{0} + \frac{d}{dx}\left(U^2\theta\right) = \frac{\tau_w}{\rho} \Rightarrow U^2 \frac{d}{dx}\theta = \frac{2\mu}{\rho}\frac{U}{\delta}$$

0 flat plate, zero pressure gradient boundary layer

$$U \frac{\cancel{2}}{15}\frac{d}{dx}\delta = \cancel{2}\mu/\rho\delta$$

$$\delta \, d\delta = \frac{15\mu}{\rho U} dx$$

$$\frac{\delta^2}{2} = \frac{15\mu x}{\rho U} + \underbrace{\cancel{\text{constant}}}_{0}$$

$\delta(x=0)=0$

$$\delta(x) = \sqrt{\frac{30\nu x}{U}}$$

$$\frac{\delta(x)}{x} = \sqrt{\frac{30\nu}{U\cdot x}} = \frac{\sqrt{30}}{\sqrt{Re_x}}$$

That gives us:

$$\frac{\delta^*}{x} = \frac{\sqrt{30}}{3} Re_x^{-1/2}$$

$$\frac{\theta}{x} = \frac{2\sqrt{30}}{15} Re_x^{-1/2}$$

Coefficient of friction

$$\frac{\tau_w}{\frac{1}{2}\rho U^2} = \frac{4\mu \cancel{U}}{\rho U^{\cancel{2}}\delta} = \frac{4\nu}{U\sqrt{\frac{30\nu x}{U}}} = \boxed{\frac{4}{\sqrt{30}} Re_x^{-1/2} = C_f}$$