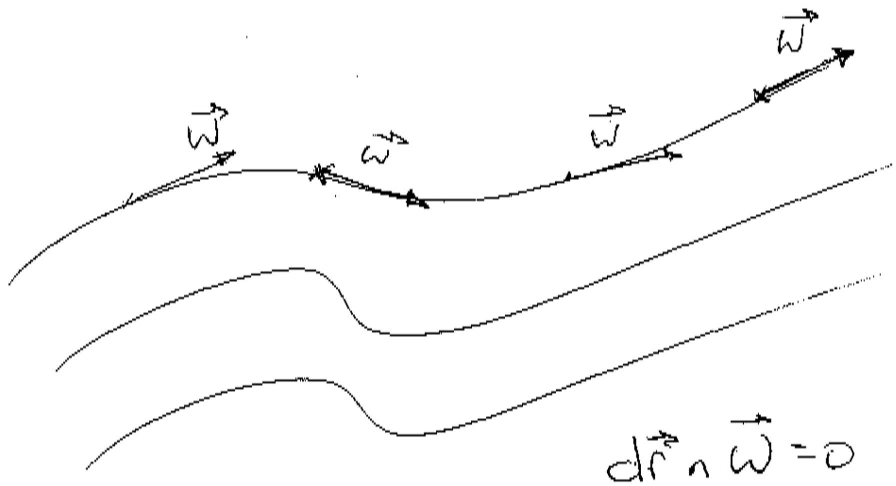
Properties of vorticity

$$\nabla \cdot \vec{\omega} = \nabla \cdot (\nabla \wedge \vec{v}) = \nabla \wedge (\nabla \cdot \vec{v}) = 0$$

$\boxed{\nabla \cdot \vec{\omega} = 0}$: Vorticity behaves like velocity in an incompressible flow.

Vortex lines: tangent to the vorticity (equivalent to streamlines for velocity)

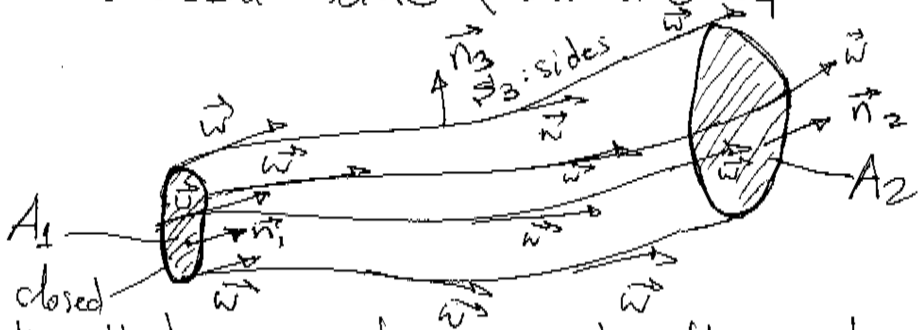


$$d\vec{r} \wedge \vec{\omega} = 0$$

or

$$\frac{dx}{\omega_x} = \frac{dy}{\omega_y} = \frac{dz}{\omega_z}$$

Vortex tubes: surface formed instantaneously by all the vortex lines passing through a closed line (bundle of vortex lines)



line that we use to generate the vortex tube.

Consider the circulation for surfaces 1 and 2

$$\Gamma_2 - \Gamma_1 = \iint_{S_2} \vec{\omega} \cdot \vec{n}\, dA - \iint_{S_1} \vec{\omega} \cdot \vec{n}\, dA$$

$$= \iint_{\substack{\text{Closed} \\ \text{Surface} \\ \text{of vortex} \\ \text{tube}}} \vec{\omega} \cdot \vec{n}\, dA$$

Since $\iint_{S_3} \vec{\omega} \cdot \vec{n}\, dA =$ because on S_3, the sides of the vortex tube $\vec{n}$ is $\perp$ to $\vec{\omega}$

Divergence or Gauss Theorem

$$= \iiint_{\substack{\text{entire vortex} \\ \text{tube volume}}} \nabla \cdot \vec{\omega}\, d\forall \equiv 0 \qquad \text{since } \nabla \cdot \vec{\omega} \equiv 0$$

$\Gamma_1 = \Gamma_2$: Helmholtz Theorem

Helmoltz Theorems

Under the following (very restrictive) assumptions,

1. Inviscid flow
2. Conservative body forces
3. Barotropic flow: $\rho(p)$
4. Non-rotating frame

Helmholtz formulated 4 theorems based on the circulation of a vortex tube:

1. Vortex lines are material lines, they move with the fluid.
2. The circulation in a vortex tube is constant along its length.
3. A vortex tube cannot end within the fluid. It must end on a solid boundary or form a loop with itself.
4. Strength of a vortex tube is constant in time

Kelvin's circulation Theorem

In an:

1. Inviscid
2. Barotropic $\rho(P)$ flow with
3. Conservative body forces and when observe
4. In a non-rotating reference frame:

The circulation of a material line C that moves with the fluid is conserved:

$$\frac{D\Gamma}{Dt} = 0 \qquad \Gamma = \oint_C \vec{v}\cdot d\vec{l} = \int_{S_C} \vec{\omega}\cdot\vec{n}\,dA$$

$$\frac{D}{Dt}\oint_C \vec{v}\cdot d\vec{l} = \oint_C \frac{D\vec{v}}{Dt}\cdot d\vec{l} + \oint_C \vec{v}\cdot\frac{D}{Dt}(d\vec{l})$$

We know $\frac{D\vec{w}}{Dt} = -\frac{1}{\rho}\nabla p + \vec{g} + \frac{1}{\rho}\nabla\cdot\overline{\overline{\tau}}'$

$$\frac{D\Gamma}{Dt} = -\oint_C \frac{1}{\rho}\nabla p\cdot d\vec{l} + \oint_C \vec{g}\cdot d\vec{l} + \oint_C \frac{1}{\rho}\nabla\cdot\overline{\overline{\tau}}'\cdot d\vec{l} + \oint_C \vec{v}\,\frac{D}{Dt}(d\vec{l})$$

$\boxed{\int\frac{\nabla P}{\rho} = \nabla \mathcal{P}}$ barotropic $\qquad \boxed{\vec{g} = -\nabla G}$ conservative $\qquad 0$ inviscid

$$-\oint_C \nabla\mathcal{P}\cdot d\vec{l} + \int -\nabla G\cdot d\vec{l} + \oint_C \vec{v}\cdot d\vec{v}$$

$$\frac{D\Gamma}{Dt} = -(\mathcal{P}_B - \mathcal{P}_A) - (G_B - G_A) + \frac{1}{2}(v_B^2 - v_A^2)$$

but in a loop $B \equiv A$ so

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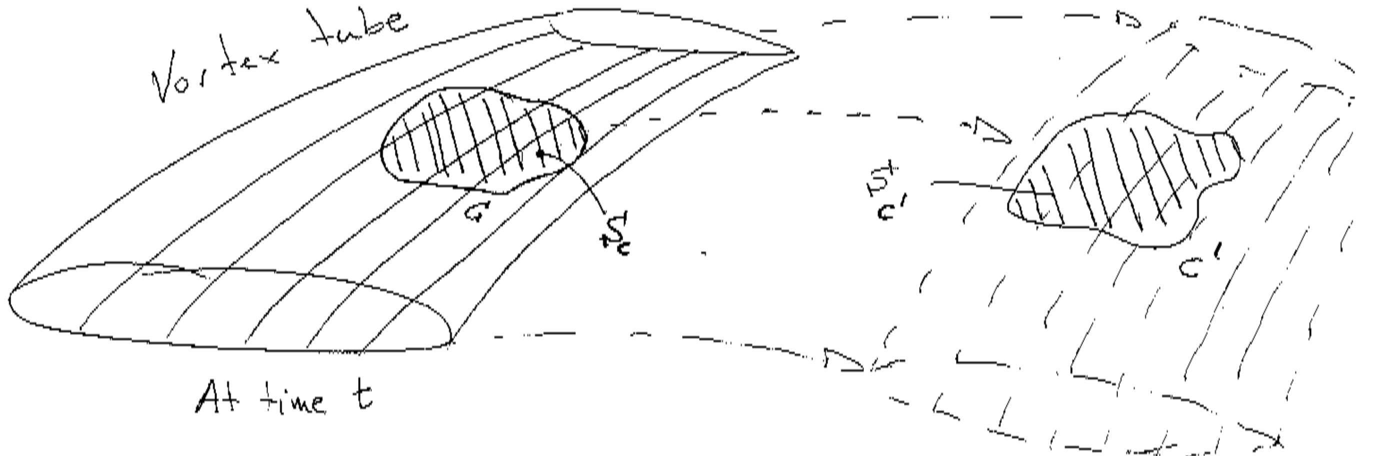
$$\frac{D\Gamma}{Dt} = 0$$

We can now prove Helmholtz 1st Theorem. We have already proven Helmholtz 2nd Theorem (and 3rd is just a corolary of the 2nd) under less restrictive assumptions. The 4th Theorem can be proven easily when we derive the vorticity equation



At time $t' = t + \Delta t$

c is on the surface of the vortex tube (arbitrary close line) $\Rightarrow$ Using Kelvin's Theo

so $\Gamma_c = \oint_{S_c} \vec{w}\cdot\vec{n}\, dA = 0$ since $\vec{\omega} \perp \vec{n}$ on S_c

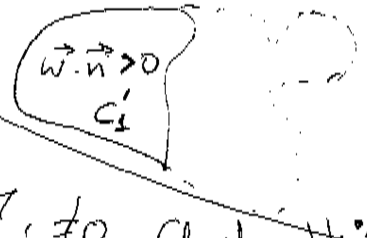
we could choose a different c_1 so that when it moves with the local convective derivative it becomes

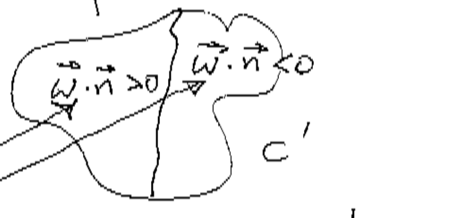
$\Gamma_{c'} = 0 = \Gamma_c$ and

$\Gamma_{c'} = \int_{S_{c'}} \vec{w}\cdot\vec{n}\, dA = 0$

this does not necessarily mean $\vec{\omega}\cdot\vec{n} = 0$ everywhe

but if





$\Gamma_{c_1} = 0$ but $\Gamma_{c_1'} \neq 0$. Clearly this is not true.

then, $\vec{w}\cdot\vec{n} = 0$ c' is o the vortext

(55) # Vorticity equation

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v} = -\frac{1}{\rho}\nabla p + \vec{g} + \frac{1}{\rho}\nabla\cdot\overline{\overline{\tau}}'$$

Take the curl of this equation:

$$\nabla\wedge\left(\qquad\qquad\right)$$

$$\frac{\partial}{\partial t}(\nabla\wedge\vec{v}) + \nabla\wedge\left[(\vec{v}\cdot\nabla)\vec{v}\right] = \nabla\wedge\left(-\frac{1}{\rho}\nabla p\right) + \nabla\wedge\vec{g} + \nabla\wedge\frac{1}{\rho}\nabla\cdot\overline{\overline{\tau}}'$$

$$\frac{\partial\vec{\omega}}{\partial t} + (\vec{v}\cdot\nabla)\vec{\omega} + \vec{\omega}(\nabla\cdot\vec{v}) - (\vec{\omega}\cdot\nabla)\vec{v} = -\left(-\frac{1}{\rho^2}\nabla\rho\wedge\nabla p + \frac{1}{\rho}\underbrace{\nabla\wedge\nabla p}_{0}\right) - \underbrace{\nabla\wedge\vec{g}}_{0} + \nabla\wedge\left(\frac{1}{\rho}\nabla\cdot\overline{\overline{\tau}}'\right)$$