Vorticity equation

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v} = -\frac{1}{\rho}\nabla p + \vec{g} + \frac{1}{\rho}\nabla\cdot\overline{\overline{\tau}}'$$

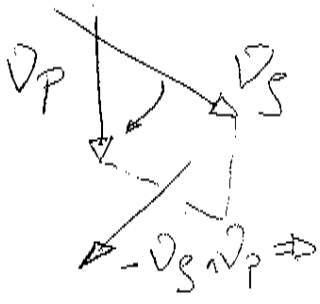
Take the curl of this equation:

$$\nabla\wedge\left(\qquad\qquad\right)$$

$$\frac{\partial}{\partial t}(\nabla\wedge\vec{v}) + \nabla\wedge\left[(\vec{v}\cdot\nabla)\vec{v}\right] = \nabla\wedge\left(-\frac{1}{\rho}\nabla p\right) + \nabla\wedge\vec{g} + \nabla\wedge\frac{1}{\rho}\nabla\cdot\overline{\overline{\tau}}'$$

$$\underbrace{\frac{\partial\vec{\omega}}{\partial t} + (\vec{v}\cdot\nabla)\vec{\omega}}_{\frac{D\vec{\omega}}{Dt}} + \underbrace{\vec{\omega}(\nabla\cdot\vec{v})}_{\text{Vorticity change due to compressibility}} - \underbrace{(\vec{\omega}\cdot\nabla)\vec{v}}_{\text{Vortex Stretching and Tilting}} = -\Big(\underbrace{-\frac{1}{\rho^2}\nabla\rho\wedge\nabla p}_{\text{Baroclinic Torque}} + \frac{1}{\rho}\underbrace{\nabla\wedge\nabla p}_{0}\Big) - \underbrace{\nabla\wedge}_{0} + \underbrace{\nabla\wedge\left(\frac{1}{\rho}\nabla\cdot\overline{\overline{\tau}}'\right)}_{\text{Viscous diffu}}$$

Rate of change of vorticity following a fluid element

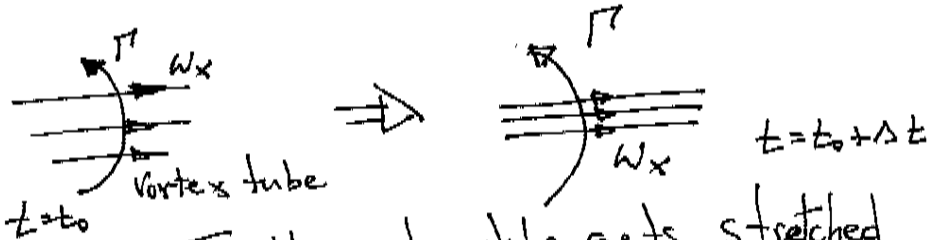


$\Rightarrow$ Rotation of fluid elements

Consider a component of the equation for $\vec{\omega}$:

$$\frac{D}{Dt}\omega_x = \underbrace{(\vec{\omega}\cdot\nabla)\omega_x}_{\underbrace{\omega_x \frac{\partial v_x}{\partial x}}_{①} + \underbrace{\omega_y \frac{\partial v_x}{\partial y} + \omega_z \frac{\partial v_x}{\partial z}}_{②}} + \cdots\cdots$$

① $\omega_x \frac{\partial v_x}{\partial x}$:

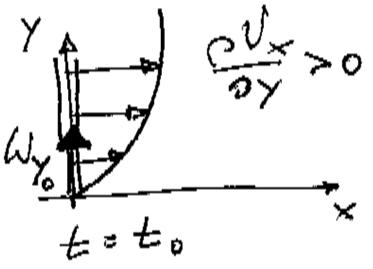


If the vortex tube gets stretched

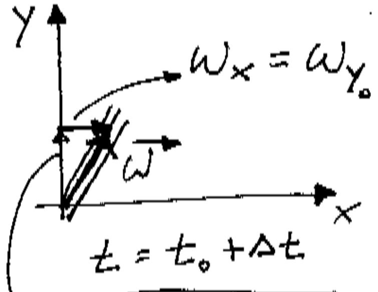
$$\left.\begin{aligned}\frac{\partial v_x}{\partial x} &> 0\\ \omega_x &> 0\end{aligned}\right\}\ \frac{D\omega_x}{Dt} > 0$$

vorticity gets more intense by stretching i the direction of vorticity

② $\omega_y \frac{\partial v_x}{\partial y}$:



$\frac{\partial v_x}{\partial y} > 0$



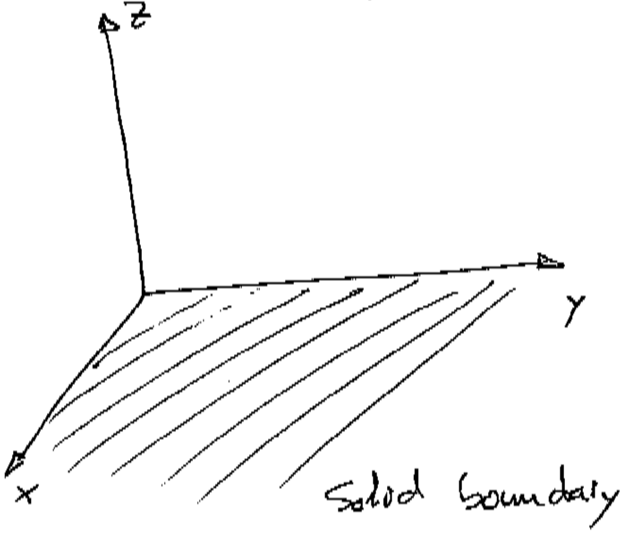
$\omega_x = \omega_{y_0} \frac{\partial v_x}{\partial y}\cdot\Delta t$: Vorticity is creat in the x direction b the shear $\frac{\partial v_x}{\partial y}$ tilti the existing vortici towards the x directio

$\omega_y = \sqrt{\|\vec{\omega}\|^2 - \omega_x^2}$
$= \sqrt{\omega_y^2 - \omega_x^2}$

(57) $$\frac{D\vec{\omega}}{Dt} = -\vec{\omega}(\nabla\cdot\vec{v}) + (\vec{\omega}\cdot\nabla)\vec{v} - \frac{1}{\rho^2}\nabla\rho\wedge\nabla p + \nabla\wedge\left(\frac{1}{\rho}\nabla\right.$$

The only two overall sources of vorticity in the flow are baroclinic torque and viscosity.

In incompressible flows, viscosity is the only source and sink of vorticity: consider incompressi flow near a boundary.



On the boundary (z=

$V_x = V_y = V_z = 0$

and

$$\frac{\partial V_x}{\partial x} = \frac{\partial V_x}{\partial y} = \frac{\partial V_y}{\partial x} = \frac{\partial V_y}{\partial y} = \frac{\partial V_z}{\partial x} = \frac{\partial V_z}{\partial y} = 0$$

Derivatives along the wall are zero.

Continuity equation shows: $\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} = 0$

so, on the boundary $(z=0)$, $\frac{\partial V_z}{\partial z} = 0$

The force due to viscous stress on the boundary is $\vec{f} = \vec{n} \cdot \bar{\bar{\tau}}'\Big|_{z=0}$ and

$\vec{n} = (0,0,1) = \vec{e}_z$ so $\vec{f}(z=0) = \Big[\mu\left(\frac{\partial V_x}{\partial z} + \frac{\partial V_z}{\partial x}\right), \mu\left(\frac{\partial V_y}{\partial z} + \frac{\partial V}{\partial y}\right.$

$$, \mu\, 2\, \frac{\partial V_z}{\partial z}\Big]_{z=0}$$

We know $\frac{\partial V_z}{\partial x} = \frac{\partial V_z}{\partial y} = \frac{\partial V_z}{\partial z} = 0$ at $z=0$ so

$$\vec{f}(z=0) = \mu \frac{\partial V_x}{\partial z}\Big|_{z=0} \vec{e}_x + \mu \frac{\partial V_y}{\partial z}\Big|_{z=0} \vec{e}_y \text{ ; tangent to the wall}$$

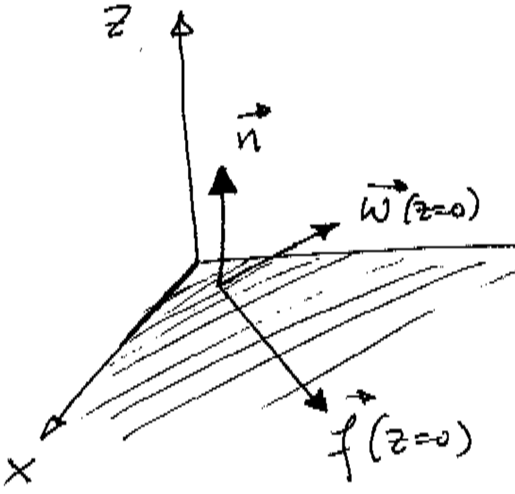
If we examine the vorticity on the wall $(z=0)$: $\omega_x = \cancel{\frac{\partial V_z}{\partial y}} - \frac{\partial V_y}{\partial z} = -\frac{\partial V_y}{\partial z}\Big|_{z=0} = -\frac{f_y^{WALL}}{\mu}$

$$\omega_y = \frac{\partial V_x}{\partial z} - \cancel{\frac{\partial V_z}{\partial x}} = \frac{\partial V_x}{\partial z}\Big|_{z=0} = \frac{f_x^{WALL}}{\mu}$$

$$\omega_z = \cancel{\frac{\partial V_y}{\partial x}} - \cancel{\frac{\partial V_x}{\partial y}} = 0$$

The vorticity at the boundary is related to the viscous stresses and is also tangent to the wall

$$\vec{f}(z=0) = \vec{n}\cdot\bar{\bar{\tau}}'\Big|_{z=0} = -\mu\,\vec{n}\wedge\vec{\omega}\Big|_{z=0}$$



Vorticity is generated at the wall due to the no-slip condition and then diffuses into the flow domain.

For incompressible flow of a Newtonian fluid: $\nabla\cdot\bar{\bar{\tau}}' = \mu\nabla^2\vec{v} = -\mu\nabla\wedge\vec{\omega}$

If we plug this into the Navier-Stokes eq. evaluated at the boundary: $(\vec{v}\cdot\nabla)\begin{matrix}v_x = 0\\ v_y = 0\end{matrix}$

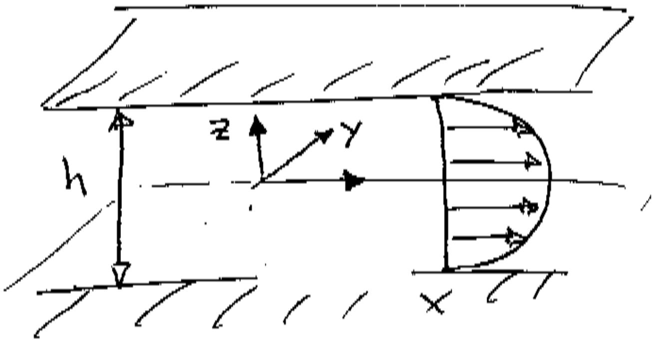
$$-\mu\left(\cancelto{0}{\frac{\partial\omega_z}{\partial y}} - \frac{\partial\omega_y}{\partial z}\right)\Bigg|_{z=0} = \frac{\partial P}{\partial x}\Bigg|_{z=0} + \rho\,\frac{\partial v_x}{\partial t}$$

$$-\mu\left(\frac{\partial\omega_x}{\partial z} - \cancelto{0}{\frac{\partial\omega_z}{\partial x}}\right)\Bigg|_{z=0} = \frac{\partial P}{\partial y}\Bigg|_{z=0} + \rho\,\frac{\partial v_y}{\partial t}$$

The diffusive flux of vorticity perpendicular to boundary is due to: pressure gradient and fluid accelera

Examples

Poisseuille Flow



$$v_x(z) = v_{max}\left[1-\left(\frac{z}{h}\right)^2\right]$$

$$v_{max} = \frac{h^2}{\mu}\frac{\Delta P}{L}$$

$$\frac{dP}{dx} = -\frac{\Delta P}{L}$$

$$\omega_y = \frac{\partial v_x}{\partial z} - \cancel{\frac{\partial v_z}{\partial x}}^{\,0} = -\frac{v_{max}}{h}\left(\frac{z}{h}\right)$$

In this case: $+\mu \dfrac{\partial \omega_y}{\partial z} = -\mu\dfrac{v_{max}}{h^2} = -\cancel{\mu}\dfrac{\cancel{\frac{h^2}{\mu}}\frac{\Delta P}{L}}{\cancel{h^2}} =$

$$= -\frac{\Delta P}{L} = \left.\frac{\partial P}{\partial x}\right|_{z=-h/2}$$

Same for the top wall $(z=h/2)$ except with opposite sign. The vorticity is generated at the wall and flows (by diffusion, no convective flux) into the fluid domain. Because **vorticity** has opposite sign at the opposite walls, it cancels at the centerli where vorticity is 0 $\omega_y(z=0)=0$