Potential Flow

Irrotational flows $\left(\vec{\omega} = \nabla \wedge \vec{v} = 0\right)$ can be studied by means of the velocity potential $\vec{v} = \nabla \phi$.

If the flow is two dimensional we can also describe the flowfield by a stream function Ψ. The stream functio[n] is such that continuity is satisfied automatically:

$$v_x = \frac{\partial \Psi}{\partial y} \; ; \quad v_y = -\frac{\partial \Psi}{\partial x}$$

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = \frac{\partial^2 \Psi}{\partial x \partial y} - \frac{\partial^2 \Psi}{\partial y \partial x} \equiv 0$$

Similarly, for a two-dimensional flow, the conditi[on] of irrotationality becomes: $\vec{\omega} = 0 \Rightarrow \omega_z = 0 \Rightarrow$

$\dfrac{\partial v_y}{\partial x} - \dfrac{\partial v_x}{\partial y} = 0$ which is satisfied identically by t[he] velocity potential: $\dfrac{\partial}{\partial x}\left(\dfrac{\partial \phi}{\partial y}\right) - \dfrac{\partial}{\partial y}\left(\dfrac{\partial \phi}{\partial x}\right) \equiv 0.$

If we apply continuity to the velocity potential, [and] irrotationality to the stream function, then we ge[t]

$$\left.\begin{aligned} \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial y}\right) &= 0 \Rightarrow \nabla^2 \phi = 0 \\ \frac{\partial}{\partial x}\left(-\frac{\partial \Psi}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial \Psi}{\partial y}\right) &= 0 \Rightarrow \nabla^2 \Psi = 0 \end{aligned}\right\} \text{Both } \phi \text{ and } \Psi \text{ satisfy Laplace's eq[uation]}$$

Solutions of Laplace's equation are called harmonic functions and there is an extensive collection of solutions from all fields of mathematical physics that can be used in fluid mechanics. (heat conduction, elasticity, electromagnetism)

If we have 2 functions ϕ and Ψ that satisfy the Cauchy-Riemann conditions:

$$\frac{\partial \phi}{\partial x} = \frac{\partial \Psi}{\partial y} \quad (= V_x)$$

$$\frac{\partial \phi}{\partial y} = -\frac{\partial \Psi}{\partial x} \quad (= V_y)$$

then we can form a complex function:

$W(z) = \phi + i\Psi$ that is analytic (differentiable in the complex pl

Because ϕ and Ψ satisfy the Cauchy-Riemann conditi W(z) satisfies Laplace's equation and so any analyt function represents an irrotational, two-dimensional flow field.

Because Laplace's equation is linear, solutions can be added to shape a flow field of interest.

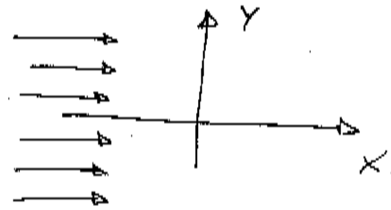
Some basic elementary solutions that have a simple physical interpretation are:

UNIFORM FLOW: (Horizontal, from left to right)

$$W(z) = U \cdot z \quad (U \cdot x + i\, U y)$$

$$\phi = Ux \; ; \; \psi = Uy$$

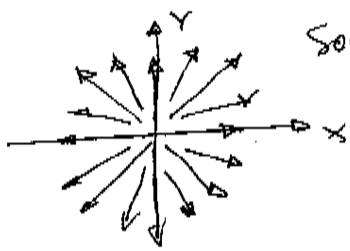
$$v_x = U \; ; \; v_y = 0$$



SOURCE FLOW (or SINK)

$$W(z) = \frac{m}{2\pi} \ln z = \frac{m}{2\pi} \ln\left(r e^{i\theta}\right)$$

$$\left.\begin{array}{ll} \text{Real part} & \phi = \frac{m}{2\pi} \ln r \\ \text{Imaginary part} & \psi = \frac{m}{2\pi}\, \theta \end{array}\right\} \quad \begin{array}{l} v_r = \frac{m}{2\pi r} \\ v_\theta = 0 \end{array}$$



SOURCE OF INTENSITY ($m > 0$)

If $m < 0$, then it's a sink and the arrows point inwards.

Doublet: combination of a source and a sink

$$W(z) = \lim_{\epsilon \to 0} \frac{m}{2\pi} \ln(z + \epsilon) - \frac{m}{2\pi} \ln(z - \epsilon) = \frac{m}{2\pi} \ln\left(\frac{z+\epsilon}{z-\epsilon}\right)$$

$$W(z) \simeq \lim_{\epsilon \to 0} \frac{m}{2\pi} \ln\left(\frac{1 + \epsilon/z}{1 - \epsilon/z}\right) \simeq \frac{m}{2\pi} \ln\left[\left(1 + \epsilon/z\right)\left(1 + \frac{\epsilon}{z}\right)\right]$$

$$W(z) \simeq \frac{m}{2\pi} \left\{ \ln\left[1 + \frac{2\epsilon}{z} + \left(\epsilon/z\right)^2 + o\left(\epsilon/z\right)^3\right] \right\} \simeq \frac{m}{2\pi} \left(\frac{2\epsilon}{z}\right)$$

(65)

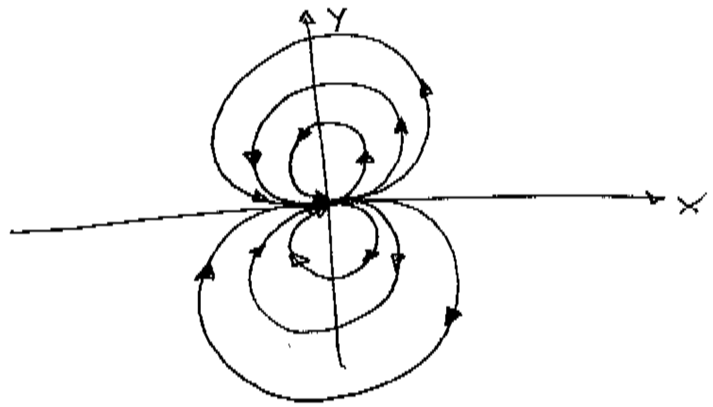
$$W(z) = \frac{m\epsilon}{\pi}\frac{1}{z}$$

$\frac{m\epsilon}{\pi}$ → Intensity of the doublet Λ

$$W(z) = \frac{\Lambda}{z} \Rightarrow \phi = Re\left(\frac{\Lambda}{x+iy}\right)$$
$$\psi = Im\left(\frac{\Lambda}{x+iy}\right)$$

$$W(z) = \frac{\Lambda(x-iy)}{(x+iy)(x-iy)} = \frac{\Lambda x - i\Lambda y}{x^2+y^2}$$

$$\underline{\phi = \frac{\Lambda x}{x^2+y^2}} \; ; \quad \underline{\psi = \frac{-\Lambda y}{x^2+y^2}}$$

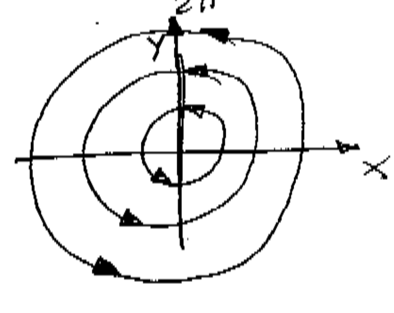


Irrotational (Potential) Vortex

$$W(z) = -\frac{i\Gamma}{2\pi}\ln z = -\frac{i\Gamma}{2\pi}\ln\left(re^{i\theta}\right)$$

$$\phi(x,y) = \frac{\Gamma}{2\pi}\theta \; ; \quad \psi(x,y) = -\frac{\Gamma}{2\pi}\ln r$$

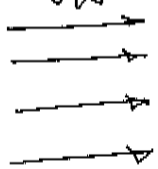
$$v_r = 0 \; ; \quad v_\theta = \frac{\Gamma}{2\pi r}$$



(66) Flow past a circular cylinder with circulation

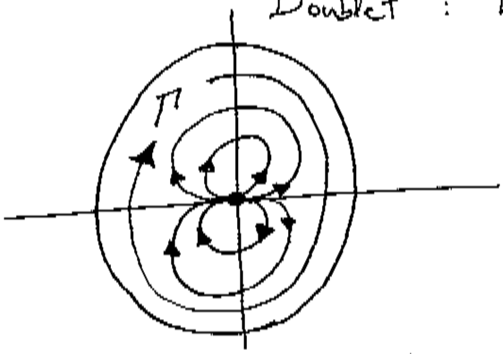
Uniform flow at infinity

U_∞



$W(z) = U_\infty z$

Doublet : $W(z) = \frac{\Lambda}{z}$



Vortex at the origin

$W(z) = +\frac{i\Gamma}{2\pi} \ln z$

$$W(z) = U_\infty z + \frac{\Lambda}{z} + i\frac{\Gamma}{2\pi} \ln z$$

Since the potential is defined except for a constant, we can rewrite it as $W(z) = U_\infty \left(z + \frac{a^2}{z}\right) + i\frac{\Gamma}{2\pi} \ln(z/a)$

$$\phi = U_\infty \left(r + \frac{a^2}{r}\right)\cos\theta - \frac{\Gamma}{2\pi}\theta$$

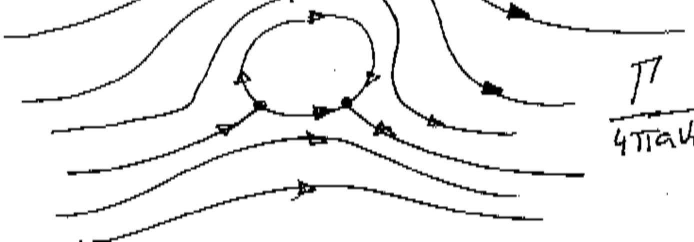
$$\psi = U_\infty \left(r - \frac{a^2}{r}\right)\sin\theta + \frac{\Gamma}{2\pi}\ln(r/a)$$

$$v_r = U_\infty\left(1 - \frac{a^2}{r^2}\right)\cos\theta$$

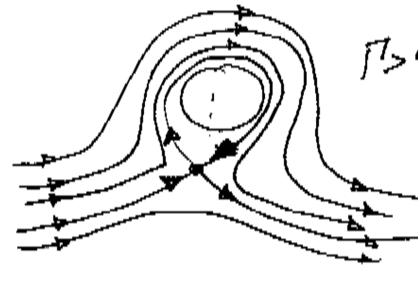
$$v_\theta = \frac{U_\infty}{r}\left(r + \frac{a^2}{r}\right)(-\sin\theta) - \frac{\Gamma}{2\pi r} = -\left[U_\infty\left(1 + \frac{a^2}{r^2}\right)\sin\theta + \frac{\Gamma}{2\pi r}\right.$$

Stagnation point : $a = r$ or $\theta = \pi/2$ or $3\pi/2$ ($v_r = 0$

and $\sin\theta = -\frac{\Gamma}{4\pi a U_\infty}$ and $r = \frac{\Gamma \pm \sqrt{\Gamma^2 - 4\pi a}}{4\pi U_\infty}$



$\frac{\Gamma}{4\pi a U_\infty} \leq 1$



$\Gamma > 4\pi a U_\infty$

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Bernouilli equation.

$$\rho\left[\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v}\right] = -\nabla p + \mu \nabla^2 \vec{v} + \rho \vec{g}$$

1. • Assume the flow is inviscid (frictionless) then

$$\rho\left[\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v}\right] = -\nabla p + \cancel{\mu \nabla^2 \vec{v}}_{\to 0} + \rho \vec{g}$$

We can write gravity as a force coming from a potential

$\vec{g} = -\nabla(gz)$ and $(\vec{v}\cdot\nabla)\vec{v} = \nabla \frac{v^2}{2} - \vec{v}\wedge\vec{w}$.

$\vec{w} = \nabla\wedge\vec{v}$

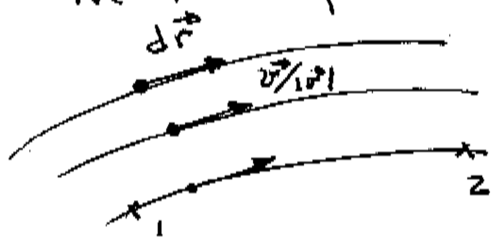
The N-S equation then becomes:

$$\cancel{\rho}\left[\frac{\partial \vec{v}}{\partial t} + \nabla \frac{v^2}{2} - \vec{v}\wedge\vec{w}\right] = -\frac{\nabla p}{\rho} - \cancel{\rho}\nabla g z$$

2. • If we can pull ρ into the gradient, that is if there is a barotropy function such that $P = P(\rho)$, th

$$\frac{\partial \vec{v}}{\partial t} + \nabla\left(\frac{v^2}{2}\right) - \vec{v}\wedge\vec{w} = -\nabla\left(P/\rho^n\right) - \nabla(gz).$$ Assuming incompr flow : $n = 1$.

3. a) If we project this equation along the stream in the flow



$$\frac{\partial \vec{v}}{\partial t}\cdot d\vec{r} + \nabla\left(\frac{v^2}{2}\right)\cdot d\vec{r} - \cancel{\vec{v}\wedge\vec{w}\cdot d\vec{r}}_{0} = -\left[\nabla(P/\rho) + \nabla(g\right.$$
$\cdot d$

$$\int_1^2 d\left(\frac{v^2}{2} + \frac{P}{\rho} + gz\right) + \frac{\partial \vec{v}}{\partial t}\, d\vec{r} = 0$$

$$\left(\frac{v^2}{2} + \frac{P}{\rho} + gz\right)_2 - \left(\frac{v^2}{2} + P/\rho + gz\right)_1 = -\int_1^2 \frac{\partial \vec{v}}{\partial t} \cdot d\vec{r}$$

- If the flow is steady: $\frac{P}{\rho} + \frac{v^2}{2} + gz =$ constant along streamlines.

b) If the flow is irrotational: $\vec{\omega} = \nabla \wedge \vec{v} = 0$, then

$\vec{v} = \nabla \phi$: $\quad \frac{\partial \nabla \phi}{\partial t} + \nabla \frac{|\nabla \phi|^2}{2} = -\nabla(P/\rho) - \nabla(gz)$]

$$\nabla\left(\frac{\partial \phi}{\partial t} + \frac{|\nabla \phi|^2}{2} + P/\rho + gz\right) = 0$$

$\frac{\partial \phi}{\partial t} + \frac{|\nabla \phi|^2}{2} + P/\rho + gz =$ constant everywhere in the flow (but different for diffe[rent] times)

- If the flow is steady:

$\nabla\left(\frac{v^2}{2} + P/\rho + gz\right) = 0 \Rightarrow \frac{P}{\rho} + \frac{v^2}{2} + gz =$ constant (everywhere, every time)