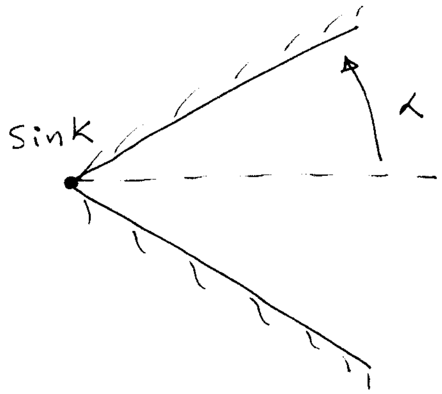
Example Potential Flow coupled to Boundary Layer



The potential (inviscid, irrotational) approximation is valid away from the walls and predict a velocity field: $\bullet\ v_r(r) = \dfrac{m}{2\pi\rho r}$; $v_\theta = 0$

Applying Bernouilli to the potential flow velocity.

$$P(r) + \frac{1}{2}\rho v^2 = P_\infty + \frac{1}{2}\rho V_\infty^2$$

$$P(r) = \left[P_\infty + \frac{1}{2}\rho\left(V_\infty^2 - \frac{m^2}{4\pi^2\rho^2 r^2}\right)\right]$$

To account for the no-slip condition at the walls, we need to introduce a θ dependency to the velocity field, so that at $\theta = \pm\alpha$ the velocity is zero, and as we move away from the wall $\theta > -\alpha$ or $\theta < \alpha$, the velocity is insensitive to θ and tends to $v_r(r) = \dfrac{m}{2\pi\rho r}$

Assuming a solution of the form: $v_r = \dfrac{m}{2\pi\rho r} f(\theta)$ and

$$\frac{P(r) - P_\infty}{\rho} = -\frac{m^2}{8\pi^2\rho^2 r^2}\cdot g(\theta) \quad (V_\infty \to 0) \quad \text{we get:}$$

r-component of N-S

$$\rho\left(\cancelto{0}{\frac{\partial v_r}{\partial t}} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + \cancelto{0}{v_z\frac{\partial v_r}{\partial z}}\right) = -\frac{\partial P}{\partial r} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_r}{\partial r}\right) - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta} + \cancelto{0}{\frac{\partial^2 v_r}{\partial z^2}} - \frac{v_r}{r^2} + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2}\right]$$

from continuity we get:

$$\cancelto{0}{\frac{\partial \rho}{\partial t}} + \frac{1}{r}\frac{\partial}{\partial r}(\rho r v_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho v_\theta) + \cancelto{0}{\frac{\partial}{\partial z}(\rho v_z)} = 0$$

since $v_\theta = 0$ at $\theta = \pm\alpha$

r-comp. N-S.

$$\rho \frac{m}{2\pi\rho r} f(\theta)\left(-\frac{m}{2\pi\rho r^2}\right) f(\theta) = -\rho \frac{2m^2}{8\pi^2\rho^2 r^3} g(\theta) + \mu\left[\frac{1}{r}\cancel{\frac{+m}{2\pi\rho r^2}} - \cancel{\frac{m}{2\pi\rho r^3}} + \frac{1}{r^2}\frac{m}{2\pi\rho r} f''(\theta)\right]$$

$$-\frac{m^2}{4\pi^2\rho r^3} f^2(\theta) = -\frac{m^2}{4\pi^2\rho r^3} g(\theta) + \mu \frac{m}{2\pi\rho r^3} f''(\theta)$$

$$\frac{2\pi\mu}{m} f''(\theta) = -f^2(\theta) + g(\theta)$$

$\frac{2\pi\mu}{m} = \frac{1}{Re} = \epsilon \ll 1$ This is a singular perturbation problem where the small parameter multiplies the highest order derivative

$$\boxed{\epsilon f'' + f^2 = g}$$

θ- comp N-S

$$\rho\left(\cancel{\frac{\partial v_\theta}{\partial t}} + v_r \cancel{\frac{\partial v_\theta}{\partial r}} + \cancel{\frac{v_\theta}{r}}\frac{\partial v_\theta}{\partial \theta} + \cancel{\frac{v_r v_\theta}{r}} + v_z \cancel{\frac{\partial v_\theta}{\partial z}}\right) = -\frac{1}{r}\frac{\partial P}{\partial \theta} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\cancel{\frac{\partial v_\theta}{\partial r}}\right) - \cancel{\frac{v_\theta}{r^2}} + \frac{1}{r^2}\cancel{\frac{\partial^2 v_\theta}{\partial \theta^2}} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} + \cancel{\frac{\partial^2 v_\theta}{\partial z^2}}\right]$$

$$0 = -\frac{1}{r}\,\frac{-m^2}{8\pi^2\rho^2 r^2}\, g'(\theta) + \mu \frac{2}{r^2}\frac{m}{2\pi\rho r} f'(\theta)$$

$$g'(\theta) = \frac{8\pi\mu}{m} f'(\theta)$$

$$\boxed{g'(\theta) = 4\epsilon f'(\theta)}$$

When $\epsilon \to 0$, we get $f^2 = g$

$$g' = 0 \Rightarrow g(\theta) = \text{constant} = C_0$$

$$f^2(\theta) = g(\theta) = C_0$$

We can impose as a condition that the flow rate between the walls is equal to the flow rate in the sink

$$\int_{-\alpha}^{\alpha} 2\pi \rho r V_r \, d\theta = m$$

$$\int_{-\alpha}^{\alpha} \cancel{2\pi\rho r} \frac{m}{\cancel{2\pi\rho r}} f(\theta)\, d\theta = m \Rightarrow \int_{-\alpha}^{\alpha} f(\theta)\, d\theta = 1 \Rightarrow \int_{-\alpha}^{\alpha} \sqrt{C_0}\, d\theta = 1 \Rightarrow$$

$$2\alpha\sqrt{C_0} = 1 \Rightarrow C_0 = \frac{1}{4\alpha^2}$$

but if we try to solve for other terms in $\epsilon, \epsilon^2, \ldots$

$$f(\theta) = f_0 + \epsilon f_1 + \epsilon^2 f_2 + \ldots$$

$$g(\theta) = g_0 + \epsilon g_1 + \epsilon^2 g_2 + \ldots$$

$$\epsilon(f_0'' + \epsilon f_1'') + (f_0 + \epsilon f_1)^2 = g_0 + \epsilon g_1$$

$$O(\epsilon) \quad \cancel{\epsilon f_0''} + 2\epsilon f_0 f_1 = g_1 \qquad f_0 = \sqrt{C_0} = \frac{1}{2\alpha}$$

$$f_1 = \frac{1}{2\epsilon f_0} \cdot g_1$$

$$\underbrace{\cancel{g_0'}}_{0} + \epsilon g_1' = 4\epsilon(f_0' + \epsilon f_1')$$

$$O(\epsilon) \quad g_1' = 4 f_0' = 0 \Rightarrow g_1 = \text{constant} = C_1$$

$$\Downarrow$$

$$f_1 = \frac{C_1}{2\epsilon \frac{1}{2\alpha}} = \frac{\alpha C_1}{\epsilon}$$

By continuity $\displaystyle\int_{-\alpha}^{\alpha} \cancel{2\pi\rho r} \frac{m}{\cancel{2\pi\rho r}} (f_0 + \epsilon f_1 + \epsilon^2 f_2 \ldots)\, d\theta = m$

$$\int_{-\alpha}^{\alpha} f_0\, d\theta = 1 \; ; \; \int_{-\alpha}^{\alpha} f_1\, d\theta = 0 \; ; \; \int_{-\alpha}^{\alpha} f_2\, d\theta = 0 \; \ldots\ldots$$

$$f_1 = \frac{\alpha C_1}{\epsilon} = 0 \Rightarrow g_1 = 0 \quad .$$

It does not work.

To obtain the behaviour inside the boundary layer we need to rescale θ so that the viscous term $\frac{\partial^2}{\partial\theta^2}$ is of the same order as the pressure gradient and the convective term inside the boundary layer:

$$\theta^* = \frac{\theta \pm \alpha}{\delta(\epsilon)}$$

$\theta = \pm\alpha \Rightarrow \theta^* = 0$ at the wall

$\theta < \alpha$ or $\theta > -\alpha \Rightarrow \theta^* \to \infty$ away from the wall.

$$\frac{d}{d\theta^*} = \delta(\epsilon)\frac{d}{d\theta}$$

$$\epsilon \cdot \frac{1}{\delta^2(\epsilon)} f''(\theta^*) + f^2 = g \Rightarrow \frac{\epsilon}{\delta^2(\epsilon)} = 1 \Rightarrow \delta(\epsilon) = \sqrt{\epsilon}$$

$$\left.\begin{aligned} \frac{1}{\delta(\epsilon)} g'(\theta^*) &= 4\epsilon \frac{1}{\delta(\epsilon)} f'(\theta^*) \\ \frac{1}{\sqrt{\epsilon}} g'(\theta^*) &= \frac{4\epsilon}{\sqrt{\epsilon}} f'(\theta^*) \end{aligned}\right\}$$

Make $f(\theta^*) = \bar{f}_0 + \epsilon\bar{f}_1 + \epsilon^2\bar{f}_2 + \dots$

$g(\theta^*) = \bar{g}_0 + \epsilon\bar{g}_1 + \epsilon^2\bar{g}_2 + \dots$

$O(\epsilon^0) \Rightarrow \bar{g}_0'(\theta^*) = 0 \Rightarrow \bar{g}_0(\theta^*) =$ constant: pressure does not change across the boundary layer, just as in the cartesian b.l.

$\bar{g}_0(\theta^*) = \text{constant} = g_0(\theta)$ outside the b.l.

$$\bar{f}_0''(\theta^*) + \bar{f}_0^2 = \text{constant} = \frac{1}{4\alpha^2}$$

When $\theta^* \to \infty \Rightarrow f(\theta^*) \to 0$

$f'(\theta^*) \to 0$

drop the 0 and the —

$$f' \frac{df'}{d\theta^*} = \left(\frac{1}{4\alpha^2} - f^2\right) f'$$

$$\frac{1}{2}\int d f'^2 = \int \left(\frac{1}{4\alpha^2} - f^2\right) \frac{df}{\cancel{d\theta^*}} \cdot \cancel{d\theta^*}$$

$$\boxed{\int \frac{df}{\left(\frac{1}{4\alpha^2}f - f^3/3\right)^{1/2}} = \sqrt{2}\int d\theta^*}$$

$$\boxed{\frac{1}{2} f'^2 = \frac{1}{4\alpha^2} f - f^3/3 + C_1} \quad C_1 \to 0$$