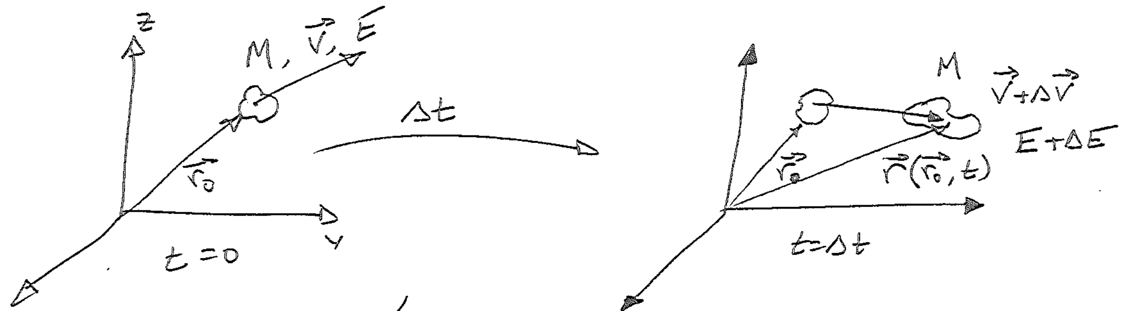
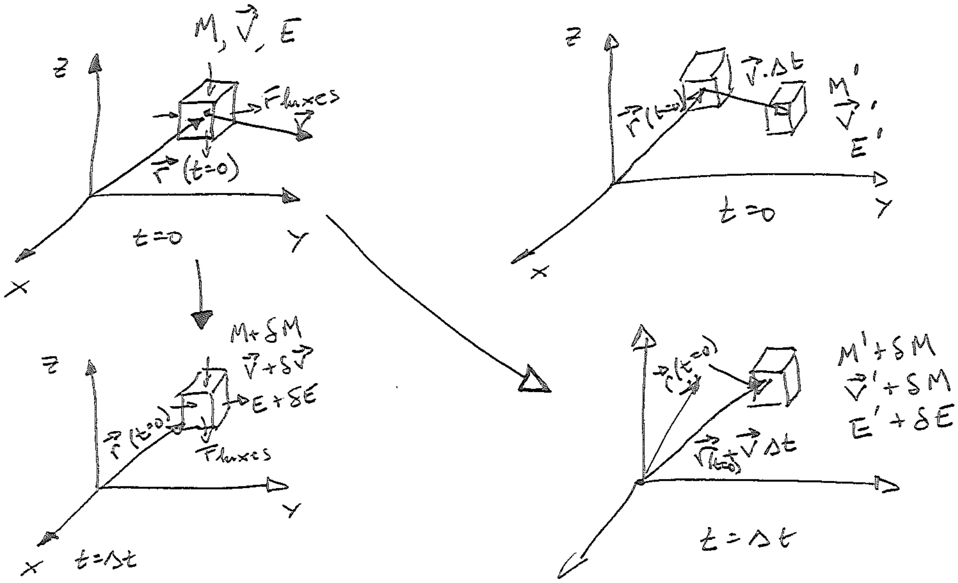
FLUID KINEMATICS

EULERIAN VS. LAGRAGIAN FRAMES:

Conservation laws are formulated in a Lagragian Frame:



Lagragian



In a Lagragian frame, we follow fluid particles along their trajectories. In an Eulerian frame, we describe the fluid variables as mathematical fields $\rho(x, y, z, t)$, $\vec{v}(x, y, z, t)$, ...

How do we relate the derivative following a fluid particle with the derivatives of the field variables?

First: a fluid particle is a lump of fluid (with a certain mass) that is initially at a certain location $(\vec{r_0})$ and occupies a certain volume (V_0) but it moves and deforms with the flow keeping its mass constant.

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$$\frac{d\phi}{dt} =$$ Lagrangian derivative, following the fluid particle

ϕ is an intrinsic variable.

$$\lim_{\Delta t\to 0} \frac{\phi(\vec r', t_0+\Delta t) - \phi(\vec r_0, t_0)}{\Delta t} = \lim_{\Delta t\to 0}\left[\frac{\phi(\vec r_0+\vec v\Delta t,\, t_0+\Delta t) - \phi(\vec r_0+\vec v\Delta t,\, t_0)}{\Delta t} + \frac{\phi(\vec r_0+\vec v\Delta t,\, t_0) - \phi(\vec r_0,\, t_0)}{\Delta t}\right] =$$

$$= \left.\frac{\partial\phi}{\partial t}\right|_{\vec r_0+\vec v\cdot\Delta t} + \lim_{\Delta t\to 0}\left[\frac{\phi\left(r_{0x}+\underbrace{v_x\Delta t}_{\Delta x},\, r_{0y}+\underbrace{v_y\Delta t}_{\Delta y},\, r_{0z}+\underbrace{v_z\Delta t}_{\Delta z},\, t_0\right) - \phi(r_{0x}, r_{0y}, r_{0z}, t_0)}{\Delta t}\right]$$

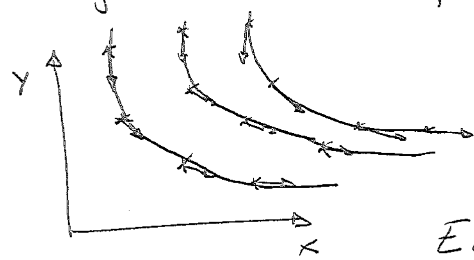
$$= \left.\frac{\partial\phi}{\partial t}\right|_{\vec r_0+\vec v\cdot\Delta t} + \underset{\underset{v_x}{\shortparallel}}{\frac{\Delta x}{\Delta t}}\left.\frac{\partial\phi}{\partial x}\right|_{(r_{0x},\, r_{0y}+v_y\Delta t,\, r_{0z}+v_z\Delta t,\, t_0)} + \overset{\overset{v_y}{\shortparallel}}{\frac{\Delta y}{\Delta t}}\left.\frac{\partial\phi}{\partial y}\right|_{(r_{0x},\, r_{0y},\, r_{0z}+v_z\Delta t,\, t_0)} + \frac{\Delta z}{\Delta t}\left.\frac{\partial\phi}{\partial z}\right|_{(r_{0x},\, r_{0y},\, r_{0z},\, t_0)}$$

$$= \underset{\substack{\downarrow\\ \text{temporal}\\ \text{derivative}}}{\frac{\partial\phi}{\partial t}} + \underbrace{(\vec v\cdot\nabla)\phi}_{\substack{\text{convective}\\ \text{derivative}}} = \underline{\frac{D\phi}{Dt}}$$

material / substantial / total } derivative (Lagrangian)

STREAMLINES, PATHLINES, STREAKLINES.

- Streamlines are tangent to the velocity field everywhere at a given instant of time (snapshot)



$v_x = f(x, y, z, t)$

$v_y = g(x, y, z, t)$

$v_z = h(x, y, z, t)$

Equation for the line tangent to $\vec{v}$ at a given point (x_0, y_0, z_0) at a given instant of time t_0 is:

$$\frac{dy}{dx} = \frac{v_y(x,y,z,t_0)}{v_x(x,y,z,t_0)} \qquad \frac{dz}{dx} = \frac{v_z(x,y,z,t_0)}{v_x(x,y,z,t_0)}$$

<u>Example</u>: $\left.\begin{array}{l} v_x = -k\frac{t}{\tau}x \\ v_y = ky \end{array}\right\}$ $\qquad \frac{dy}{dx} = \frac{ky}{-k\frac{t_0}{\tau}x}$ Streamlines at $t = t_0$

$$\int \frac{dy}{\cancel{k}y} = -\int \frac{dx}{\cancel{k}\frac{t_0}{\tau}x} \Rightarrow \underbrace{\ln y = -\frac{\tau}{t_0}\ln x + C}$$

This is a generic streamline

The streamline that goes through x_0, y_0 ⇒

$$\ln y_0 = -\frac{\tau}{t_0}\ln x_0 + C' \Rightarrow C' = \ln y_0 + \frac{\tau}{t_0}\ln x_0$$

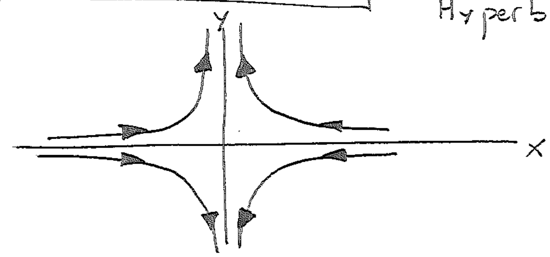
$$\ln y = -\frac{\tau}{t_0}\ln x + \ln y_0 + \frac{\tau}{t_0}\ln x_0$$

$$\ln(y/y_0) = -\frac{\tau}{t_0}\ln(x/x_0)$$

$$e \qquad e$$

$$y/y_0 = (x/x_0)^{-\tau/t_0} \Rightarrow \frac{y}{y_0} = \left(\frac{x_0}{x}\right)^{\tau/t_0}$$

For $t_0 = \tau \Rightarrow \frac{y}{y_0} = \frac{x_0}{x} \Rightarrow \boxed{x \cdot y = x_0 \cdot y_0}$ Equilateral Hyperbolas



For $t_0 = 2\tau \rightarrow \frac{y}{y_0} = \left(\frac{x_0}{x}\right)^{1/2}$

- Pathlines or trajectories: $\frac{d\vec{r}}{dt} = \vec{v}(x, y, z, t)$: trajectory of a fluid particle.

$\frac{dx}{dt} = v_x$; $\frac{dy}{dt} = v_y$; $\frac{dz}{dt} = v_z \Rightarrow$ Parametric equations (t is the parameter)

If we eliminate the parameter t, we get 2 implicit equations

$$dt = \underbrace{\frac{dx}{v_x} = \frac{dy}{v_y} = \frac{dz}{v_z}}_{} \Rightarrow \boxed{\frac{dy}{dx} = \frac{v_y}{v_x} ; \quad \frac{dz}{dx} = \frac{v_z}{v_x}}$$

: formally identical to the streamlines but completely different in general

t has been eliminated, cannot be in these eq.

The only way this will be the equal to the streamlines is if:
- steady $\vec{v} \neq f(t)$
- $\vec{v} = \underbrace{f(t)}_{} \cdot \vec{v}'(x, y, z)$

all three components have the same time dependency.

Previous example:

$$\frac{dx}{dt} = -k\frac{t}{\tau}x \Rightarrow \frac{dx}{x} = -k\frac{t}{\tau}dt$$

$$\ln x = -\frac{kt^2}{2\tau} + C_1$$

$x = x_0$ @ $t = t_0 \Rightarrow C_1 = \ln x_0 + k\frac{t_0^2}{2\tau}$

$$\ln(x/x_0) = -\frac{k}{2\tau}(t^2 - t_0^2)$$

$$\boxed{x = x_0 e^{-\frac{k(t^2 - t_0^2)}{2\tau}}}$$

$$\frac{dy}{dt} = ky \Rightarrow \frac{dy}{y} = k\,dt$$

$$\ln y = kt + C_2$$

$y = y_0$ @ $t = t_0 \Rightarrow C_2 = \ln y_0 - kt_0$

$$\ln(y/y_0) = k(t - t_0) \Rightarrow \boxed{y = y_0 e^{k(t - t_0)}}$$

Pathlines or trajectories.

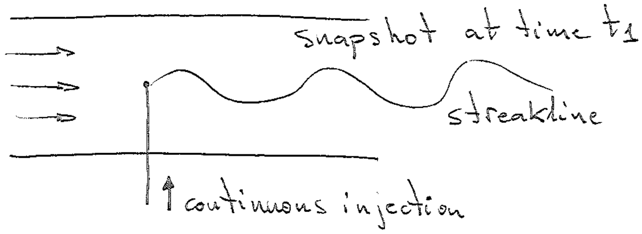
If we want to express the pathline in terms of an implicit equation; I need to remove time: $t = t_0 + \frac{1}{k} \ln(y/y_0)$

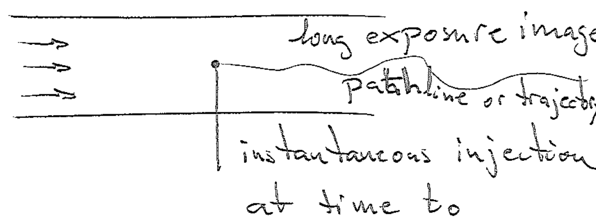
$$x = x_0 e^{-\frac{k(t^2 - t_0^2)}{2\tau}} \Rightarrow x = x_0 e^{-\frac{k\{[t_0 + \frac{1}{k}\ln(y/y_0)]^2 - t_0^2\}}{2\tau}}$$

$$\ln\left(\frac{x}{x_0}\right) = -\frac{k}{2\tau}\left[\cancel{t_0^2} + \frac{2t_0}{k}\ln(y/y_0) + \frac{1}{k^2}\ln^2(y/y_0) - \cancel{t_0^2}\right]$$

$$\boxed{\ln\left(\frac{x}{x_0}\right) = -\frac{t_0}{\tau}\ln(y/y_0) - \frac{1}{2k\tau}\ln^2(y/y_0)}$$

- Streaklines are the location at a certain time t_1 of all particles that have gone through a point (x_0, y_0, z_0) over time: $t_0 \in (-\infty, t_1]$





Streamfunction

In two dimensional, or three dimensional axisymmetric, flows, we can define a scalar function that fully characterizes the velocity field.

For incompressible flows, the function is defined as

$$v_x = \frac{\partial \Psi}{\partial y} \qquad v_y = -\frac{\partial \Psi}{\partial x}$$

not to get confussed with

$$v_x = \frac{\partial \phi}{\partial x} \quad \text{the velocity potential.} \qquad v_y = \frac{\partial \phi}{\partial y}$$

If both are defined (flow is 2-D and irrotational)

then we have that $\frac{\partial v_x}{\partial y} = \frac{\partial v_y}{\partial x}$ ⟶ irrotational

$$\frac{\partial^2 \phi}{\partial y \partial x} = \frac{\partial^2 \phi}{\partial x \partial y}$$

⟹ ϕ and Ψ are harmonic functions

$$\frac{\partial v_x}{\partial x} = -\frac{\partial v_y}{\partial y}$$ ⟵ continuity

$$\frac{\partial^2 \Psi}{\partial x \partial y} = \frac{\partial^2 \Psi}{\partial y \partial x}$$

continuity ⟶ $\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \boxed{\Delta \phi = 0}$

irrotational ⟶ $\frac{\partial v_x}{\partial y} = \frac{\partial v_y}{\partial x} \Rightarrow 0 = \frac{\partial^2 \Psi}{\partial y^2} - \left(-\frac{\partial^2 \Psi}{\partial x^2}\right) \Rightarrow \frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} = 0 \Rightarrow \boxed{\Delta \Psi = 0}$

Both ϕ and Ψ are solutions to Laplace's equation ⟹ which is the definition of harmonic functions.