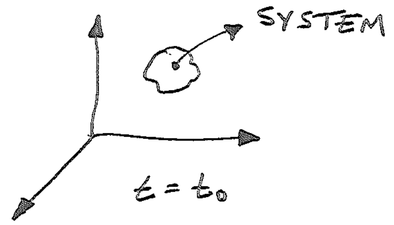
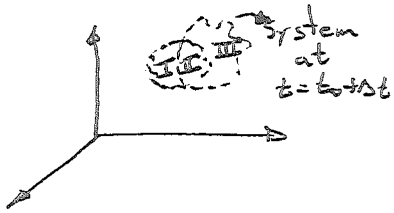
# CONSERVATION LAWS

## INTEGRAL FORMULATION - CONTROL VOLUME

We define a system to which we apply the three conservation laws we use in mechanics:




1) $$\frac{D M_{sys}}{Dt} = 0$$

2) $$\frac{D(M_{sys}\vec{V}_{CG})}{Dt} = \sum \vec{F}_{ext}$$

3) $$\frac{D E_{sys}}{Dt} = \dot{Q} - \dot{W}$$

How do we relate this to the properties of an arbitrary Control Volume that we define at each instant of time? ⇒ Reynolds Transport Theorem.

Let $B$ be a property of the system $\begin{cases} M, \text{ mass} \\ M\vec{V}, \text{ momentum} \\ E, \text{ energy} \end{cases}$

then, $$\frac{DB}{Dt} = \lim_{\Delta t\to 0} \frac{B_{sys}(t+\Delta t) - B_{sys}(t)}{\Delta t} = \lim_{\Delta t\to 0} \frac{B_{III}(t_0+\Delta t) + B_{II}(t_0+\Delta t)}{\Delta t} - \frac{B_{II}(t_0) + B_{I}(t_0)}{\Delta t}$$

An extensive property $B$ can be related to the intensive property $b$ as $B_{sys} = \int_{V_{sys}} \rho\, b\, dV$

$b$: property per unit mass

$$\frac{D\int_{V_{sys}}\rho b\,dV}{Dt} = \lim_{\Delta t\to 0}\frac{\int_{III}\rho b\,dV^{(t_0+\Delta t)} + \int_{II}\rho b\,dV^{(t_0+\Delta t)} - \int_{II}\rho b\,dV^{(t_0)} - \int_{I}\rho b\,dV}{\Delta t} =$$

$$= \frac{\partial}{\partial t}\int_{II}\rho b\,dV + \lim_{\Delta t\to 0}\frac{1}{\Delta t}\left[\int_{III}\rho b(t_0+\Delta t)\,dV - \int_{I}\rho b(t_0)\,dV\right]$$


Arrows are the local velocity at the boundary of the volume at $t = t_0$.


$$dV = dA(\vec{v}\Delta t\cdot\vec{n})$$

$$= \frac{\partial}{\partial t}\int_{II}\rho b\,dV + \lim_{\Delta t\to 0}\frac{1}{\cancel{\Delta t}}\left[\int_{III}\rho b\;dA(\vec{v}\cancel{\Delta t}\cdot\vec{n}) - \int_{I}\rho b\,dA(-\vec{v}\cancel{\Delta t}\cdot\vec{n})\right]$$

$$= \frac{\partial}{\partial t}\int_{V}\rho b\,dV + \int_{S}\rho b\,\vec{v}\cdot\vec{dA}$$

We typically will choose a volume that is fixed at the volume occupied by the system at time $t_0$.

$$\frac{DB_{sys}}{Dt} = \frac{\partial}{\partial t}\int_{C.V.}\rho b\,dV + \int_{C.S.}\rho b\,\vec{v}\cdot\vec{dA}$$

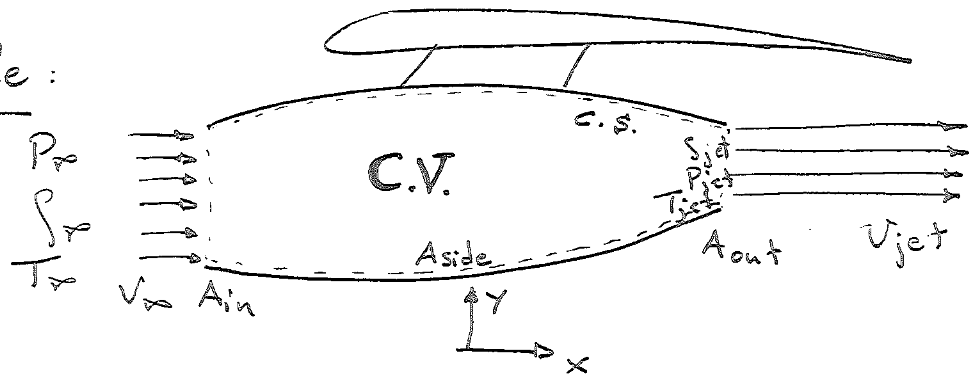
: 3-D equivalent to Leibniz Theorem

$$\frac{d}{dt}\int_{a(t)}^{b(t)}F(x,t)\,dx = \frac{\partial}{\partial t}\int_{a}^{b}F(x,t)\,dx + F(b,t)\cdot\frac{db}{dt} - F(a,t)\frac{da}{dt}$$

# CONSERVATION OF MASS

$$\frac{DM_{sys}}{Dt} = 0 \Rightarrow \frac{D}{Dt}\int_{sys} \rho \, dV = \boxed{\frac{\partial}{\partial t}\int_{c.v} \rho \, dV + \int_{c.s.} \rho \, \vec{v} \cdot d\vec{A} = 0}$$

Example:




$$\frac{DM_{sys}}{Dt} = 0 \Rightarrow \frac{\partial}{\partial t}\int_{c.v.} \rho \, dV + \int_{c.s} \rho \, \vec{v} \cdot d\vec{A} = 0$$

If the flow is steady then the amount of mass inside the engine (control volume) is always the same $\frac{\partial}{\partial t}\int_{c.v.} \rho \, dV = 0$

$$\int_{A_{in}} \rho \, \vec{v} \cdot d\vec{A} + \int_{A_{out}} \rho \, \vec{v} \cdot d\vec{A} + \int_{A_{side}} \rho \, \vec{v} \cdot d\vec{A} = 0$$

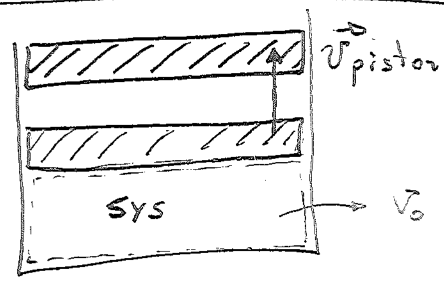
We can consider viscous flow $\vec{v}_{wall} = \vec{0}$ or inviscid $\vec{v}_{wall} \cdot \vec{n}_{wall} = 0$,

either way $\int_{A_{side}} \rho \, \vec{v} \cdot dA \, \vec{n} = 0$

$$\int_{A_{in}} \rho \, V_\infty \vec{i} \cdot dA(-\vec{i}) + \int_{A_{out}} \rho \, V_{jet} \, \vec{i} \cdot dA \, \vec{i} = 0$$

$$-\rho_\infty V_\infty A_{in} + \rho_{jet} V_{jet} A_{out} = 0 \qquad \boxed{\rho_\infty V_\infty A_{in} = \rho_{jet} V_{jet} A_{out} = \dot{m}}$$

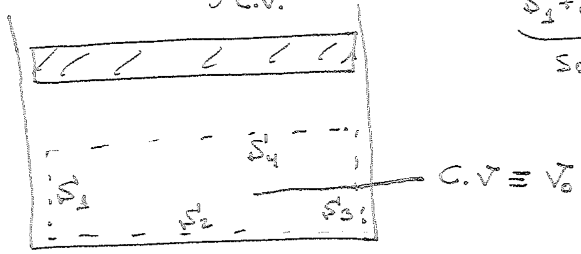
# Alternative example




$$\frac{D M_{sys}}{Dt} = 0 = \frac{\partial}{\partial t}\int_{C.V.} \rho \, dV + \int_{C.S.} \rho \, \vec{v}\cdot\vec{n}\, dA$$

Let's choose the original volume occupied by the system as the control volume:

$$\frac{\partial}{\partial t}\int_{C.V.} \rho \, dV + \underbrace{\cancelto{0}{\int_{S_1+S_2+S_3} \rho \, \vec{v}\cdot\vec{n}\, dA}}_{\text{SOLID WALLS}} + \int_{S_4} \rho \, \vec{v}\cdot\vec{n}\, dA = 0$$




Let's assume that at any given instant the density of the gas inside the ~~piston~~ cylinder is uniform:

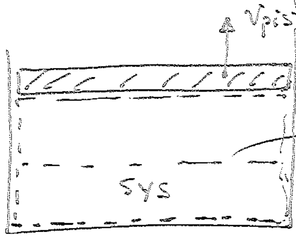
$$V \cdot \frac{\partial}{\partial t}\rho_{cyl} + \int_{S_4} \rho \, \vec{v}\cdot\vec{n}\, dA = 0$$

I don't know what $\rho$ or $\vec{v}$ are at $S_4$ but I do know that whatever mass is in the volume outside my control volume has gone through $S_4$: mass flux = density × volume flux

$$V \frac{\partial \rho(t)}{\partial t} + \rho(t) \cdot \frac{d V_{cyl}}{dt} = 0 \Rightarrow \frac{1}{\rho}\frac{\partial \rho}{\partial t} = -\frac{1}{V}\frac{d V_{cyl}}{dt} = -\frac{1}{V}\cdot A_{cyl}\cdot v_{piston}$$

If the cylinder is straight: $A_{cyl}$ is constant $\Rightarrow$ $\boxed{\frac{1}{\rho}\frac{\partial \rho}{\partial t} = -\frac{1}{h}\cdot v_{piston}}$

Let's try again with a deformable control volume:




Control Volume is the interior of the cylinder.

$$\frac{\partial}{\partial t}\underbrace{\int_{c.v.}\rho\, dV}_{\text{Mass inside the control volume, which does not change over time}} + \int_{c.s.}\rho\,\vec{v}\cdot\vec{n}\, dA = 0$$

if $\rho$ is uniform

$$\frac{\partial}{\partial t}\int_{c.v}\rho\, dV = \left(\frac{\partial\rho}{\partial t}\right)\int_{c.v} dV + \rho\frac{\partial}{\partial t}\int_{c.v} dV = 0$$

$$\boxed{\frac{1}{\rho}\frac{\partial\rho}{\partial t} = -\frac{1}{V}\frac{\partial V}{\partial t}}$$

$$\underbrace{\frac{\partial}{\partial t}\int_{c.v.}\rho\, dV}_{=0} + \underbrace{\int_{S_1+S_2+S_3+S_4}\rho\,\vec{v}\cdot\vec{n}\, dA}_{} = 0$$

$$0 + \cancel{\underbrace{\int_{S_1+S_2+S_3}}_{\text{SOLID WALLS}}\rho\,\vec{v}\cdot\vec{n}\, dA} \to 0 \;+ \int_{S_4}\rho\,\vec{v}\cdot\vec{n}\, dA = 0$$

The velocity of the fluid in contact with the piston is equal to the velocity of the piston

$$\int_{S_4}\rho\,\vec{v}\cdot\vec{n}\, dA = 0$$

Physically this is the flux of mass through the piston, which is a solid surface and therefore no mass goes through it.

The way to reconcile this is to account for the moving boundary of the control volume. The flux is not proportional to the velocity of the fluid, but the velocity of the fluid RELATIVE TO THE CONTROL VOLUME BOUNDARY

$$\int\rho(\underbrace{\vec{v}}_{\vec{v}_{piston}} - \underbrace{\vec{v}_c}_{\vec{v}_{piston}})\cdot\vec{n}\, dA = 0$$