Conservation of Energy

1^{ST} Law of Thermodynamics

$$\frac{DE_{sys}}{Dt} = \dot{W} - \dot{Q}$$

$$\frac{\partial}{\partial t}\int_{c.v.} \rho e \, dV + \int_{c.s.} \rho e \, (\vec{v}\cdot\vec{n})\, dA = + \int_{c.s.} \overbrace{\left(-p\vec{n} + \bar{\bar{\tau}}'\cdot\vec{n}\right)}^{d\vec{F}_{surf}} \cdot \vec{v}\, d$$

$$+ \underbrace{\int_{c.v.} \rho\vec{g}\cdot\vec{v}\, dA}_{d\vec{F}_{vol}\cdot\vec{v}} \; \underbrace{- \int_{c.v.} \dot{Q}_{\substack{rad \\ comb}}\, dV}_{\text{Volumetric heat fluxes}} \; \underbrace{- \int_{c.s.} \vec{q}\cdot\vec{n}\, dA}_{\text{Surface heat fluxes}}$$

Work done by gravitational force ⟺

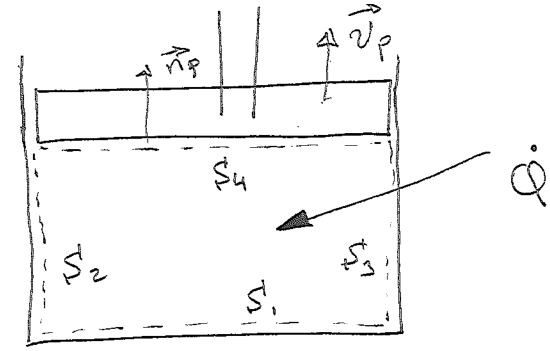
What do we account for in e?

$e = u + \frac{1}{2}v^2$: typically e accounts for internal energy, which in a liquid or calorically perfect gas is only a function of composition and temperature, and kinetic ene

We can also include potential energy:

$e = u + \frac{1}{2}v^2 + gz$, but then we cannot

EXAMPLE



$$\frac{\partial}{\partial t}\int_{C.V.} \rho e \, dV + \int_{C.S} \rho e (\vec{v}-\vec{v}_c)\cdot\vec{n}\, dA = \int_{C.S} -p\,\vec{v}\cdot\vec{n}\,dA + \int_{C.S} \vec{v}\cdot\bar{\bar{\tau}}\cdot\vec{n}\,dA + \int_{C.V.}\rho\vec{g}\cdot\vec{v} + \dot{Q}$$

$$\frac{\partial}{\partial t}\int \rho e\,dV + \int_{S_1+S_2+S_3} \rho e \underbrace{(\vec{v}-\vec{v}_c)}_{0}\cdot\vec{n}\,dA + \int_{S_4}\rho e\,\cancel{(\vec{v}-\vec{v}_c)\cdot\vec{n}}\,dA = -\int_{S_1+S_2+S_3}\cancel{p\,\vec{v}\cdot\vec{n}\,dA}$$

$\vec{v}\cdot\vec{n} \to 0$ SOLID WALL

PISTON IS also a SOLID WALL $\to 0$

$(\vec{v}=0) \to 0$

$$-\int_{S_4} p\,\underbrace{\vec{v}}_{\vec{v}_p}\cdot\vec{n}\,dA + \int_{S_4} \vec{v}\cdot\underbrace{\bar{\bar{\tau}}\cdot\vec{n}}_{\text{viscous force}}\,dA + \int_{S_1+S_2+S_3}\cancel{\vec{v}\cdot\bar{\bar{\tau}}\cdot\vec{n}\,dA} + \vec{g}\cdot\underbrace{\int_{C.V}\rho\vec{v}\,dV}_{M_{gas}\cdot\vec{V}_{C.G.}} + \dot{Q}$$

$(\vec{v}=0) \to 0$

$\vec{v}_p$ is normal to the piston

$\vec{v}_p\cdot(\bar{\bar{\tau}}\cdot\vec{n})$ is the work done by the normal component of the viscous force which is very small (compared the work done by the pressure).

$$e = u + \frac{1}{2}v^2$$

If we assume that the thermodynamic properties are uniform inside the piston, then:

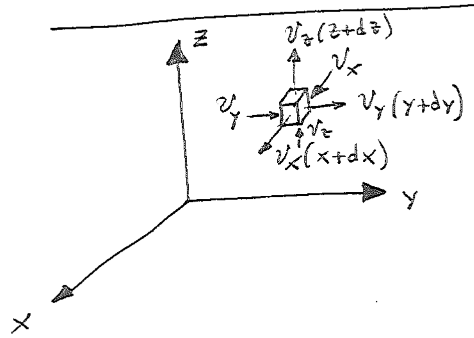
negligible kinetic energy in the gas.

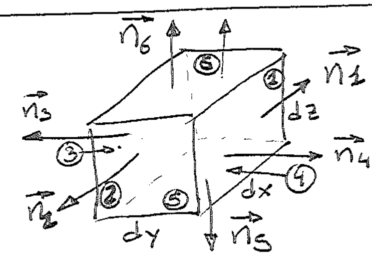
$$\frac{\partial}{\partial t}\int \rho\left(u+\frac{1}{2}v^2\right)dV = \frac{d}{dt}\rho V_{cyl}\cdot u + \cancel{\frac{\partial}{\partial t}\int_{C.V}\rho\frac{1}{2}v^2\,dV}$$

$$\frac{d}{dt}M_{cyl}\cdot u = -P\cdot\underbrace{(\vec{v}_p\cdot\vec{n}_p)\,A_p} + \cancel{M_{gas}\,\vec{g}\cdot\vec{V}_{cg}} + \dot{Q}$$

neglecting this

DIFFERENTIAL FORMULATION OF THE CONSERVATION LAWS: NAVIER-STOKES EQUATIONS





Conservation of Mass for an infinitesimally small volume (cube)

$$\frac{\partial}{\partial t}\int_{c.v} \rho \, dV + \int_{c.s.} \rho \, \vec{v} \cdot \vec{n} \, dA = 0$$

$$\frac{\partial}{\partial t}\rho \, dx\,dy\,dz \; \overbrace{- \rho(x)\, v_x(x)\, dy\,dz}^{S_1} + \overbrace{\rho(x+dx)\, v_x(x+dx)\, dy\,dz}^{S_2} - \overbrace{\rho(y)\, v_y(y)\, dx\,dz}^{S_3} + \overbrace{\rho(y+dy)\, v_y(y+dy)\, dx\,dz}^{S_4} - \underbrace{\rho(z)\, v_z(z)\, dx\,dy}_{S_5} + \underbrace{\rho(z+dz)\, v_z(z+dz)\, dx\,dy}_{S_6} = 0$$

$$\frac{\partial}{\partial t}\rho \, \cancel{dx\,dy\,dz} + \left[\frac{\partial}{\partial x}(\rho v_x)\, \cancel{dx} + o(dx)^2\right] \cancel{dy\,dz} + \left[\frac{\partial}{\partial y}(\rho v_y)\, \cancel{dy} + o(dy)^2\right] \cancel{dx\,dz} + \frac{\partial}{\partial z}(\rho v_z)\, \cancel{dz} + o(dz)^2\Big] \cancel{dx\,dy} = 0$$

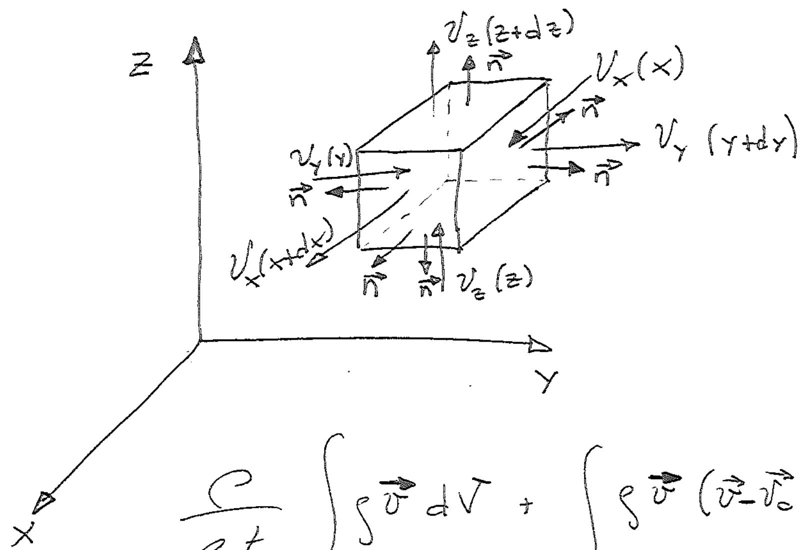
$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0$$

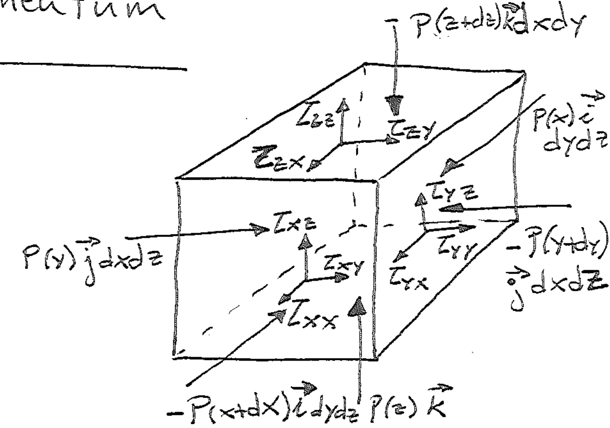
$$\underbrace{\frac{\partial \rho}{\partial t} + \vec{v} \cdot \nabla \rho}_{\frac{D\rho}{Dt}} + \rho \nabla \cdot \vec{v} = 0 \quad \Rightarrow \quad \boxed{\frac{D\rho}{Dt} + \rho \nabla \cdot \vec{v} = 0}$$

If the density of a material particle does not change with time. $\frac{D\rho}{Dt} = 0 \Rightarrow$

$\Rightarrow \boxed{\nabla \cdot \vec{v} = 0}$

Conservation of Momentum





$$\frac{\partial}{\partial t}\int_{C.V.}\rho\vec{v}\,dV + \int_{C.S.}\rho\vec{v}\,(\vec{v}-\vec{v}_o)\cdot\vec{n}\,dA = -\int_{C.S.}p\,\vec{n}\,dA + \int_{C.S.}\bar{\bar{\tau}}'\cdot\vec{n}\,dA + \int_{C.V.}\rho\vec{g}\,dV$$

$$\frac{\partial}{\partial t}\left(\rho\,\vec{v}\,\cancel{dx\,dy\,dz}\right) + \left[\rho(x+dx)\,\vec{v}(x+dx)\,v_x(x+dx) - \rho(x)\,\vec{v}(x)\,v_x(x)\right]\cancel{dydz}$$

$$\lim_{dx\to 0}[\;] = \frac{\partial}{\partial x}\left(\rho\vec{v}\,v_x\right)\cancel{dx} + o(dx)^2$$

$$+\left[\rho(y+dy)\,\vec{v}(y+dy)\;v_y(y+dy) - \rho(y)\,\vec{v}(y)\,v_y(y)\right]dxdz$$

$$+\left[\rho(z+dz)\,\vec{v}(z+dz)\;v_z(z+dz) - \rho(z)\,\vec{v}(z)\,v_z(z)\right]dxdy$$

$$= -\overbrace{\left[P(x+dx)-P(x)\right]}^{-\left[\frac{\partial P}{\partial x}\cancel{dx}+o(dx)^2\right]}\vec{i}\;\cancel{dydz} - \overbrace{\left[P(y+dy)-P(y)\right]}\vec{j}\,dxdz - \overbrace{\left[P(z+dz)-P(z)\right]}\vec{k}\,dxdy$$

$$+\left[\left(\tau_{xx}\vec{i}+\tau_{xy}\vec{j}+\tau_{xz}\vec{k}\right)(x+dx)\;dydz - \left(\tau_{xx}\vec{i}+\tau_{xy}\vec{j}+\tau_{xz}\vec{k}\right)(x)\;dydz\right]$$

$$+\left(\qquad\right)(y+dy)\;dxdz - \left(\qquad\right)(y)\;dxdz$$

$$+\left(\qquad\right)(z+dz)\;dxdy - \left(\qquad\right)(z)\;dxdy$$

$$+\rho\vec{g}\;\cancel{dxdydz}$$

$$\lim_{dx\to 0}[\quad] = \left[\frac{\partial}{\partial x}\left(\tau_{xx}\vec{i}+\tau_{xy}\vec{j}+\tau_{xz}\vec{k}\right)\cancel{dx} + o(dx)^2\right]\cancel{dydz}$$

$$\lim_{dy\to 0}[\quad] = \left[\frac{\partial}{\partial y}\left(\tau_{yx}\vec{i}+\tau_{yy}\vec{j}+\tau_{yz}\vec{k}\right)\cancel{dy} + o(dy)^2\right]\cancel{dxdz}$$

$$\lim_{dz\to 0}[\quad] = \left[\frac{\partial}{\partial z}\left(\tau_{zx}\vec{i}+\tau_{zy}\vec{j}+\tau_{zz}\vec{k}\right)\cancel{dz} + o(dz)^2\right]\cancel{dxdy}$$

$$\frac{\partial}{\partial t}(\rho\vec{v}) + \frac{\partial}{\partial x}(\rho\vec{v}v_x) + \frac{\partial}{\partial y}(\rho\vec{v}v_y) + \frac{\partial}{\partial z}(\rho\vec{v}v_z) = -\nabla p +$$

$$+\left(\frac{\partial}{\partial x}\tau_{xx} + \frac{\partial}{\partial y}\tau_{yx} + \frac{\partial}{\partial z}\tau_{zx}\right)\vec{i} + \left(\frac{\partial}{\partial x}\tau_{xy} + \frac{\partial}{\partial y}\tau_{yy} + \frac{\partial}{\partial z}\tau_{zy}\right)\vec{j} +$$

$$+\left(\frac{\partial}{\partial x}\tau_{xz} + \frac{\partial}{\partial y}\tau_{yz} + \frac{\partial}{\partial z}\tau_{zz}\right)\vec{k} + \rho\vec{g}$$

$$\rho\frac{\partial\vec{v}}{\partial t} + \underbrace{\vec{v}\frac{\partial\rho}{\partial t} + \vec{v}\left(\frac{\partial\rho v_x}{\partial x} + \frac{\partial\rho v_y}{\partial y} + \frac{\partial\rho v_z}{\partial z}\right)}_{\vec{v}\cdot\left(\frac{\partial\rho}{\partial t} + \nabla\cdot\rho\vec{v}\right)} + \underbrace{\rho v_x\frac{\partial\vec{v}}{\partial x} + \rho v_y\frac{\partial\vec{v}}{\partial y} + \rho v_z\frac{\partial\vec{v}}{\partial z}}_{\rho(\vec{v}\cdot\nabla)\vec{v}} =$$

$$\left(\frac{\partial\rho}{\partial t} + \nabla\cdot\rho\vec{v}\right) = 0$$ Continuity

$$= -\nabla p + \nabla\cdot\bar{\bar{\tau}}' + \rho\vec{g}$$

$$\boxed{\rho\left[\frac{\partial\vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v}\right] = -\nabla p + \nabla\cdot\bar{\bar{\tau}}' + \rho\vec{g}}$$

ρ, p, $\vec{v}$ are unknowns in these equations, but we need a constitutive equation for $\bar{\bar{\tau}}'$ as a function of the deformation (strain) of the fluid.

$$\bar{\bar{\tau}}' = f(\dot{\bar{\bar{\varepsilon}}})$$

In tensor form this translates into

$$\tau_{ij}' = K_{ijkl}\,\dot{\varepsilon}_{kl}$$

K_{ijkl}: 4th order tensor (3 components in each direction)

$3^4 = 81$ possible coefficients.

The simplest assumption is that the stress tensor is linearly proportional to the rate of strain.

K_{ijkl} is composed of 81 constants.

If we further assume that the fluid is isotropic, that is to say that the stress developed by the fluid due to the rate of strain in a certain direction does not depend on the direction (the stress/rate of strain relationship does not change under a rotation of the reference frame).

$$K_{ijkl} = \lambda\, \delta_{ij}\, \delta_{kl} + \mu\, \delta_{ik}\, \delta_{jl} + \gamma\, \delta_{il}\, \delta_{jk}$$

3 independent variables (vs. 81 possible)

We can show that $\tau'_{ij} = \tau'_{ji}$, the stress tensor is symmetric and then

$$K_{ijkl} = K_{jikl} = \lambda\, \delta_{ji}\, \delta_{kl} + \mu\, \delta_{jk}\, \delta_{il} + \gamma\, \delta_{jl}\, \delta_{ik}$$

$$= \lambda\, \delta_{ij}\delta_{kl} + \mu\, \delta_{ik}\delta_{jl} + \gamma\, \delta_{jk}\, \delta_{il}$$

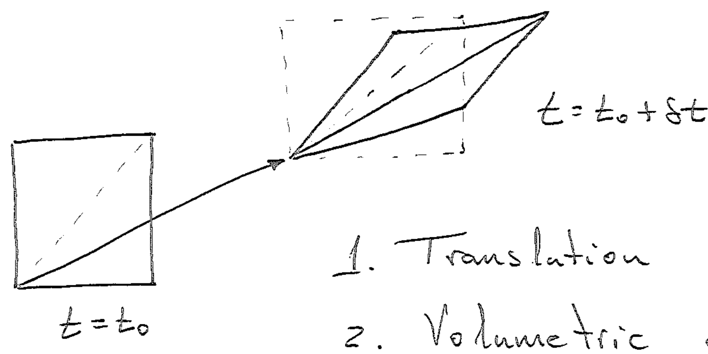
$$\boxed{\mu = \gamma}$$

$$\tau'_{ij} = \lambda\, \delta_{ij}\, \dot{\epsilon}_{kk} + 2\mu\, \dot{\epsilon}_{ij}$$

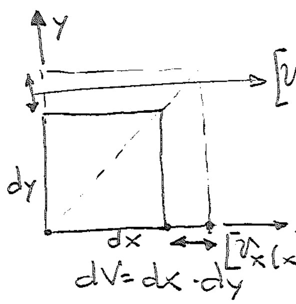
$$\overline{\overline{\tau}}' = 2\mu \frac{1}{2}\left(\nabla \vec{v} + \nabla \vec{v}^{T}\right) + \lambda\, \nabla \cdot \vec{v}\ \overline{\overline{I}}$$

Now we need to relate the rate of strain to some fluid variable used in our analysis.

F. M. White: "Viscous flow" 2^{nd} Edition 1991. McGraw Hill
Section 1.3.3.



1. Translation $\vec{v} \cdot \delta t$
2. Volumetric deformation : $\frac{1}{V}\frac{\delta V}{\delta t}$



$[v_y(y+dy) - v_y(y)]\,\delta t = \frac{\partial v_y}{\partial y}\,dy\,\delta t$

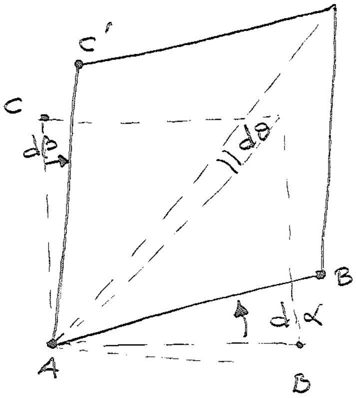
$[v_x(x+dx) - v_x(x)]\,\delta t = \frac{\partial v_x}{\partial x}\,\delta x\,\delta t$

$$\delta\, dV = \left[dx + \frac{\partial v_x}{\partial x}\,dx\,\delta t\right]\left[dy + \frac{\partial v_y}{\partial y}\,dy\,\delta t\right] - dx\,dy$$

$$\frac{1}{dV}\,\frac{\delta\, dV}{\delta t} = \frac{1}{\cancel{dx\,dy}}\;\frac{\frac{\partial v_x}{\partial x}\,\cancel{dx\,dy}\,\cancel{\delta t} + \frac{\partial v_y}{\partial y}\,\cancel{dx\,dy}\,\cancel{\delta t} + \frac{\partial v_x}{\partial x}\frac{\partial v_y}{\partial y}\,\cancel{dx\,dy}\,(\delta t)^2}{\cancel{\delta t}}$$

$$\frac{1}{V}\,\frac{DV}{Dt} = \underline{\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y}} + \cancel{\frac{\partial v_x}{\partial x}\,\frac{\partial v_y}{\partial y}\,\delta t} \rightarrow 0 \qquad \delta t \rightarrow 0$$

$$\boxed{\frac{1}{V}\,\frac{DV}{Dt} = \nabla \cdot \vec{v}} = \dot{\epsilon}_{xx} + \dot{\epsilon}_{yy} + \dot{\epsilon}_{zz}$$



3. Rotation

$$d\theta = \frac{\pi}{4} - \left[\frac{(\pi/2 - d\alpha - d\beta)}{2} + d\alpha\right] = \frac{d\alpha - d\beta}{2}$$

4. Angular deformation is given by the change of the right angle in the original element, substracting the effect of rotation.

rate of change of angle at AB $\frac{d\alpha - d\theta}{dt} = \frac{d\alpha - \frac{d\alpha - d\beta}{2}}{dt} = \frac{1}{dt}\frac{d\alpha + d\beta}{2} = \dot{\epsilon}_{xy}$

rate of change of angle at AC: $\frac{d\beta + d\theta}{dt} = \frac{1}{2}\frac{d\beta + \frac{d\alpha - d\beta}{2}}{dt} = \frac{1}{2}\frac{d\alpha + d\beta}{dt} = \dot{\epsilon}_{yx}$

If $d\alpha = d\beta \Rightarrow$ there is no rotation only deformation

If $d\alpha = -d\beta \Rightarrow$ there is no deformation, just pure rotation.

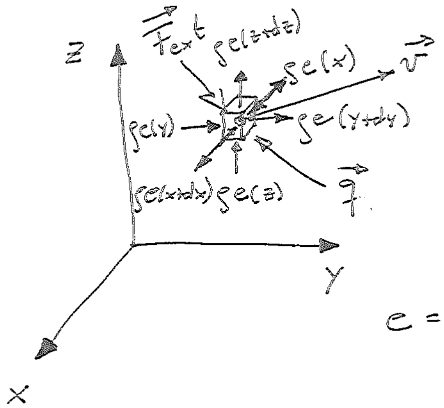
$$d\alpha = \lim_{dt \to 0} \frac{\frac{\partial v_y}{\partial x}\cancel{dx}\,dt}{\cancel{dx} + \frac{\partial v_x}{\partial x}\cancel{dx\,dt}} \simeq \frac{\partial v_y}{\partial x}dt$$

$$d\beta = \lim_{dt \to 0} \frac{\frac{\partial v_x}{\partial y}\cancel{dy}\,dt}{\cancel{dy} + \frac{\partial v_y}{\partial y}\cancel{dy\,dt}} \simeq \frac{\partial v_x}{\partial y}dt$$

The angular deformation **is** the rate of change of the right angle at $\hat{ABC}$ that is equally split between $\dot{\epsilon}_{xy}$ and $\dot{\epsilon}_{yx}$ (the rate of deformation tensor is symmetric).

$$\overline{\overline{\epsilon}} = \frac{1}{2}\begin{pmatrix} \frac{\partial v_x}{\partial x} + \frac{\partial v_x}{\partial x} & \frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y} & \frac{\partial v_z}{\partial x} + \frac{\partial v_x}{\partial v_z} \\ \frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} & \frac{\partial v_y}{\partial y} + \frac{\partial v_y}{\partial y} & \frac{\partial v_z}{\partial y} + \frac{\partial v_y}{\partial z} \\ \frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x} & \frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y} & \frac{\partial v_z}{\partial z} + \frac{\partial v_z}{\partial z} \end{pmatrix}$$

Conservation of Energy in Differential form



$$\frac{\partial}{\partial t}\int_{c.v} \rho e \, dV + \int_{c.s} \rho e(\vec v - \vec v_c)\cdot \vec n \, dA = -\int_{c.s} p\vec n \cdot \vec v \, dA + \int_{c.s} \vec v \cdot \bar{\bar{\tau}}' \cdot \vec n \, dA + \int_{c.v.} \rho \vec g \cdot \vec v \, dV + \int_{c.v.} \dot Q_{\substack{rad \\ che}} \, dV - \int_{c.s.} \vec q \cdot \vec n \, dA$$

$$e = u + \frac{1}{2} v^2$$

$$\frac{\partial \rho e}{\partial t} + \nabla\cdot(\rho \vec v e) = -\nabla\cdot(p\vec v) + \nabla\cdot(\vec v \cdot \bar{\bar{\tau}}') + \rho g \vec v + \dot q_{\substack{rad \\ che}} - \nabla\cdot\vec q$$

$$\underbrace{\rho\frac{\partial e}{\partial t} + \rho \vec v\cdot\nabla e + \underbrace{e\frac{\partial \rho}{\partial t} + e\nabla\cdot\rho\vec v}_{e\left[\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\vec v)\right] = 0} = -p(\nabla\cdot\vec v) - \vec v\cdot\nabla p + \bar{\bar{\tau}}' : \nabla\vec v + \vec v\cdot(\nabla\cdot\bar{\bar{\tau}}') + \rho g \vec v + \dot q_{\substack{rad \\ che}} - \nabla\cdot\vec q}_{\text{Conservation of total energy}}$$

Mechanical Energy: multiply the momentum conservation equation by the velocity

$$\frac{D\rho\vec v}{Dt}\cdot\vec v = \sum F_{ext}\cdot\vec v$$

$$\frac{D}{Dt}\left(\frac{1}{2}\rho v^2\right) = \dot W_{ext}$$

$$\vec v\cdot\left[\rho\frac{\partial \vec v}{\partial t} + \rho(\vec v\cdot\nabla)\vec v\right] = \left(-\nabla p + \nabla\cdot\bar{\bar{\tau}}' + \rho\vec g\right)\cdot\vec v$$

$$\rho\frac{\partial \frac{1}{2}v^2}{\partial t} + \rho(\vec v\cdot\nabla)\left(\frac{1}{2}v^2\right) = -\vec v\cdot\nabla p + \vec v\cdot\left(\nabla\cdot\bar{\bar{\tau}}'\right) + \rho\vec g\cdot\vec v$$

Substracting Mechanical Energy from the total energy we get:

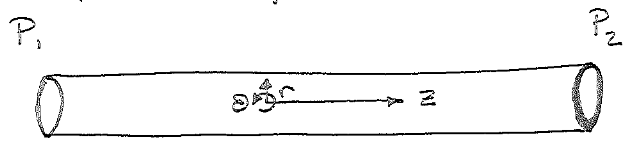
$$\underbrace{\rho\frac{\partial u}{\partial t} + \rho\vec{v}\cdot\nabla u}_{\text{The rate of change of the internal energy of the fluid is due to:}} = -p(\nabla\cdot\vec{v}) + \bar{\bar{\tau}}':\nabla\vec{v} + \underbrace{\dot{q}_{\substack{rad\\che}} - \nabla\cdot\vec{q}}_{\text{Heat addition}}$$

$-p(\nabla\cdot\vec{v})$: work done by pressure on a volumetric deformation

$\bar{\bar{\tau}}':\nabla\vec{v}$: viscous dissipation ϕ (positive definite)

Pipe flow

Fully develop, laminar, steady, incompressible



Continuity: $\cancel{\frac{\partial\rho}{\partial t}}^{\,0} + \nabla\cdot\rho\vec{v} = 0$

$$\rho\left[\frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \frac{1}{r}\frac{\partial}{\partial\theta}v_\theta + \cancel{\frac{\partial}{\partial z}v_z}\right] = 0$$

Fully develop means $\frac{\partial\vec{v}}{\partial z} = 0$

$$\frac{\partial}{\partial r}(r v_r) = -\frac{\partial}{\partial\theta}v_\theta$$

$$r\frac{\partial v_r}{\partial r} + v_r = -\frac{\partial}{\partial\theta}v_\theta$$

@ $r=0$ $\quad v_r = v_\theta = 0$

@ $r=R$ $\quad v_r = v_\theta = 0$

@ $r=0 \Rightarrow \frac{\partial v_\theta}{\partial\theta} = 0$

@ $r=R \Rightarrow \frac{\partial v_\theta}{\partial\theta} = 0 \Rightarrow \frac{\partial v_r}{\partial r} = 0$

Only solution is $v_r = v_\theta = 0$

Conservation of momentum

$$\rho \cancel{\frac{\partial v_r}{\partial t}} + \rho\left(\cancel{v_r \frac{\partial v_r}{\partial r}} + \cancel{\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}} + \cancel{v_z \frac{\partial v_r}{\partial z}} - \cancel{\frac{v_\theta^2}{r}}\right) = -\frac{\partial P}{\partial r} + \mu\left[\cancel{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_r}{\partial r}\right)} + \right.$$

$$\left. + \frac{1}{r^2}\cancel{\frac{\partial^2 v_r}{\partial \theta^2}} + \cancel{\frac{\partial^2 v_r}{\partial z^2}} - \cancel{\frac{v_r}{r^2}} - \frac{2}{r^2}\cancel{\frac{\partial v_\theta}{\partial \theta}}\right]$$

$$\rho \cancel{\frac{\partial v_\theta}{\partial t}} + \rho\left(\cancel{v_r \frac{\partial v_\theta}{\partial r}} + \cancel{\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}} + \cancel{v_z \frac{\partial v_\theta}{\partial z}} + \cancel{\frac{v_r v_\theta}{r}}\right) = -\frac{1}{r}\frac{\partial P}{\partial \theta} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\cancel{\frac{\partial v_\theta}{\partial r}}\right)\right.$$

$$\left. + \frac{1}{r^2}\cancel{\frac{\partial^2 v_\theta}{\partial \theta^2}} + \cancel{\frac{\partial^2 v_\theta}{\partial z^2}} + \frac{2}{r}\cancel{\frac{\partial v_r}{\partial \theta}} - \cancel{\frac{v_\theta}{r^2}}\right]$$

$$\rho \cancel{\frac{\partial v_z}{\partial t}} + \rho\left(\cancel{v_r \frac{\partial v_z}{\partial r}} + \cancel{\frac{v_\theta}{r}}\frac{\partial v_z}{\partial \theta} + v_z \cancel{\frac{\partial v_z}{\partial z}}\right) = -\frac{\partial P}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \right.$$

$$\left. + \frac{1}{r^2}\cancel{\frac{\partial^2 v_z}{\partial \theta^2}} + \cancel{\frac{\partial^2 v_z}{\partial z^2}}\right]$$

$$\boxed{\nabla^2\left(\frac{\partial v_z}{\partial \theta}\right) = 0 \quad \text{with} \quad \frac{\partial v_z}{\partial \theta} = 0 \ @\ r=0 \text{ and } r=R}$$

Resulting equations are $\frac{\partial P}{\partial r} = \frac{1}{r}\frac{\partial P}{\partial \theta} = 0$

$$\boxed{0 = -\frac{\partial P}{\partial z} + \mu \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)}$$

If I take the derivative $\frac{\partial}{\partial z}$

$$\frac{\partial^2 P}{\partial z^2} = \mu \frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial}{\partial r}\underbrace{\left(\frac{\partial v_z}{\partial z}\right)}_{0}\right] = 0$$

Therefore $\frac{\partial P}{\partial z} = \frac{dP}{dz} = \text{constant}$

$$\frac{1}{r}\frac{\partial}{\partial r}\left[r\left(\frac{\partial v_z}{\partial r}\right)\right] = \frac{1}{\mu}\frac{dP}{dz} = \text{constant}$$

$$\frac{d}{dr}\left(r\frac{dv_z}{dr}\right) = \frac{1}{\mu}\frac{dP}{dz}\, r$$

$$\cancel{r}\frac{dv_z}{dr} = \frac{1}{\mu}\frac{dP}{dz}\frac{\cancel{r^2}}{2} + \frac{C_1}{r}$$

$$\frac{dv_z}{dr} = \frac{1}{\mu}\frac{dP}{dz}\frac{r}{2} + \cancel{\frac{C_1}{r}} \to 0$$

$$v_z(r) = \frac{1}{\mu}\frac{dP}{dz}\frac{r^2}{4} + C_2$$

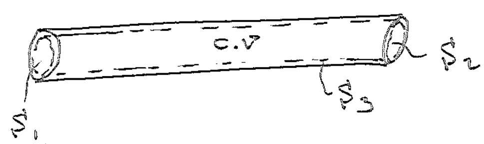
$$\left.\frac{dv_z}{dr}\right|_{r=0} = 0 \quad \underline{\text{Symmetry}}$$

$$v_z(r) = \frac{1}{\mu}\frac{dP}{dz}\frac{r^2}{4} + C_2$$

$$v_z(r=R) = 0 \Rightarrow \quad C_2 = -\frac{1}{\mu}\frac{dP}{dz}R^2/4$$

$$v_z(r) = \frac{1}{4\mu}\underbrace{\frac{dP}{dz}}_{-}\underbrace{\left(r^2-R^2\right)}_{-} \qquad // \qquad v_z(r) = \frac{1}{4\mu}\underbrace{\left(-\frac{dP}{dz}\right)}_{+}\underbrace{\left(R^2-r^2\right)}_{+}$$

Conservation of energy



$$\underbrace{\cancel{\frac{\partial}{\partial t}\int_{C.V}\rho e\, dV}}_{\text{steady}} + \int_{C.S.}\rho e(\vec{v}-\vec{v}_0)\cdot\vec{n}\, dA = -\int_{C.S} p\,\vec{n}\cdot\vec{v}\, dA + \int_{C.S}\vec{v}\cdot\bar{\bar{\tau}}'\cdot\vec{n}\, dA + \underbrace{\cancel{\int_{C.V}\rho\vec{g}\cdot\vec{v}\, dV}}_{\substack{\text{negligible}\\ \text{no increase in height}}} + \underbrace{\cancel{\int_{C.V}\dot{Q}_{rad/che}\, dV - \int_{C.S}\vec{q}\cdot\vec{n}\, dA}}_{\text{Adiabatic}}$$

$$\rho_1 e_1 \underbrace{\int_{S_1}\vec{v}\cdot\vec{n}\, dA}_{-Q} + \rho_2 e_2 \underbrace{\int_{S_2}\vec{v}\cdot\vec{n}\, dA}_{Q} + \underbrace{\cancel{\int_{S_3}\rho e\vec{v}\cdot\vec{n}\, dA}}_{0\ \text{SOLID WALL}} = -P_1\underbrace{\int_{S_1}\vec{v}_1\cdot\vec{n}\, dA}_{-Q} - P_2\underbrace{\int_{S_2}\vec{v}\cdot\vec{n}\, dA}_{Q} - \underbrace{\cancel{\int_{S_3} p\,\vec{v}\cdot\vec{n}\, dA}}_{0\ \text{SOLID WALL}}$$

$$+ \underbrace{\cancel{\int_{S_1}\vec{v}\cdot\bar{\bar{\tau}}'\cdot\vec{n}\, dA}}_{\text{negligible}} + \underbrace{\cancel{\int_{S_2}\vec{v}\cdot\bar{\bar{\tau}}'\cdot\vec{n}\, dA}}_{\text{negligible}} + \underbrace{\cancel{\int_{S_3}\vec{v}\cdot\bar{\bar{\tau}}'\cdot\vec{n}\, dA}}_{0\ \text{SOLID WALL}}$$

$$\rho_2 e_2 - \rho_1 e_1 = P_1 - P_2$$

If I consider $e = u + \frac{1}{2}v^2$ then

$$\rho_1 u_1 \int_{S_1}\vec{v}\cdot\vec{n}\, dA + \rho_1 \cancel{\int_{S_1}\frac{1}{2}v^2\,\vec{v}\cdot\vec{n}\, dA} + \rho_2 u_2\int_{S_2}\vec{v}\cdot\vec{n}\, dA + \rho_2\cancel{\int_{S_2}\frac{1}{2}v^2\,\vec{v}\cdot\vec{n}\, dA} =$$

equal but opposite signs ($\vec{v}\cdot\vec{n}$)

therefore: $\boxed{u_2 - u_1 = \frac{P_1 - P_2}{\rho}}$