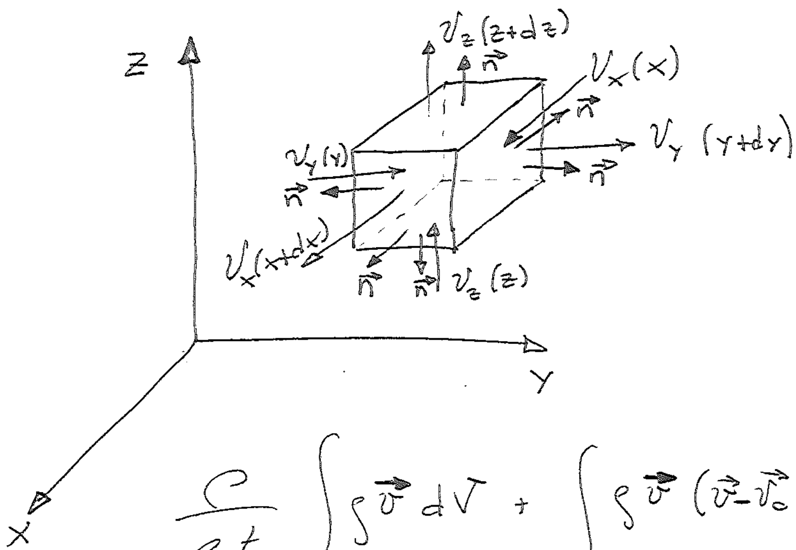
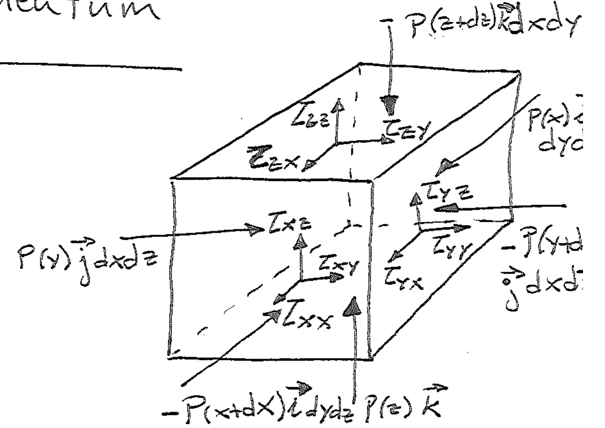
Conservation of Momentum





$$\frac{\partial}{\partial t}\int_{C.V.}\rho\vec{v}\,dV+\int_{C.S.}\rho\vec{v}\,(\vec{v}-\vec{v}_0)\cdot\vec{n}\,dA=-\int_{C.S.}p\,\vec{n}\,dA+\int_{C.S.}\bar{\bar{\tau}}'\cdot\vec{n}\,dA+\int_{C.V.}\rho\vec{g}\ldots$$

$$\frac{\partial}{\partial t}\left(\rho\,\vec{v}\,\cancel{dx\,dy\,dz}\right)+\Big[\rho(x+dx)\,\vec{v}(x+dx)\,v_x(x+dx)-\rho(x)\,\vec{v}(x)\,v_x(x)\Big]\,\cancel{dy\,dz}$$

$$\lim_{dx\to 0}[\;]=\frac{\partial}{\partial x}\left(\rho\vec{v}v_x\right)\cancel{dx}+o(dx)^2$$

$$+\Big[\rho(y+dy)\,\vec{v}(y+dy)\,v_y(y+dy)-\rho(y)\,\vec{v}(y)\,v_y(y)\Big]\,dx\,dz$$

$$+\Big[\rho(z+dz)\,\vec{v}(z+dz)\,v_z(z+dz)-\rho(z)\,\vec{v}(z)\,v_z(z)\Big]\,dx\,dy$$

$$=-\overbrace{\Big[P(x+dx)-P(x)\Big]}^{-\left[\frac{\partial P}{\partial x}\cancel{dx}+o(dx)^2\right]}\vec{i}\,\cancel{dy\,dz}-\overbrace{\Big[P(y+dy)-P(y)\Big]}\vec{j}\,dx\,dz-\overbrace{\Big[P(z+dz)-P(z)\Big]}\vec{k}\,dx\,dy$$

$$+\Big[\left(\tau_{xx}\vec{i}+\tau_{xy}\vec{j}+\tau_{xz}\vec{k}\right)(x+dx)\;dy\,dz-\left(\tau_{xx}\vec{i}+\tau_{xy}\vec{j}+\tau_{xz}\vec{k}\right)(x)\;dy\,dz$$

$$+\Big(\qquad\Big)(y+dy)\;dx\,dz-\Big(\qquad\Big)(y)\;dx\,dz$$

$$+\Big(\qquad\Big)(z+dz)\;dx\,dy-\Big(\qquad\Big)(z)\;dx\,dy$$

$$+\rho\vec{g}\,\cancel{dx\,dy\,dz}$$

$$\lim_{dx\to 0}[\quad]=\left[\frac{\partial}{\partial x}\left(\tau_{xx}\vec{i}+\tau_{xy}\vec{j}+\tau_{xz}\vec{k}\right)\cancel{dx}+o(dx)^2\right]\cancel{dy\,dz}$$

$$\lim_{dy\to 0}[\quad]=\left[\frac{\partial}{\partial y}\left(\tau_{yx}\vec{i}+\tau_{yy}\vec{j}+\tau_{yz}\vec{k}\right)\cancel{dy}+o(dy)^2\right]\cancel{dx\,dz}$$

$$\lim[\quad]=\left[\frac{\partial}{\partial z}\left(\tau_{zx}\vec{i}+\tau_{zy}\vec{j}+\tau_{zz}\vec{k}\right)\cancel{dz}+o(dz)^2\right]\cancel{dx\,dy}$$

$$\frac{\partial}{\partial t}(\rho\vec{v}) + \frac{\partial}{\partial x}(\rho\vec{v}v_x) + \frac{\partial}{\partial y}(\rho\vec{v}v_y) + \frac{\partial}{\partial z}(\rho\vec{v}v_z) = -\nabla p +$$

$$+\left(\frac{\partial}{\partial x}\tau_{xx} + \frac{\partial}{\partial y}\tau_{yx} + \frac{\partial}{\partial z}\tau_{zx}\right)\vec{i} + \left(\frac{\partial}{\partial x}\tau_{xy} + \frac{\partial}{\partial y}\tau_{yy} + \frac{\partial}{\partial z}\tau_{zy}\right)\vec{j} +$$

$$+\left(\frac{\partial}{\partial x}\tau_{xz} + \frac{\partial}{\partial y}\tau_{yz} + \frac{\partial}{\partial z}\tau_{zz}\right)\vec{k} + \rho\vec{g}$$

$$\rho\frac{\partial\vec{v}}{\partial t} + \underbrace{\vec{v}\frac{\partial\rho}{\partial t} + \vec{v}\left(\frac{\partial\rho v_x}{\partial x} + \frac{\partial\rho v_y}{\partial y} + \frac{\partial\rho v_z}{\partial z}\right)}_{\vec{v}\cdot\left(\frac{\partial\rho}{\partial t} + \nabla\cdot\rho\vec{v}\right)} + \underbrace{\rho v_x\frac{\partial\vec{v}}{\partial x} + \rho v_y\frac{\partial\vec{v}}{\partial y} + \rho v_z\frac{\partial\vec{v}}{\partial z}}_{\rho(\vec{v}\cdot\nabla)\vec{v}} =$$

$\left(\frac{\partial\rho}{\partial t} + \nabla\cdot\rho\vec{v}\right) = 0$ Continuity

$$= -\nabla p + \nabla\cdot\overline{\overline{\tau}}' + \rho\vec{g}$$

$$\boxed{\rho\left[\frac{\partial\vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v}\right] = -\nabla p + \nabla\cdot\overline{\overline{\tau}}' + \rho\vec{g}}$$

ρ, p, $\vec{v}$ are unknowns in these equations, but we need a constitutive equation for $\overline{\overline{\tau}}'$ as a function of the deformation (strain) of the fluid.

$$\overline{\overline{\tau}}' = f(\dot{\overline{\overline{\varepsilon}}})$$

In tensor form this translates into

$$\tau'_{ij} = K_{ijkl}\,\dot{\varepsilon}_{kl}$$

4^{th} order tensor (3 components in each direction)

$3^4 = 81$ possible coefficients.

The simplest assumption is that the stress tensor is linearly proportional to the rate of strain.

K_{ijkl} is composed of 81 constants.

If we further assume that the fluid is isotropic, that is to say that the stress developed by the fluid due to the rate of strain in a certain direction does not depend on the direction (the stress/rate of strain relationship does not change under rotation of the reference frame).

$$K_{ijkl} = \lambda\, \delta_{ij}\, \delta_{kl} + \mu\, \delta_{ik}\, \delta_{jl} + \gamma\, \delta_{il}\, \delta_{jk}$$

3 independent variables (vs. 81 possible)

We can show that $\tau'_{ij} = \tau'_{ji}$, the stress tensor is symmetric and then

$$K_{ijkl} = K_{jikl} = \lambda\, \delta_{ji}\, \delta_{kl} + \mu\, \delta_{jk}\, \delta_{il} + \gamma\, \delta_{jl}\, \delta_{ik}$$

$$\lambda\, \delta_{ij} \delta_{kl} + \mu\, \delta_{ik} \delta_{jl} + \gamma\, \delta_{jk}\, \delta_{il}$$

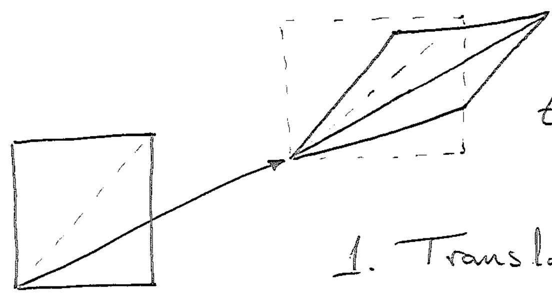
$$\boxed{\mu = \gamma}$$

$$\tau'_{ij} = \lambda\, \delta_{ij}\, \dot{\epsilon}_{kk} + 2\mu\, \dot{\epsilon}_{ij}$$

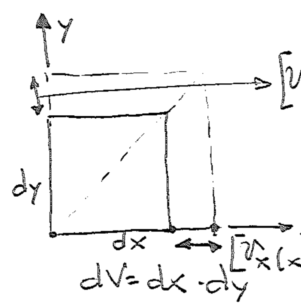
$$\overline{\overline{\tau}}' = 2\mu \frac{1}{2}\left(\nabla \vec{v} + \nabla \vec{v}^{T}\right) + \lambda\, \nabla \cdot \vec{v}\, \overline{\overline{I}}$$

Now we need to relate the rate of strain to some fluid variable used in our analysis.

F.M. White: "Viscous flow" 2^{nd} Edition 1991. McGraw Hill Section 1.3.3.



1. Translation $\vec{v} \cdot \delta t$
2. Volumetric deformation : $\dfrac{1}{V}\dfrac{\delta V}{\delta t}$



$$[v_y(y+dy) - v_y(y)]\,\delta t = \frac{\partial v_y}{\partial y}\, dy\, \delta t$$

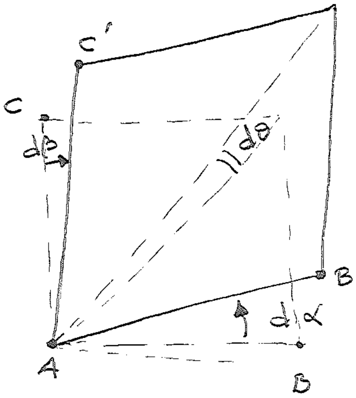
$$[v_x(x+dx) - v_x(x)]\,\delta t = \frac{\partial v_x}{\partial x}\, dx\, \delta t$$

$$\delta\, dV = \left[dx + \frac{\partial v_x}{\partial x} dx\, \delta t\right]\left[dy + \frac{\partial v_y}{\partial y} dy\, \delta t\right] - dx\,dy$$

$$\frac{1}{dV}\frac{\delta\, dV}{\delta t} = \frac{1}{\cancel{dx\,dy}}\;\frac{\frac{\partial v_x}{\partial x}\cancel{dx\,dy}\,\cancel{\delta t} + \frac{\partial v_y}{\partial y}\cancel{dx\,dy}\,\delta t + \frac{\partial v_x}{\partial x}\frac{\partial v_y}{\partial y}\cancel{dx\,dy}\,\delta t}{\cancel{\delta t}}$$

$$\frac{1}{V}\frac{DV}{Dt} = \underline{\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y}} + \cancel{\frac{\partial v_x}{\partial x}\frac{\partial v_y}{\partial y}\,\delta t} \to 0 \qquad \delta t \to 0$$

$$\boxed{\frac{1}{V}\frac{DV}{Dt} = \nabla \cdot \vec{v}} = \dot{\epsilon}_{xx} + \dot{\epsilon}_{yy} + \dot{\epsilon}_{zz}$$



3. Rotation

$$d\theta = \pi/4 - \left[\frac{(\pi/2 - d\alpha - d\beta)}{2} + d\alpha\right] = \frac{d\alpha - d\beta}{2}$$

4. Angular deformation is given by the chang of the right angle in the original element, substracting the effect of rotatio

rate of change of angle at AB $\quad \frac{d\alpha - d\theta}{dt} = \frac{d\alpha - \frac{d\alpha - d\beta}{2}}{dt} = \frac{1}{dt}\,\frac{d\alpha + d\beta}{2} = \dot{\epsilon}_{xy}$

rate of change of angle at AC : $\quad \frac{d\beta + d\theta}{dt} = \frac{1}{2}\,\frac{d\beta + \frac{d\alpha - d\beta}{2}}{dt} = \frac{1}{2}\,\frac{d\alpha + d\beta}{dt} = \dot{\epsilon}_{yx}$

If $d\alpha = d\beta \Rightarrow$ there is no rotation only deformation

If $d\alpha = -d\beta \Rightarrow$ there is no deformation, just pure rotation.

$$d\alpha = \lim_{dt \to 0} \frac{\frac{\partial v_y}{\partial x}\cancel{dx}\,dt}{\cancel{dx} + \frac{\partial v_x}{\partial x}\cancel{dx\,dt}} \simeq \frac{\partial v_y}{\partial x}\,dt$$

$$d\beta = \lim_{dt \to 0} \frac{\frac{\partial v_x}{\partial y}\cancel{dy}\,dt}{\cancel{dy} + \frac{\partial v_y}{\partial y}\cancel{dy\,dt}} \simeq \frac{\partial v_x}{\partial y}\,dt$$

The angular deformation is the rate of change of the right angle at $\hat{ABC}$ that is equally split between $\dot{\epsilon}_{xy}$ and $\dot{\epsilon}_{yx}$ (the rate of deformation tensor is symmetric).

$$\overline{\overline{\epsilon}} = \frac{1}{2}\begin{pmatrix} \frac{\partial v_x}{\partial x} + \frac{\partial v_x}{\partial x} & \frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y} & \frac{\partial v_z}{\partial x} + \frac{\partial v_x}{\partial v_z} \\ \frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} & \frac{\partial v_y}{\partial y} + \frac{\partial v_y}{\partial y} & \frac{\partial v_z}{\partial y} + \frac{\partial v_y}{\partial z} \\ \frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x} & \frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y} & \frac{\partial v_z}{\partial z} + \frac{\partial v_z}{\partial z} \end{pmatrix}$$