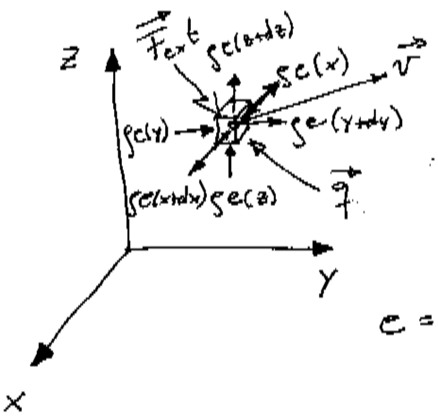
Conservation of Energy in Differential form



$$\frac{\partial}{\partial t}\int_{c.v}\rho e\,dV + \int_{c.s}\rho e(\vec v-\vec v_c)\cdot\vec n\,dA = -\int_{c.s}p\vec n\cdot\vec v\,dA + \int_{c.s}\vec v\cdot\bar{\bar\tau}'\cdot\vec n\,dA + \int_{c.v}\rho\vec g\cdot\vec v\,dV + \int_{c.v}\dot Q_{\substack{rad\\che}}\,dV - \int_{c.s}\vec q\cdot\vec n\,dA$$

$$e = u + \frac{1}{2}v^2$$

$$\frac{\partial \rho e}{\partial t} + \nabla\cdot(\rho\vec v e) = -\nabla\cdot(p\vec v) + \nabla\cdot(\vec v\cdot\bar{\bar\tau}') + \rho g\vec v + \dot q_{\substack{rad\\che}} - \nabla\cdot\vec q$$

$$\rho\frac{\partial e}{\partial t} + \rho\vec v\cdot\nabla e + \underbrace{e\frac{\partial\rho}{\partial t} + e\nabla\cdot\rho\vec v}_{e\left[\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\vec v)\right] = 0} = -p(\nabla\cdot\vec v) - \vec v\cdot\nabla p + \bar{\bar\tau}':\nabla\vec v + \vec v\cdot(\nabla\cdot$$

$$+\rho g\vec v + \dot q_{\substack{rad\\che}} - \nabla\cdot\vec q$$

Conservation of total energy

Mechanical Energy: multiply the momentum conservation equa[tion] by the velocity

$$\frac{D\rho\vec v}{Dt}\cdot\vec v = \sum F_{ext}\cdot\vec v$$

$$\frac{D}{Dt}\left(\frac{1}{2}\rho v^2\right) = \dot W_{ext}$$

$$\vec v\cdot\left[\rho\frac{\partial\vec v}{\partial t} + \rho(\vec v\cdot\nabla)\vec v\right] = \left(-\nabla p + \nabla\cdot\bar{\bar\tau}' + \rho\vec g\right)\cdot\vec v$$

$$\rho\frac{\partial\frac{1}{2}v^2}{\partial t} + \rho(\vec v\cdot\nabla)\left(\frac{1}{2}v^2\right) = -\vec v\cdot\nabla p + \vec v\cdot(\nabla\cdot\bar{\bar\tau}') + \rho\vec g\cdot\vec v$$

Substracting Mechanical Energy from the total energy we get:

$$\underbrace{\rho \frac{\partial u}{\partial t} + \rho \vec{v}\cdot\nabla u}_{\text{The rate of change of the internal energy of the fluid is due to:}} = -p(\nabla\cdot\vec{v}) + \bar{\bar{\tau}}' : \nabla\vec{v} + \underbrace{\dot{q}_{\substack{rad\\che}} - \nabla\cdot\vec{q}}_{\text{Heat addition}}$$

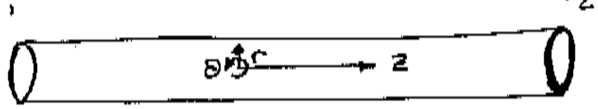
$-p(\nabla\cdot\vec{v}) \rightarrow$ Work done by pressure on a volumetric deformation

$\bar{\bar{\tau}}' : \nabla\vec{v} \rightarrow$ viscous dissipation ϕ (positive definite)

Pipe flow

Fully develop, laminar, steady, incompressible

P_1 $\qquad\qquad\qquad\qquad$ P_2



Continuity: $\cancel{\frac{\partial \rho}{\partial t}} + \nabla\cdot\rho\vec{v} = 0$

$$\rho\left[\frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \frac{1}{r}\frac{\partial}{\partial \theta} v_\theta + \cancel{\frac{\partial}{\partial z} v_z}\right] = 0$$

Fully develop means $\frac{\partial \vec{v}}{\partial z} = 0$

$$\frac{\partial}{\partial r}(r v_r) = -\frac{\partial}{\partial \theta} v_\theta$$

$$r\frac{\partial v_r}{\partial r} + v_r = -\frac{\partial}{\partial \theta} v_\theta$$

@ $r=0$ $\quad v_r = v_\theta = 0$

@ $r=R$ $\quad v_r = v_\theta = 0$

@ $r=0 \Rightarrow \frac{\partial v_\theta}{\partial \theta} = 0$

@ $r=R \Rightarrow \frac{\partial v_\theta}{\partial \theta} = 0 \Rightarrow \frac{\partial v_r}{\partial r} = 0$

Only solution is $v_r = v_\theta = 0$

Conservation of momentum

$$\rho \cancel{\frac{\partial v_r}{\partial t}} + \rho\left(\cancel{v_r\frac{\partial v_r}{\partial r}} + \cancel{\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}} + \cancel{v_z\frac{\partial v_r}{\partial z}} - \cancel{\frac{v_\theta^2}{r}}\right) = -\frac{\partial P}{\partial r} + \mu\left[\cancel{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_r}{\partial r}\right)} + \cancel{\frac{1}{r^2}\frac{\partial^2 v_r}{\partial\theta^2}} + \cancel{\frac{\partial^2 v_r}{\partial z^2}} - \cancel{\frac{v_r}{r^2}} - \cancel{\frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}}\right]$$

$$\rho \cancel{\frac{\partial v_\theta}{\partial t}} + \rho\left(\cancel{v_r\frac{\partial v_\theta}{\partial r}} + \cancel{\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}} + \cancel{v_z\frac{\partial v_\theta}{\partial z}} + \cancel{\frac{v_r v_\theta}{r}}\right) = -\frac{1}{r}\frac{\partial P}{\partial \theta} + \mu\left[\cancel{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_\theta}{\partial r}\right)} + \cancel{\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial\theta^2}} + \cancel{\frac{\partial^2 v_\theta}{\partial z^2}} + \cancel{\frac{2}{r}\frac{\partial v_r}{\partial \theta}} - \cancel{\frac{v_\theta}{r^2}}\right]$$

$$\rho \cancel{\frac{\partial v_z}{\partial t}} + \rho\left(\cancel{v_r\frac{\partial v_z}{\partial r}} + \cancel{\frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta}} + \cancel{v_z\frac{\partial v_z}{\partial z}}\right) = -\frac{\partial P}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \cancel{\frac{1}{r^2}\frac{\partial^2 v_z}{\partial\theta^2}} + \cancel{\frac{\partial^2 v_z}{\partial z^2}}\right]$$

$$\boxed{\nabla^2\left(\frac{\partial v_z}{\partial \theta}\right) = 0 \quad \text{with} \quad \frac{\partial v_z}{\partial \theta} = 0 \ @ \ r=0 \text{ and } r=R}$$

Resulting equations are $\dfrac{\partial P}{\partial r} = \dfrac{1}{r}\dfrac{\partial P}{\partial \theta} = 0$

$$\boxed{0 = -\frac{\partial P}{\partial z} + \mu\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)}$$

If I take the derivative $\dfrac{\partial}{\partial z}$

$$\frac{\partial^2 P}{\partial z^2} = \mu\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial}{\partial r}\left(\underbrace{\frac{\partial v_z}{\partial z}}_{0}\right)\right] = 0$$

Therefore $\dfrac{\partial P}{\partial z} = \dfrac{dP}{dz} = \text{Constant}$

$$\frac{1}{r}\frac{\partial}{\partial r}\left[r\left(\frac{\partial v_z}{\partial r}\right)\right] = \frac{1}{\mu}\frac{dP}{dz} = \text{constant}$$

$$\frac{d}{dr}\left(r\frac{dv_z}{dr}\right) = \frac{1}{\mu}\frac{dP}{dz}\, r$$

$$\cancel{r}\frac{dv_z}{dr} = \frac{1}{\mu}\frac{dP}{dz}\frac{r^{\cancel{2}}}{2} + C_1/r$$

$$\frac{dv_z}{dr} = \frac{1}{\mu}\frac{dP}{dz}\, r/2 + \cancel{C_1/r} \to \infty$$

$$v_z(r) = \frac{1}{\mu}\frac{dP}{dz}\frac{r^2}{4} + C_2$$

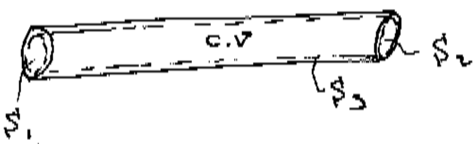
$$\left.\frac{dv_z}{dr}\right|_{r=0} = 0 \quad \underline{\text{Symmetry}}$$

$$v_z(r) = \frac{1}{\mu}\frac{dP}{dz}\frac{r^2}{4} + C_2$$

$$v_z(r=R) = 0 \Rightarrow \quad C_2 = -\frac{1}{\mu}\frac{dP}{dz}R^2/4$$

$$v_z(r) = \frac{1}{4\mu}\underset{-}{\frac{dP}{dz}}\underset{-}{\left(r^2-R^2\right)} \quad // \quad v_z(r) = \frac{1}{4\mu}\underset{+}{\left(-\frac{dP}{dz}\right)}\underset{+}{\left(R^2-r^2\right)}$$

Conservation of energy



$$\underbrace{\cancel{\frac{\partial}{\partial t}\int_{C.V}\rho e\,dV}}_{\text{steady}} + \int_{C.S.}\rho e(\vec{v}-\vec{v}_c)\cdot\vec{n}\,dA = -\int_{C.S}p\vec{n}\cdot\vec{v}\,dA + \int_{C.S}\vec{v}\cdot\bar{\bar{\tau}}'\cdot\vec{n}\,dA + \underbrace{\cancel{\int_{C.V}\rho\vec{g}\cdot\vec{v}\,dV}}_{\substack{\text{negligible}\\ \text{no increase in height}}} + \underbrace{\cancel{\int_{C.V}\dot{Q}_{rad}}}_{\text{Adiabatic}} - \int_{C.}\ldots$$

Steady

$$\rho_1 e_1\underbrace{\int_{S_1}\vec{v}\cdot\vec{n}\,dA}_{-Q} + \rho_2 e_2\underbrace{\int_{S_2}\vec{v}\cdot\vec{n}\,dA}_{Q} + \underbrace{\cancel{\int_{S_3}\rho e\vec{v}\cdot\vec{n}\,dA}}_{0\ \text{SOLID WALL}} = -P_1\underbrace{\int_{S_1}\vec{v}_1\cdot\vec{n}\,dA}_{-Q} - P_2\underbrace{\int_{S_2}\vec{v}\cdot\vec{n}\,dA}_{Q} - \underbrace{\cancel{\int_{S_3}p\vec{v}\cdot\vec{n}\,dA}}_{0\ \text{SOLID WALL}}$$

$$+ \underbrace{\cancel{\int_{S_1}\vec{v}\cdot\bar{\bar{\tau}}'\cdot\vec{n}\,dA}}_{\text{negligible}} + \underbrace{\cancel{\int_{S_2}\vec{v}\cdot\bar{\bar{\tau}}'\cdot\vec{n}\,dA}}_{\text{negligible}} + \underbrace{\cancel{\int_{S_3}\vec{v}\cdot\bar{\bar{\tau}}'\cdot\vec{n}}}_{0\ \text{SOLID WALL}}$$

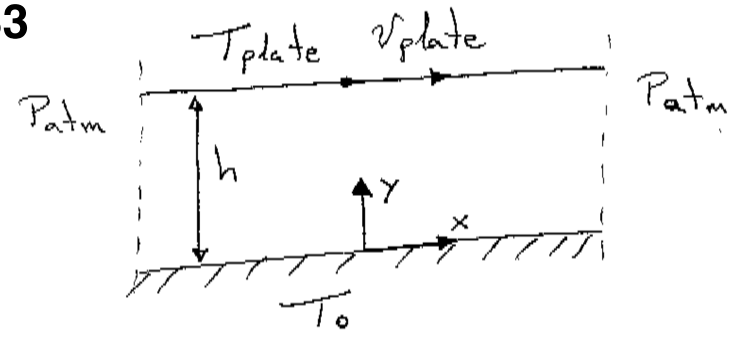
$$\rho_2 e_2 - \rho_1 e_1 = P_1 - P_2$$

If I consider $e = u + \frac{1}{2}v^2$ then

$$\rho_1 u_1\int_{S_1}\vec{v}\cdot\vec{n}\,dA + \rho_1\cancel{\int_{S_1}\frac{1}{2}v^2\vec{v}\cdot\vec{n}\,dA} + \rho_2 u_2\int_{S_2}\vec{v}\cdot\vec{n}\,dA + \rho_2\cancel{\int_{S_2}\frac{1}{2}v^2\vec{v}\cdot\vec{n}\,dA} =$$

equal but opposite signs ($\vec{v}$.i

therefore: $\boxed{u_2 - u_1 = \frac{P_1 - P_2}{\rho}}$



Steady, incompressible flow between two infinite plates

$$\frac{\partial}{\partial t} = 0, \quad v_z = 0, \quad \frac{\partial}{\partial x} = 0$$

- Continuity:

$$\underbrace{\cancel{\frac{\partial \rho}{\partial t}}}_{\text{STEADY}} + \underbrace{\cancel{\frac{\partial}{\partial x}\rho v_x} + \frac{\partial}{\partial y}\rho v_y + \cancel{\frac{\partial}{\partial z}\rho v_z}}_{\text{INFINITE PLATES}} = 0 \Rightarrow$$

$$\Rightarrow \frac{\partial}{\partial y}\rho v_y = 0 \Rightarrow \underbrace{\frac{\partial v_y}{\partial y} = 0 \quad @\ y=0 \Rightarrow v_y = 0}_{\boxed{v_y = 0}} \Rightarrow$$

- Conservation of momentum:

$$\rho\left(\underbrace{\cancel{\frac{\partial v_x}{\partial t}}}_{\text{STEADY}} + v_x \cancel{\frac{\partial v_x}{\partial x}} + \cancel{v_y} \frac{\partial v_x}{\partial y} + \cancel{v_z} \cancel{\frac{\partial v_x}{\partial z}}\right) = -\underset{0}{\underset{\parallel}{\frac{\partial P}{\partial x}}} + \mu\left(\cancel{\frac{\partial^2 v_x}{\partial x^2}} + \frac{\partial^2 v_x}{\partial y^2} + \cancel{\frac{\partial^2 v_x}{\partial z^2}}\right)$$

$$\boxed{0 = \mu \frac{d^2 v_x}{d y^2}}$$

$$\rho\left(\cancel{\frac{\partial v_y}{\partial t}} + v_x \cancel{\frac{\partial v_y}{\partial x}} + \cancel{v_y} \cancel{\frac{\partial v_y}{\partial y}} + \cancel{v_z} \cancel{\frac{\partial v_y}{\partial z}}\right) = -\frac{\partial P}{\partial y} + \mu\left(\cancel{\frac{\partial^2 v_y}{\partial x^2}} + \cancel{\frac{\partial^2 v_y}{\partial y^2}} + \cancel{\frac{\partial^2 v_y}{\partial z^2}}\right)$$

$$\boxed{\frac{\partial P}{\partial y} = 0}$$

$$\frac{d^2 v_x}{dy^2} = 0 \Rightarrow \frac{d v_x}{d y} = C_1 \Rightarrow v_x = C_1 \cdot y + C_2$$

$$\left.\begin{aligned} & v_x\ (y=0) = 0 \Longrightarrow C_2 = 0 \\ & v_x\ (y=h) = v_{plate} \Rightarrow C_1 = \frac{v_{plate}}{h} \end{aligned}\right\} \quad \boxed{v_x = v_{plate} \cdot (y/h)}$$

- Conservation of Energy

STEADY   INFINITE PLATES   Continuity

$$\rho\left(\cancel{\frac{\partial e}{\partial t}} + \cancel{v_x \frac{\partial e}{\partial x}} + v_y \frac{\partial e}{\partial y} + \cancel{v_z \frac{\partial e}{\partial z}}\right) = -p\cancelto{0}{\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right)} - \left(\cancel{v_x \frac{\partial p}{\partial x}} + \cancel{v_y \frac{\partial p}{\partial y}} + \cancel{v_z \frac{\partial p}{\partial z}}\right) + \left[2\mu\begin{pmatrix} \cancel{\frac{\partial v_x}{\partial x}} & \frac{1}{2}\left(\cancel{\frac{\partial v_y}{\partial x}} + \frac{\partial v_x}{\partial y}\right) & \frac{1}{2}\left(\cancel{\frac{\partial v_z}{\partial x}} + \cancel{\frac{\partial v_x}{\partial z}}\right) \\ \frac{1}{2}\left(\frac{\partial v_x}{\partial y} + \cancel{\frac{\partial v_y}{\partial x}}\right) & \cancel{\frac{\partial v_y}{\partial y}} & \frac{1}{2}\left(\cancel{\frac{\partial v_z}{\partial y}} + \cancel{\frac{\partial v_y}{\partial z}}\right) \\ \frac{1}{2}\left(\cancel{\frac{\partial v_x}{\partial z}} + \cancel{\frac{\partial v_z}{\partial x}}\right) & \frac{1}{2}\left(\cancel{\frac{\partial v_y}{\partial z}} + \cancel{\frac{\partial v_z}{\partial y}}\right) & \cancel{\frac{\partial v_z}{\partial z}} \end{pmatrix} + \lambda\cancelto{0}{\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right)}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right]$$

continuity

$$: \begin{pmatrix} \cancel{\frac{\partial v_x}{\partial x}} & \frac{\partial v_x}{\partial y} & \cancel{\frac{\partial v_x}{\partial z}} \\ \cancel{\frac{\partial v_y}{\partial x}} & \cancel{\frac{\partial v_y}{\partial y}} & \cancel{\frac{\partial v_y}{\partial z}} \\ \cancel{\frac{\partial v_z}{\partial x}} & \cancel{\frac{\partial v_z}{\partial y}} & \cancel{\frac{\partial v_z}{\partial z}} \end{pmatrix} + \left\{\begin{pmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{pmatrix}\left[2\mu\begin{pmatrix} \cancel{\frac{\partial v_x}{\partial x}} & \frac{1}{2}\left(\cancel{\frac{\partial v_y}{\partial x}} + \frac{\partial v_x}{\partial y}\right) & \frac{1}{2}\left(\cancel{\frac{\partial v_z}{\partial x}} + \cancel{\frac{\partial v_x}{\partial z}}\right) \\ \frac{1}{2}\left(\frac{\partial v_x}{\partial y} + \cancel{\frac{\partial v_y}{\partial x}}\right) & \cancel{\frac{\partial v_y}{\partial y}} & \frac{1}{2}\left(\cancel{\frac{\partial v_z}{\partial y}} + \cancel{\frac{\partial v_y}{\partial z}}\right) \\ \frac{1}{2}\left(\cancel{\frac{\partial v_x}{\partial z}} + \cancel{\frac{\partial v_z}{\partial x}}\right) & \frac{1}{2}\left(\cancel{\frac{\partial v_y}{\partial z}} + \cancel{\frac{\partial v_z}{\partial y}}\right) & \cancel{\frac{\partial v_z}{\partial z}} \end{pmatrix}\right.\right.$$

$$\left.\left. + \lambda\cancel{\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right)}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right]\right\} \cdot \begin{pmatrix} v_x \\ \cancel{v_y} \\ \cancel{v_z} \end{pmatrix} + k\left(\cancel{\frac{\partial^2 T}{\partial x^2}} + \frac{\partial^2 T}{\partial y^2} + \cancel{\frac{\partial^2 T}{\partial z^2}}\right)$$

Continuity
Incompressible   INFINITE PLATES

$$0 = 2\mu \cdot \frac{1}{2}\left(\frac{\partial v_x}{\partial y}\right)\left(\frac{\partial v_x}{\partial y}\right) + 2\mu \cdot \frac{1}{2}\left(\frac{\partial^2 v_x}{\partial y^2}, \cancel{\frac{\partial}{\partial x}\left(\frac{\partial v_x}{\partial y}\right)}, 0\right)\begin{pmatrix} v_x \\ \cancel{v_y} \\ \cancel{v_z} \end{pmatrix}$$

$$- k\frac{\partial^2 T}{\partial y^2}$$

$$\boxed{0 = \mu\left(\frac{\partial v_x}{\partial y}\right)^2 + \mu v_x \frac{\partial^2 v_x}{\partial y^2} + k\frac{\partial^2 T}{\partial y^2}}$$

$$\frac{dv_x}{dy} = \frac{v_{plate}}{h}\ ; \quad \frac{d^2 v_x}{dy^2} = 0 \Rightarrow \quad \frac{d^2 T}{dy^2} = -\frac{\mu}{k}\left(\frac{v_{plate}}{h}\right)^2$$

$$T(y=0) = T_0$$

$$T(y=h) = T_{plate}$$

$$\frac{dT}{dy} = -\frac{\mu}{K}\left(\frac{V_{plate}}{h}\right)^2 y + C_3$$

$$T = -\frac{\mu}{K}\left(\frac{V_{plate}}{h}\right)^2 y^2/2 + C_3 y + C_4$$

$@ y = 0 \Rightarrow T = T_0 = C_4$

$@ y = h \Rightarrow T = T_{plate} = -\frac{\mu}{K}\left(\frac{V_{plate}}{h}\right)^2 \frac{h^2}{2} + C_3 h + T_0$

$$C_3 = \frac{T_{plate} - T_0}{h} + \frac{\mu}{K}\left(\frac{V_{plate}}{h}\right)^2 \frac{h}{2}$$

$$\boxed{T(y) = -\frac{\mu}{K}\left(\frac{V_{plate}}{h}\right)^2 \frac{y^2 - yh}{2} + \frac{T_{plate} - T_0}{h} y + T_0}$$