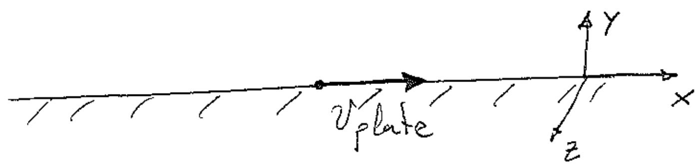
Impulsively started plate: STOKES' 1ST PROBLEM

Semi-infinite expanse of fluid in contact with a plate that starts to move at constant speed V_{plate} at t

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial z} = 0$$

No pressure gradient

Incompressible



Continuity:

$$\underbrace{\cancel{\frac{\partial \rho}{\partial t}}}_{0} + \cancel{\frac{\partial \rho V_x}{\partial x}} + \frac{\partial \rho V_y}{\partial y} + \cancel{\frac{\partial \rho V_z}{\partial z}} = 0$$

$$\underbrace{\frac{\partial V_y}{\partial y} = 0 \qquad V_y = 0 \ @ \ y = 0}_{\boxed{V_y = 0}}$$

Navier-Stokes (Conservation of momentum)

x-component:
$$\rho\left(\frac{\partial V_x}{\partial t} + V_x \cancel{\frac{\partial V_x}{\partial x}} + \cancel{V_y \frac{\partial V_x}{\partial y}} + V_z \cancel{\frac{\partial V_x}{\partial z}}\right) = -\cancel{\frac{\partial p}{\partial x}} + \mu\left(\cancel{\frac{\partial^2 V_x}{\partial x^2}} + \frac{\partial^2 V_x}{\partial y^2} + \cancel{\frac{\partial^2 V_x}{\partial z^2}}\right) + \cancel{\rho g_x}$$

$$\boxed{\frac{\partial V_x}{\partial t} = \nu \frac{\partial^2 V_x}{\partial y^2}}$$

At $t=0$ $V_x(y>0, 0) = 0$

At $y=0$ $V_x(0,t) = V_{plate}$

as $y \to \infty$ $V_x(y,t) \to 0$

There are no length or time scales characteristic to this problem. That makes it a candidate for a self-similar solution.

We define a self-similarity variable $\eta = \frac{y}{\delta(t)}$, leaving t unchanged

Change variables from $(y,t) \to (\eta, t)$

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \eta}\cdot \underbrace{\frac{\partial \eta}{\partial t}}_{= -\frac{y}{\delta^2}\cdot\delta' = -\eta\frac{\delta'}{\delta}} + \frac{\partial}{\partial t}\cdot \underbrace{\frac{\partial t}{\partial t}}_{=1} \;;\quad \frac{\partial}{\partial y} = \frac{\partial}{\partial \eta}\cdot \underbrace{\frac{\partial \eta}{\partial y}}_{=1/\delta} + \frac{\partial}{\partial t}\cdot \underbrace{\frac{\partial t}{\partial y}}_{=0}$$

y, t are independent variables

We hypothesize that $v_x(y,t)$ is only a function of the self-similar variable η $v_x(\eta)$.

$$\frac{dv_x}{d\eta}\left(-\eta\,\delta'/\delta\right) + 1\cdot \cancel{\frac{\partial v_x}{\partial t}} = \nu \frac{1}{\delta^2}\frac{d^2 v_x}{d\eta^2}$$

If the only independent variable in this equation is η

$$-\eta\,\frac{dv_x}{d\eta}\underbrace{\left(\frac{\delta\cdot\delta'}{\nu}\right)}_{\text{constant}} = \frac{d^2 v_x}{d\eta^2}$$

$$\delta\cdot\frac{d\delta}{dt} = C_1\cdot\nu \Rightarrow \frac{d(\delta^2)}{dt} = 2C_1\nu \Rightarrow \delta^2(t) = 2C_1\nu t + C_2$$

at $t=0$ $\delta(t)=0 \Rightarrow C_2=0$

$\delta(t) = \sqrt{2C_1\nu t}$

We make the resulting equation non-dimensional with $\hat{v}_x = v_x/v_{plate}$.

$$\frac{d^2\hat{v}_x}{d\eta^2} + C_1\,\eta\,\frac{d\hat{v}_x}{d\eta} = 0\,.$$ It is convenient to set $C_1 = 2$

$$\frac{d\hat{v}_x}{d\eta} = \xi \Rightarrow \frac{d\xi}{d\eta} + 2\eta\xi = 0$$

$$\int\frac{d\xi}{\xi} = \int -2\eta\, d\eta$$

$$\ln\xi = -\eta^2 + C_1$$

The boundary conditions for the problem are:

$$v_x(y>0,\ t=0)=0 \Rightarrow \hat{v}_x(\eta\to\infty)\to 0$$

$$v_x(y=0,\ t>0)=v_{plate} \Rightarrow \hat{v}_x(\eta=0)=1$$

$$v_x(y\to\infty)\to 0 \Rightarrow \hat{v}_x(\eta\to\infty)\to 0$$

$$\xi=\frac{d\hat{v}_x}{d\eta}=e^{-\eta^2}\cdot e^{c_1}=c_1' e^{-\eta^2}$$

$$\hat{v}_x=c_1'\int_0^{\eta} e^{-\eta'^2}\,d\eta'+c_2$$

$$\hat{v}_x(\eta=0)=c_1'\cancel{\int_0^0 e^{-\eta'^2}\,d\eta'}+c_2=1$$

0

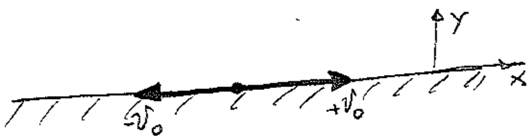
$$\hat{v}_x(\eta\to\infty)=c_1'\underbrace{\int_0^{\infty} e^{-\eta'^2}\,d\eta'}_{\sqrt{\pi}/2}+1\to 0 \Rightarrow c_1'=-\frac{2}{\sqrt{\pi}}$$

$$v_x(y,t)=v_{plate}\left[1-\underbrace{\frac{2}{\sqrt{\pi}}\int_0^{\frac{y}{2\sqrt{\nu t}}} e^{-\eta^2}\,d\eta}_{\mathrm{erf}\left(\frac{y}{2\sqrt{\nu t}}\right)}\right]$$

$$\boxed{v_x(y,t)=\left[1-\mathrm{erf}\left(y/2\sqrt{\nu t}\right)\right]v_{plate}}$$

Oscillatory moving plate: Stokes' 2^{nd} Problem

Semi-infinite expanse of fluid in contact with a plate that moves periodically with an oscillatory motion $V_{plate} = v_0 \sin(wt)$



$$\frac{\partial}{\partial x} = \frac{\partial}{\partial z} = 0$$

No pressure gradient.

Incompressible.

Continuity

$$\cancel{\frac{\partial \rho}{\partial t}} + \cancel{\frac{\partial \rho v_x}{\partial x}} + \frac{\partial \rho v_y}{\partial y} + \cancel{\frac{\partial \rho v_z}{\partial z}} = 0$$

$$\rho \frac{\partial v_y}{\partial y} = 0 \; ; \; v_y = 0 \text{ @ } y=0 \Rightarrow \boxed{v_y = 0}$$

Conservation of momentum

$$\rho\left(\frac{\partial v_x}{\partial t} + v_x \cancel{\frac{\partial v_x}{\partial x}} + \cancel{v_y} \frac{\partial v_x}{\partial y} + v_z \cancel{\frac{\partial v_x}{\partial z}}\right) = -\cancel{\frac{\partial P}{\partial x}} + \mu\left(\cancel{\frac{\partial^2 v_x}{\partial x^2}} + \frac{\partial^2 v_x}{\partial y^2} + \cancel{\frac{\partial^2 v_x}{\partial z^2}}\right) + \cancel{\rho g}$$

$$\boxed{\frac{\partial v_x}{\partial t} = \nu \frac{\partial^2 v_x}{\partial y^2}}$$

@ $y=0$ $\quad v_x = v_{plate} = v_0 \sin(wt)$

@ $y \to \infty$ $\quad v_x$ is unknown but finite

There is a time scale for this problem $\left(\frac{1}{w}\right)$, therefore it is not a good candidate for a self-similar solution.

Since the forcing is periodic, it is a good candidate for Fourier series solution.

$$v_x = Re\left\{\sum_{n=0}^{\infty} A_n(y) \; e^{i w_n t}\right\}$$

At $y=0$ $\quad v_x(y=0,t) = v_0 \sin(wt) = \sum_{n=0}^{\infty} A_n(y=0)\, e^{i w_n t}$

$$i w_n A_n \cancel{e^{i w_n t}} = \nu A_n'' \cancel{e^{i w_n t}}$$

$$i\omega_n A_n = \nu A_n'' \qquad \sqrt{\frac{i\omega_n}{\nu}} = \left(e^{i\pi/2}\right)^{1/2}\sqrt{\omega_n/\nu} = \frac{-1-i}{\sqrt{2}}\sqrt{\frac{\omega_n}{\nu}} + \frac{1+i}{\sqrt{2}}\sqrt{\omega_n/\nu}$$

$$\left(e^{i\pi/2}\right)^{1/2} \to e^{i\pi/4}, \qquad e^{i(\pi/4+\pi)}$$

$$A_n'' - \frac{i\omega_n}{\nu}A_n = 0 \Rightarrow A_n(y) = a_n e^{\frac{-1-i}{\sqrt{2}}\sqrt{\frac{\omega_n}{\nu}}y} + b_n e^{\frac{1+i}{\sqrt{2}}\sqrt{\frac{\omega_n}{\nu}}y}$$

$$V_x(y,t) = \sum a_n e^{\frac{-1-i}{\sqrt{2}}\sqrt{\frac{\omega_n}{\nu}}y} e^{i\omega_n t} + \sum b_n e^{\frac{1+i}{\sqrt{2}}\sqrt{\frac{\omega_n}{\nu}}y} e^{i\omega_n t}$$

$$V_x(0,t) = V_0 \sin(\omega t)$$

$V_x(y\to\infty,t)$ is finite $\Rightarrow$ $b_n = 0 \ \forall n$ since $e^{\frac{1+i}{\sqrt{2}}\sqrt{\omega_n/\nu}\,y} \xrightarrow[y\to\infty]{} \infty$

$$V_x(y,t) = \sum_{n=0}^{\infty} a_n e^{i\left(\omega_n t - \sqrt{\frac{\omega_n}{2\nu}}y\right)} \cdot e^{-\sqrt{\frac{\omega_n}{2\nu}}y}$$

$$V_x(y=0,t) = \sum_{n=0}^{\infty} a_n e^{i\omega_n t} = V_0 \sin\omega t$$

$$a_1 = iV_0 \qquad a_n = 0 \quad n \neq 1$$

$$\omega_1 = \omega$$

$$\boxed{V_x(y,t) = V_0 e^{-\sqrt{\frac{\omega}{2\nu}}y}\sin\left(\omega t - \sqrt{\frac{\omega}{2\nu}}y\right)}$$