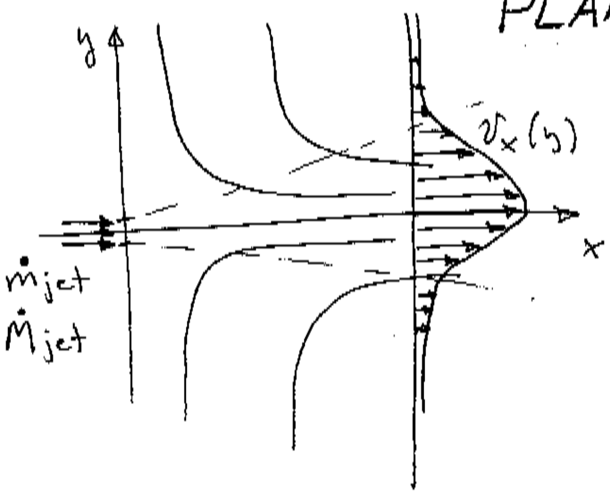
PLANE SLOT JET



We know the flux of mass and momentum in the jet.
The pressure gradient along the jet is zero $\frac{\partial P}{\partial x} = 0$

Incompressible, steady flow.

$$\psi = 6\alpha\nu x^{1/3} f(\eta) \quad \text{where} \quad \eta = \alpha \frac{y}{x^{2/3}}$$

Continuity is satisfied by construction

$$\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} = \frac{\partial \Psi}{\partial x \partial y} - \frac{\partial \Psi}{\partial y \partial x} \equiv 0$$

<u>x-momentum</u>:

$$\cancel{\frac{\partial V_x}{\partial t}}^{0} + V_x \frac{\partial V_x}{\partial x} + V_y \frac{\partial V_x}{\partial y} = -\frac{1}{\rho}\cancel{\frac{\partial P}{\partial x}}^{0} + \nu\left(\frac{\partial^2 V_x}{\partial x^2} + \frac{\partial^2 V_x}{\partial y^2}\right)$$

$$\frac{\partial \Psi}{\partial y}\frac{\partial^2 \Psi}{\partial x \partial y} - \frac{\partial \Psi}{\partial x}\frac{\partial^2 \Psi}{\partial y^2} = \nu\left(\frac{\partial^3 \Psi}{\partial y \partial x^2} + \frac{\partial^3 \Psi}{\partial y^3}\right)$$

<u>y-momentum</u>

$$\cancel{\frac{\partial V_y}{\partial t}}^{0} + V_x \frac{\partial V_y}{\partial x} + V_y \frac{\partial V_y}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial y} + \nu\left(\frac{\partial^2 V_y}{\partial x^2} + \frac{\partial^2 V_y}{\partial y^2}\right)$$

This equation will provide the pressure gradient $\frac{\partial P}{\partial y}$.

$$\frac{\partial \Psi}{\partial x} = 6\alpha\nu \frac{1}{3} x^{-2/3} f(\eta) + 6\alpha\nu x^{1/3} f'(\eta)\left(-\frac{2}{3}\alpha y x^{-5/3}\right)$$

$$\frac{\partial^2 \Psi}{\partial x^2} = 6\alpha\nu \frac{1}{3}\left(-\frac{2}{3}\right) x^{-5/3} f(\eta) + 6\alpha\nu \frac{1}{3} x^{-2/3} f'(\eta)\left(-\frac{2}{3}\alpha y x^{-5/3}\right)$$
$$+ 6\alpha\nu\left(\frac{4}{3}\right) x^{-7/3} f'(\eta)\left(-\frac{2}{3}\alpha y\right) + 6\alpha\nu x^{-4/3}\left(f'' \frac{4}{9}\alpha^2 y^2 x^{-5/3}\right)$$

$$\frac{\partial^2 \Psi}{\partial x \partial y} = 2\alpha\nu x^{-2/3} f'(\eta)\cdot\alpha x^{-2/3} - 4\alpha^2\nu x^{-4/3} f'(\eta) - 4\alpha^2\nu x^{-4/3} f''(\eta)\, y\frac{\alpha}{x^{2/3}}$$

$$\frac{\partial \Psi}{\partial y} = 6\alpha\nu x^{1/3} f'(\eta)\cdot\alpha x^{-2/3}$$

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$$\frac{\partial^2 \Psi}{\partial y^2} = 6\alpha^2 \nu\, x^{-1/3} f''(\eta) \cdot \alpha x^{-2/3}$$

$$\frac{\partial^3 \Psi}{\partial y^3} = 6\alpha^3 \nu\, x^{-1} f'''(\eta) \cdot \alpha x^{-2/3}$$

$$\frac{\partial^3 \Psi}{\partial x^2 \partial y} = -\frac{4}{3}\alpha\nu x^{-5/3} f' \alpha x^{-2/3} + 4\alpha\nu x^{-5/3}\left(\eta f'' \alpha x^{-2/3} + f' \alpha x^{-2/3}\right) + \frac{8}{3}\alpha\nu x^{-5/3}\left(\eta^2 f''' \cdot \alpha x^{-1/3} + 2\eta f'' \alpha \ldots\right.$$

$$6\alpha^2 \nu x^{-1/3} f' \cdot \left(\cancel{2}\alpha^2 \nu x^{-4/3} f' - \frac{\cancel{4}}{2}\alpha^2 \nu x^{-4/3} f' - 4\alpha^2 \nu x^{-4/3} \eta f''\right) -$$

$$-\left(2\alpha\nu x^{-2/3} f - 4\alpha\nu x^{-2/3} \eta f'\right) 6\alpha^3 \nu x^{-1} f'' = \nu\left(6\alpha^4 \nu x^{-5/3} f' \ldots\right.$$

$$\left.\cancel{+ \frac{8}{3}\alpha^2 \nu x^{-7/3} f' + \frac{28}{3}\alpha^2 \nu x^{-7/3} \eta f'' + \frac{8}{3}\alpha^2 \nu x^{-7/3} \eta^2 f'''}\right)$$

We need to drop this term for the self similarit[y] solution to work: $\frac{\partial^2}{\partial x^2} << \frac{\partial^2}{\partial y^2}$ within the b.l

$$\cancel{6}\,\cancel{\alpha^4}\,\cancel{\nu^2}\,\cancel{x^{-5/3}}\left(-2f'^2 - \cancel{4\eta f'' f'}\right) - 6\alpha^4 \nu^2 x^{-5/3} f''\left(2f - \cancel{4\eta f'}\right) = \cancel{6\alpha^4\nu^2 x^{-5/3}} \ldots$$

$$f''' + 2f'^2 \cancel{+ 4\eta f'' f'} + 2ff'' \cancel{- 4\eta f'' f'} = 0$$

$$\frac{d^3 f}{d\eta^3} + 2\,\frac{d(ff')}{d\eta} = 0$$

$$\frac{d^2 f}{d\eta^2} + 2ff' = C_1$$

Boundary conditions:
- Far field: $\eta \to \infty$ $\quad v_x = 0 \Rightarrow f'(\eta) \to 0$ as $\eta \to \infty$

  this also represents that the wall $x = 0 \Rightarrow \eta \to \infty$ is a streamline

- The axis is a symmetry line: $v_y = \frac{\partial \Psi}{\partial x} = 0 \Rightarrow \underline{f(0) = 0}$

  and $\left.\frac{\partial v_x}{\partial y}\right|_{y=0} = \left.\frac{\partial^2 \Psi}{\partial y^2}\right|_{\eta=0} = 0 \Rightarrow \underline{f''(0) = 0}$

$$\cancel{f''(0)} + 2\,\cancel{f(0)}\,f'(0) = C_1 = 0$$

$$\frac{d^2f}{d\eta^2} + \frac{df^2}{d\eta} = 0 \Rightarrow \quad \frac{df}{d\eta} + f^2(\eta) = C_2$$

at $\eta \to \infty$ $f'(\eta) \to 0 \Rightarrow \quad C_2 = f^2(\eta)\Big|_{\eta\to\infty} > 0$

$$\frac{df}{d\eta} = C_2 - f^2(\eta) \Rightarrow \quad \int \frac{df}{C_2 - f^2} = \int d\eta$$

$$\frac{1}{\sqrt{C_2}} \tanh^{-1}\left(\frac{f(\eta)}{\sqrt{C_2}}\right) = \eta + C_3$$

at $\eta = 0 \Rightarrow f(\eta) = 0 \Rightarrow \quad C_3 = 0$

$$f(\eta) = \sqrt{C_2}\,\tanh\left(\sqrt{C_2}\cdot\eta\right)$$

$$f'(\eta) = C_2\left[1 - \tanh^2\left(\sqrt{C_2}\,\eta\right)\right] \xrightarrow[\eta\to\infty]{} C_2\left[1 - \left(\frac{e^{\infty} - e^{-\infty}}{e^{\infty} + e^{-\infty}}\right)^2\right] = 0$$

$$\Psi(x,y) = 6\alpha\nu\, x^{1/3}\sqrt{C_2}\,\tanh\left(\sqrt{C_2}\,\alpha\,\frac{y}{x^{2/3}}\right)$$

We can assimilate $\sqrt{C_2}$ into α

$$v_x = \frac{\partial\Psi}{\partial y} = 6\alpha\nu\, x^{1/3}\left[1 - \tanh^2\left(\alpha\, y/x^{2/3}\right)\right]\alpha\, x^{-2/3}$$

$$v_y = -\frac{\partial\Psi}{\partial x} = 6\alpha\nu\,\frac{1}{3}\,x^{-2/3}\tanh\left(\alpha\frac{y}{x^{2/3}}\right) + 6\alpha\nu\, x^{1/3}\left[1 - \tanh^2\left(\frac{\alpha y}{x^{2/3}}\right)\right]\Big(\cdots$$

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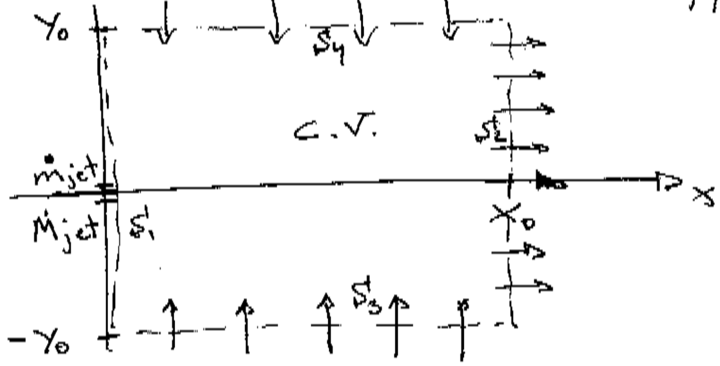
Using a control volume approach:



Conservation of mass: $\cancel{\frac{\partial}{\partial t}}\int_{c.v}\rho\, dV + \int_{c.s}\rho\vec v\cdot\vec n\, dA = 0$

$$\rho\int_{S_1}\vec v\cdot\vec n\, dA + \int_{S_2}\rho\underbrace{\vec v\cdot\vec n}_{V_x=\frac{\partial\Psi}{\partial y}}\underbrace{dA}_{dy} + \int_{S_3}\rho\underbrace{\vec v\cdot\vec n\, dA}_{-V_y=\frac{\partial\Psi}{\partial x}dx} + \int_{S_4}\rho\underbrace{\vec v\cdot\vec n\, dA}_{V_y=-\frac{\partial\Psi}{\partial x}dx} = 0$$

$$\rho\int_{S_1}\vec v\cdot\vec n\, dA + \rho\int_{-Y_0}^{Y_0}\left.\frac{\partial\Psi}{\partial y}\right|_{x=X_0} dy + \int_0^{X_0}\left.\frac{\partial\Psi}{\partial x}\right|_{Y=Y_0} dx - \int_0^{X_0}\left.\frac{\partial\Psi}{\partial x}\right|_{Y=-Y_0} dx = 0$$

$$\rho\int_{S_1}\vec v\cdot\vec n\, dA + 6\alpha\nu X_0^{1/3}\left[\tanh\left(\alpha\frac{Y_0}{X_0^{2/3}}\right) - \tanh\left(\frac{-\alpha Y_0}{X_0^{2/3}}\right)\right] +$$

$$+ 6\alpha\nu X_0^{1/3}\tanh\left(\frac{-\alpha Y_0}{X_0^{2/3}}\right) - 6\alpha\nu\cdot 0\,\underbrace{\cancel{\tanh(-\infty)}}_{(-1)} +$$

$$-\left[6\alpha\nu X_0^{1/3}\tanh\left(\frac{\alpha Y_0}{X_0^{2/3}}\right) - \overset{0}{6\alpha\nu\cdot 0\cdot\underbrace{\cancel{\tanh(\infty)}}_{1}}\right] = 0$$

$$\rho\int_{S_1}\vec v\cdot\vec n\, dA + \overset{0}{\cancel{6\alpha\nu X_0^{1/3}\, 2\tanh\left(\frac{\alpha Y_0}{X_0^{2/3}}\right) - 6\alpha\nu X_0^{1/3}\, 2\tanh\left(\frac{\alpha Y_0}{X_0^{2/3}}\right)}} = 0$$

$$\dot M_{jet} = \int_{S_1}\rho\vec v\cdot\vec n\, dA = 0$$

The solution considers the mass injected by the jet negligible. It considers the jet only as source of momentum.

$$\cancel{\frac{\partial}{\partial t}\int_{c.v.} \rho \vec{v}\, dV}^{0} + \int_{c.s.} \rho \vec{v}(\vec{v}\cdot\vec{n})\, dA = -\int_{c.s.} p\vec{n}\, dA + \int_{c.s.} \bar{\bar{\tau}}\cdot\vec{n}\, dA + \cancel{\int_{c.v.} \rho \vec{g}\, dV} \text{ neglig.}$$

$$\underbrace{\int_{S_1} \rho\vec{v}(\vec{v}\cdot\vec{n})dA}_{-\dot{M}_{jet}} + \int_{S_2} \rho \underbrace{\vec{v}}_{v_x\vec{i}+v_y\vec{j}} (\vec{v}\cdot\vec{n})\Big|_{x=x_0} dA + \underbrace{\cancel{\int_{S_3} \rho\vec{v}(\vec{v}\cdot\vec{n})\Big|_{y=-y_0} dA + \int_{S_4} \rho\vec{v}(\vec{v}\cdot\vec{n})\Big|_{y=y_0} dA}}_{\text{by symmetry}} = \underbrace{-\cancel{\int_{S_1} p\vec{n}\,dA} - \cancel{\int_{S_2} p\vec{n}}}_{\text{no pressure gradient along}}$$

$(\vec{v}\cdot\vec{n}) = v_x$ on S_2, $= v_y$ on S_3, S_4; $\cancel{v_y}$ by symmetry

$$\underbrace{-\cancel{\int_{S_3} p\vec{n}\,dA} - \cancel{\int_{S_4} p\vec{n}\,dA}}_{\text{by symmetry}} + \underbrace{\cancel{\int_{S_1\,(-y_0)}^{y_0} \bar{\bar{\tau}}\cdot\vec{n}\,dA}}_{0} + \underbrace{\cancel{\int_{S_2\,(-y_0)}^{y_0} \bar{\bar{\tau}}\cdot\vec{n}\,dA}}_{0 \text{ by symmetry}} + \underbrace{\cancel{\int_{S_3} \bar{\bar{\tau}}\cdot\vec{n}\,dA\Big|_{-y_0}} + \cancel{\int_{S_4} \bar{\bar{\tau}}\cdot\vec{n}\,dA\Big|_{y_0}}}_{0 \text{ as } \eta\to\infty,\ y_0\to\infty}$$

$$\int_0^{y_0} \bar{\bar{\tau}}\cdot\vec{n}\,dA = -\int_{-y_0}^{0} \bar{\bar{\tau}}\cdot\vec{n}\,dA$$

We get that $\dot{M}_{jet} = \int_{S_2} \rho v_x \vec{i}\, v_x\, dA$; the jet acts as a source of momentum to compensate the outflo[w] at $x = x_0$

$$\dot{M}_{jet} = \int_{-y_0}^{y_0} \rho\, 36\alpha^2 \nu^2 x^{2/3}\left[1-\tanh^2\left(\alpha y/x^{2/3}\right)\right]^2 \frac{\alpha^2}{x^{4/3}}\Big|_{x=x_0} dy$$

if we change variables to $\eta = \alpha\frac{y}{x^{2/3}}$; $d\eta = \frac{\alpha}{x^{2/3}}dy$

$$\dot{M}_{jet} = \int_{-\eta_0}^{\eta_0} \rho\, 36\alpha^{\cancel{4}}\nu^2 \cancel{x^{2/3}}\left[1-\tanh^2(\eta)\right]^2 \frac{\cancel{x^{2/3}}}{\cancel{\alpha}}\frac{d\eta}{\cancel{x^{4/3}}}$$

$$\dot{M}_{jet} = \int_{-\eta_0}^{\eta_0} \rho\, 36\alpha^3\nu^2\left(1-\tanh^2\eta\right)^2 d\eta$$ which is independent of $x \Rightarrow$ momentum is conserved!!

$\tanh\eta = z$
$d\eta(1-\tanh^2\eta) = dz$

$$\dot{M}_{jet} = \int_{-z_0}^{z_0} 36\alpha^3\rho\nu^2(1-z^2)\,dz = 36\alpha\rho\nu^2\left[z-\frac{z^3}{3}\right]_{-z_0}^{z_0}$$

$$\underline{\underline{\dot{M}_{jet}}} = 36\alpha^3\rho\nu^2\underbrace{\left[\tanh\eta - \frac{\tanh^3\eta}{3}\right]_{\eta\to-\infty}^{\eta\to\infty}}_{\{1-(-1)-[\frac{1}{3}-(-\frac{1}{3})]\}} = \underline{\underline{48\alpha^3\rho\nu^2}}$$