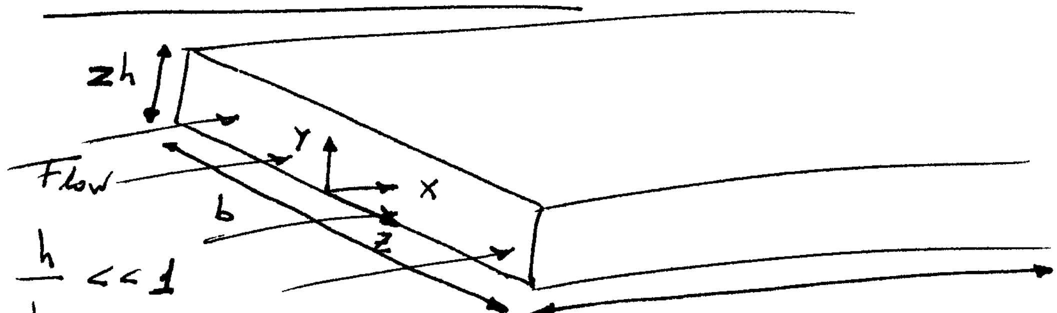
Turbulent Boundary Layer

Canonical Flows: $\begin{cases} \text{Channel Flow} \\ \text{Pipe Flow} \\ \text{Boundary layer} \end{cases}$

$\Updownarrow$

Free Shear Flows: Canonical Cases $\begin{cases} \text{Jet} \\ \text{Mixing Layer} \\ \text{Wake} \end{cases}$

CHANNEL FLOW



$\frac{h}{b} << 1$

The Reynolds number for this flow can be defined in several ways:

$$Re_{max} = \frac{U_0 \cdot 2h}{\nu} \; ; \; Re_{AVE} = \frac{U_{AVE} \cdot h}{\nu}$$

and any combination in between $\frac{U_0 \cdot h}{\nu}$ or $\frac{U_{AVE} \cdot 2h}{\nu}$

The averaged momentum equation in the axial direction is just:

$$0 = \nu \frac{d^2 \langle V_x \rangle}{dy^2} - \frac{d}{dy} \langle v'_x v'_y \rangle - \frac{1}{\rho} \frac{\partial \langle P \rangle}{\partial x}$$

Which can be rewritten as:

$$\frac{d\tau}{dy} = \frac{dP_W}{dx}$$

because $\frac{dP_W}{dx} = \frac{\partial \langle p \rangle}{\partial x}$ and we define

$$\tau = \underbrace{\rho \nu \frac{d\langle V_x \rangle}{dy}}_{\text{viscous stress}} \underbrace{- \rho \langle V_x' V_y' \rangle}_{\text{Reynolds stresses}}$$

Because of symmetry $-\frac{dP_W}{dx} = \frac{\tau_W}{h}$ (Balance of forces in a control volume)

and therefore $\tau(y) = \tau_W \left(1 - \frac{y}{h}\right)$

Law of the Wall

According to the Boundary Layer principle due to the Prandtl (1925), for High Reynolds numbers $R_e \gg 1$, there is a very thin layer where the flow is dominated by viscous effects. In this region $\frac{y}{\delta(h)} \ll 1$ and the flow is independent of h or U_0. This is called the "inner region".

$$\frac{d\langle V_x\rangle}{dy} = \frac{u_\tau}{y}\,\Phi_I\left(\frac{y}{\delta_v}\right)$$

: In the inner region the only length scale is $\delta_v = \frac{\nu}{u_\tau}$ and the only velocity scale is $u_\tau = \sqrt{\frac{h}{\rho}\left(-\frac{dP_w}{dx}\right)}$ based on $\underline{\underline{\tau_w}}$

We saw previously that the general solution to this problem can be written, based on dimensional analysis, as

$$\frac{d\langle V_x\rangle}{dy} = \frac{u_\tau}{y}\,\Phi\left(\frac{y}{\delta_v}, \frac{y}{h}\right)$$

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In the "inner layer" $\frac{y}{h} \to 0$ and

$$\Phi_I\left(\frac{y}{\delta_v}\right) = \lim_{\frac{y}{h}\to 0} \Phi\left(\frac{y}{\delta_v}, \frac{y}{h}\right)$$

Defining u^+ as $\frac{\langle V_x \rangle}{u_\tau}$ we get

$\frac{du^+}{dy^+} = \frac{1}{y^+}\Phi_I(y^+)$ and integrating we get

$\boxed{u^+ = f(y^+)}$ Law of the Wall.

There is an "outer region" where $y^+ = \frac{y}{\delta_v} \gg 1$

and therefore $\frac{d\langle V_x\rangle}{dy} = \frac{u_\tau}{y}\Phi\left(\frac{y}{\delta_v}\to\infty, \frac{y}{h}\right)$

so $\frac{d\langle V_x\rangle}{dy} = \frac{u_\tau}{y}\Phi_0\left(\frac{y}{h}\right)$

We can integrate this to define Velocity Defect Law:

$$\int_y^h \frac{d\langle V_x\rangle}{dy} = \langle V_x(h)\rangle - \langle V_x(y)\rangle = \int_y^h \frac{u_\tau}{y}\Phi_0 d$$

$$\boxed{\frac{\langle V_x(h)\rangle - \langle V_x(y)\rangle}{u_\tau} = F(y/h)}$$ Velocity Defect Law

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In every singular perturbation problem, there is an overlap region where the solutions obtained asymptotically for the inner and outer regions must match:

$$\lim_{y/\delta_v \to \infty} \Phi_I(y/\delta_v) = \lim_{y/h \to 0} \Phi_o(y/h)$$

$$\frac{y}{u_\tau} \frac{d\langle V_x \rangle}{dy} = \Phi_I(y/\delta_v) = \Phi_o(y/h)$$

for a range $\delta_v \ll y \ll h$

This equation can only be satisfied if both

$$\left.\begin{array}{l} \Phi_I(y/\delta_v) \xrightarrow[y/\delta_v \to \infty]{} \text{constant} \\ \Phi_o(y/h) \xrightarrow[y/h \to 0]{} \text{constant} \end{array}\right\} \quad \frac{y}{u_\tau} \frac{d\langle V_x \rangle}{dy} = \frac{1}{\kappa}$$

Millikan (1939)

$$\int d\left(\frac{\langle V_x \rangle}{u_\tau}\right) = \frac{1}{\kappa} \int \frac{dy}{y} \Rightarrow u^+ = \frac{1}{\kappa} \ln y^+ + \underset{A}{\text{constant}}$$

$$\Downarrow$$

$A \simeq 5.2$

$\kappa \simeq 0.41$

$$\frac{V_0 - \langle V_x \rangle}{u_\tau} = -\frac{1}{\kappa} \ln\left(\frac{y}{h}\right) + B_1$$