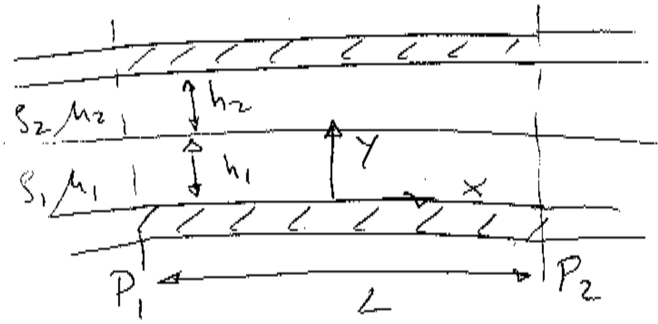
(35) TWO FLUID EXAMPLE



- Steady
- Fully developed
- Incompressible
- No z-dependency

Continuity (Conservation of mass)

$$\cancel{\frac{\partial \rho}{\partial t}} + \nabla \cdot (\cancel{\rho} \vec{v}) = 0$$

incompressible

Within fluid 1

$$\cancel{\frac{\partial v_x^1}{\partial x}} + \frac{\partial v_y^1}{\partial y} + \cancel{\frac{\partial v_z^1}{\partial z}} = 0$$

$$\frac{\partial v_y^1}{\partial y} = 0 \Rightarrow v_y^1\Big|_{y=0} = 0 \Rightarrow v_y^1 = 0 \text{ everywh}$$

Within fluid 2

$$\cancel{\frac{\partial v_x^2}{\partial x}} + \frac{\partial v_y^2}{\partial y} + \cancel{\frac{\partial v_z^2}{\partial z}} = 0$$

$$\frac{\partial v_y^2}{\partial y} = 0 \Rightarrow v_y^2\Big|_{y=h_1+h_2} = 0 \Rightarrow v_y^2 = 0 \text{ everywher}$$

Conservation of momentum

$$\rho\left[\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v}\right] = -\nabla p + \mu \nabla^2 \vec{v} + \rho$$

x-axis: $\rho\left(\underbrace{\cancel{\frac{\partial v_x}{\partial t}}}_{\text{steady}} + v_x \underbrace{\cancel{\frac{\partial v_x}{\partial x}}}_{\text{Fully developed}} + \underbrace{\cancel{v_y}}_{0} \frac{\partial v_x}{\partial y} + v_z \underbrace{\cancel{\frac{\partial v_x}{\partial z}}}_{\text{no z dependency}}\right) = -\frac{\partial p}{\partial x} + \mu\left(\underbrace{\cancel{\frac{\partial^2 v_x}{\partial x^2}}}_{\text{fully developed}} + \frac{\partial^2 v_x}{\partial y^2} + \underbrace{\cancel{\frac{\partial^2 v_x}{\partial z^2}}}_{\text{no z depende}}\right)$

For fluid 1: $\quad 0 = -\frac{\partial P}{\partial x} + \mu_1 \frac{\partial^2 v_x^1}{\partial y^2} \Rightarrow \frac{d^2 v_x^1}{dy^2} = \frac{P_2 - P_1}{\mu_1 L} \Rightarrow$

For fluid 2: $\quad 0 = -\frac{\partial P}{\partial x} + \mu_2 \frac{\partial^2 v_x^2}{\partial y^2} \Rightarrow \frac{d^2 v_x^2}{dy^2} = \frac{P_2 - P_1}{\mu_2 L} \Rightarrow$

(36)

$$\frac{dV_x^1}{dy} = -\frac{P_1-P_2}{\mu_1 L}\, y + C_1 \Rightarrow V_x^1(y) = -\frac{P_1-P_2}{\mu_1 L}\, y^2/2 + C_1 y + C_2$$

$$\frac{dV_x^2}{dy} = -\frac{P_1-P_2}{\mu_2 L}\, y + C_3 \Rightarrow V_x^2(y) = -\frac{P_1-P_2}{\mu_2 L}\, y^2/2 + C_3 y + C_4$$

at $y = 0 \Rightarrow V_x^1 = 0 \Rightarrow C_2 = 0$

at $y = h_1 + h_2 \Rightarrow V_x^2(h_1+h_2) = 0 \Rightarrow C_4 = \frac{P_1-P_2}{\mu_2 L}\frac{(h_1+h_2)^2}{2} - C_3(h_1+h_2)$

2 extra boundary conditions: $V_x^1(y=h_1) = V_x^2(y=h_1)$ and

$$\mu_1 \frac{dV_x^1}{dy}(y=h_1) = \mu_2 \frac{dV_x^2}{dy}(y=h_1)$$

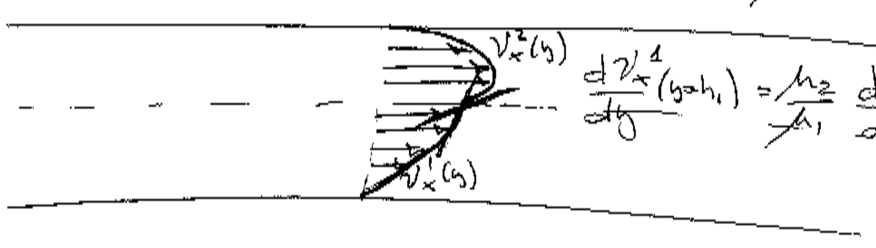
$$V_x^1(h_1) = -\frac{P_1-P_2}{\mu_1 L}\frac{h_1^2}{2} + \cancel{C_1 h_1} = -\frac{P_1-P_2}{\mu_2 L}\frac{h_1^2}{2} + \cancel{C_3 h_1} + C_4$$

$$\cancel{-\mu_1 \frac{P_1-P_2}{\mu_1 L} h_1} + C_1 = \cancel{-\mu_2 \frac{P_1-P_2}{\mu_2 L} h_1} + C_3$$

$$C_1 = C_3$$

$$C_4 = \frac{P_1-P_2}{\mu_2 L}\frac{(h_1+h_2)^2}{2} - C_3(h_1+h_2) = \underline{\underline{\frac{P_1-P_2}{L}\frac{h_1^2}{2}\left(\frac{1}{\mu_2} - \frac{1}{\mu_1}\right)}}$$

$$C_3 = C_1 = \frac{P_1-P_2}{2L}\left(\frac{h_1+h_2}{\mu_2} + \frac{h_1^2}{\mu_1(h_1+h_2)} - \frac{h_1^2}{\mu_2(h_1+h_2)}\right)$$



$$\frac{dV_x^1}{dy}(y=h_1) = \frac{\mu_2}{\mu_1}\frac{dV_x^2}{dy}(y=h_1)$$