Waveguide Theory

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Week 12

- Course Website:
  http://courses.washington.edu/me557/sensors
- Reading Materials:

- Week 12 reading materials can be found:

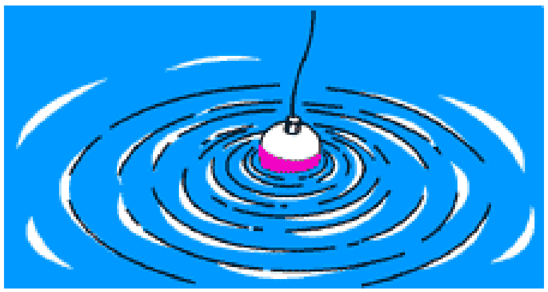
http://courses.washington.edu/me557/reading/

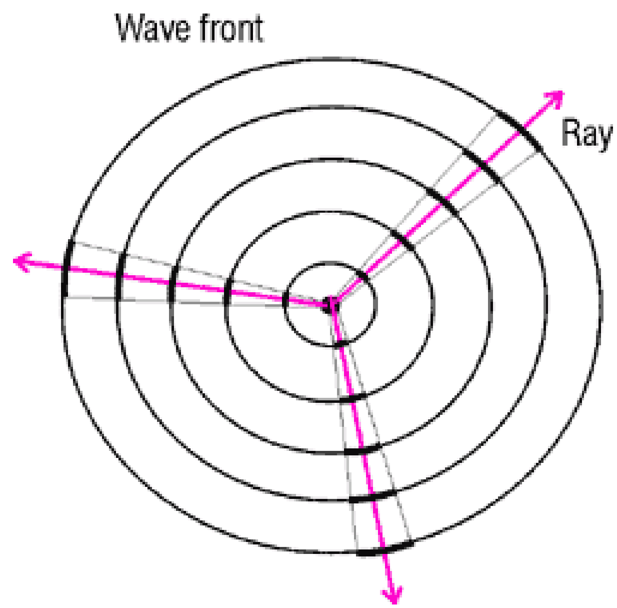
- Proposal meeting this Wednesday (1-6PM, Delta 319)
- Work on Lab 2 (arrange time to meet with TA)
- HW 3 due today (If you need more time, let me know)
- Proposal due Next week (please follow the instruction on our website)
- Final presentation is on 12/23,, final report due 1/7/20

This Week

- Waveguide structures and materials
- Waveguide modes
- Field equations
- Waveguide modes, $n_{eff,}$ dispersion equation
- Guided modes in symmetric and asymmetric slab waveguides
- General formalisms for step-index planar waveguides

Light rays and light waves

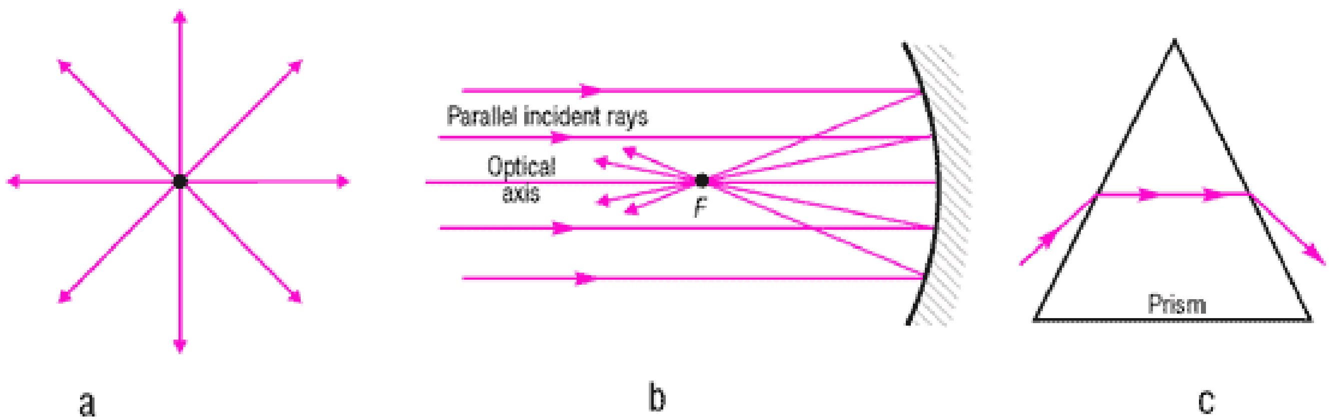




Wave from the bubble

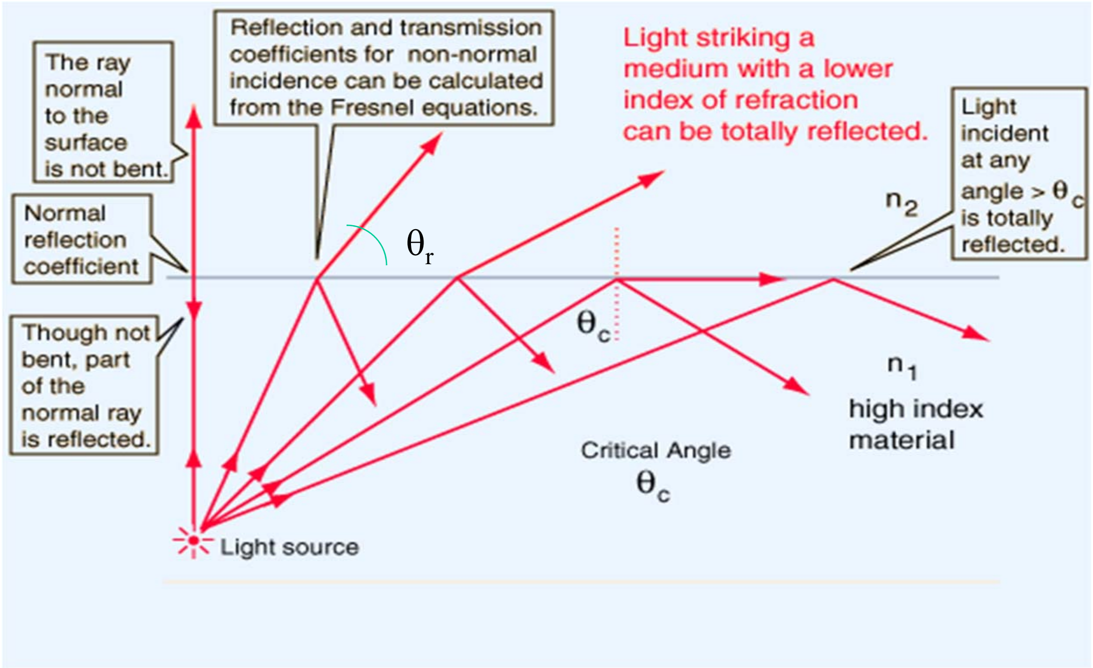
Light rays and wavefronts

Geometric construct of a light ray we can illustrate propagation, reflection, and refraction of light



Typical light rays in (a) *propagation,* (b) *reflection, and* (c) *refraction*

Physical Mechanic: Total Internal Reflection



Hyperphysics

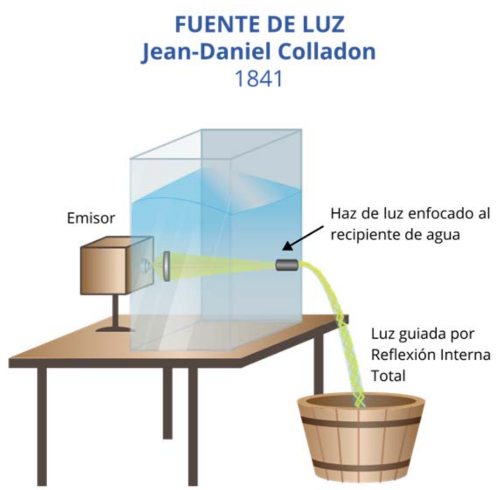
Refraction of light at a dielectric interface is governed by Snell's law: $n_1 \sin\theta_i = n_2 \sin\theta_r$.

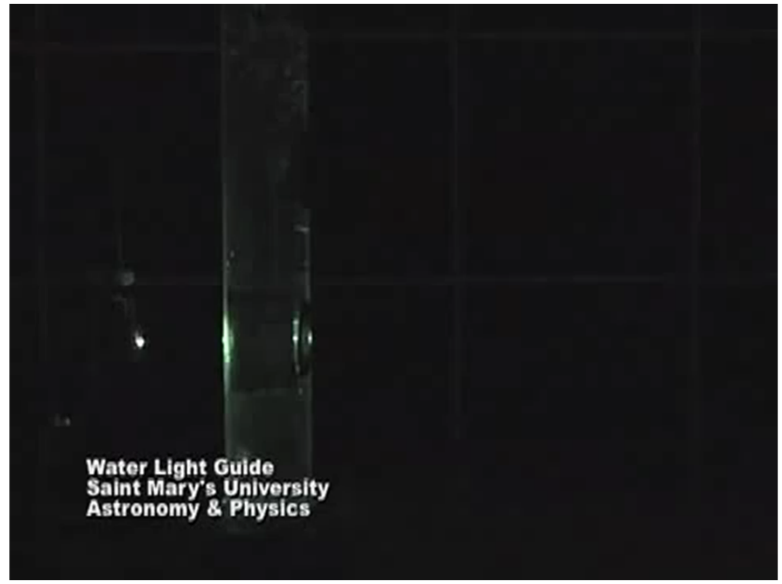
When $n_1 > n_2$, light bends away from the normal ($\theta_r > \theta_i$).

At a critical angle $\theta_i = \theta_c$, θ_r becomes 90^o (parallel to interface).

Total internal reflection occurs when $\theta_i > \theta_c$.

History of Total Internal Reflection





- First demonstration of light remained confined to a falling stream of water (TIR) in 1841 by Daniel Colladon in Geneva.
- Demonstrated internal reflection to follow a specific path to John Tyndall (1870 experiment in London).

Geometrical-Optics Explanation

- Ray picture valid only within geometrical-optics approximation.
- Useful for a physical understanding of waveguiding mechanism.
- It can be used to show that light remains conned to a waveguide for only a few specific incident angles if one takes into account the Goos-Hanchen shift (extra phase shift at the interface).
- **The angles corresponds to waveguide modes in wave optics.**
- For thin waveguides, only a single mode exists.
- One must resort to wave-optics description for thin waveguides (thickness d ~ λ).

Waveguide Structure

- Metallic waveguide (hallow metal waveguide, coaxial cable, micro strip)
- Dielectric waveguide (optical fiber, integrated waveguide)

Differences Between Metallic and Dielectric Waveguides

Losses are from real and imaginary parts of epsilon- real part from
Snell's law and imaginary part from conductivity (not so for dielectric).

- At millimeter wave frequencies and above, **metal is not a good conductor, so metal waveguides can have increasing attenuation**. At these wavelengths dielectric waveguides can have lower losses than metal waveguides. Optical fiber is a form of dielectric waveguide used at optical wavelengths.
- One difference between dielectric and metal waveguides is that at **a metal surface the electromagnetic waves are tightly confined**; **at high frequencies the electric and magnetic fields penetrate a very short distance into the metal (smaller the skin depth).** In contrast, **the surface of the dielectric waveguide is an interface between two dielectrics, so the fields of the wave penetrate outside the dielectric in the form of an evanescent (non-propagating) wave**.

Skin effect in conductor

We can derive a practical formula for skin depth :

$$\delta = \sqrt{\frac{2\rho}{(2\pi f)(\mu_o \mu_r)}}$$

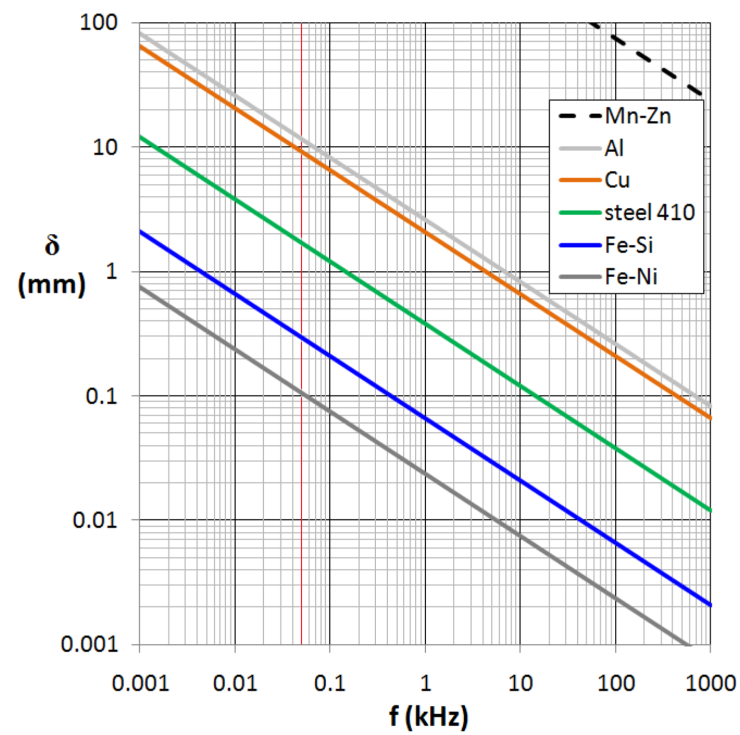
Where

δ = the skin depth in meters

μ_r = the relative permeability of the medium

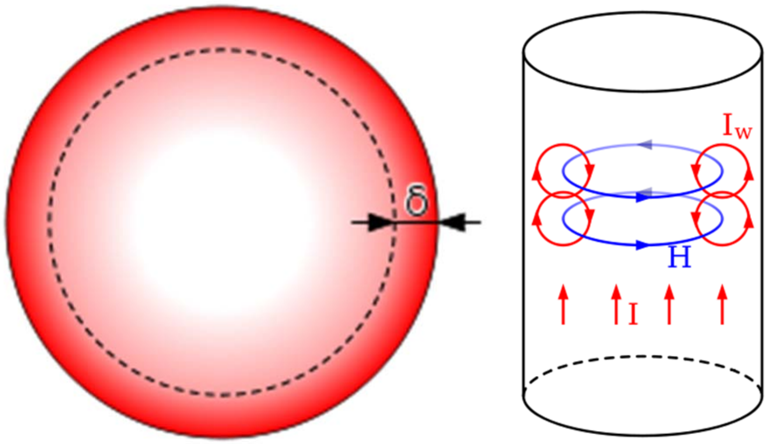
ρ = the resistivity of the medium in Ω·m, also equal to the reciprocal of its conductivity: $\rho = 1/\sigma$ (for copper, $\rho = 1.68\times10^{-8}$ Ω·m)

f = the frequency of the current in Hz



Skin effect

Skin effect is the tendency of an alternating electric current (AC) to become distributed within a conductor such that the current density is largest near the surface of the conductor, and decreases with greater depths in the conductor. The electric current flows mainly at the "skin" of the conductor, between the outer surface and a level called the skin depth. The skin effect causes the effective resistance of the conductor to increase at higher frequencies where the skin depth is smaller, thus reducing the effective cross-section of the conductor. The skin effect is due to opposing eddy currents induced by the changing magnetic field resulting from the alternating current. At 60 Hz in copper, the skin depth is about 8.5 mm. At high frequencies the skin depth becomes much smaller. Increased AC resistance due to the skin effect can be mitigated by using specially woven litz wire. Because the interior of a large conductor carries so little of the current, tubular conductors such as pipe can be used to save weight and cost.



Distribution of current flow in a cylindrical conductor, shown in cross section. For alternatin current, most (63%) of the electric current flows between the surface and the skin depth, δ, which depends on the frequency of the current and the electrical and magnetic properties of the conductor

Penetration Depth (dielectric and slight conductive)

According to Beer-Lambert lawType equation here., the intensity of an **electromagnetic wave inside a material falls off exponentially from the surface** as

$$I(z) = I_o e^{-\alpha z}$$

If δ_p denotes the penetration we have $\delta_p = 1/\alpha$ "Penetration depth" is one term that describes the decay of electromagnetic waves inside of a material. The above definition refers to the depth δ_p at which the intensity or power of the field decays to 1/e of its surface value. In many contexts one is concentrating on the field quantities themselves: the electric and magnetic fields in the case of electromagnetic waves. Since the power of a wave in a particular medium is proportional to the square of a field quantity, one may speak of a penetration depth at which the magnitude of the electric (or magnetic) field has decayed to 1/e of its surface value, and at which point the power of the wave has thereby decreased to 1/e or about 13% of its surface value:

$$\delta_e = \frac{1}{\alpha/2} = \frac{2}{\alpha} = 2\delta_p$$ (slightly conductive)

$$\delta_p = 1/\alpha = \sqrt{\left(\frac{2}{\omega\mu\sigma}\right)} = \delta$$ (highly conductive)

Note that δ is identical to the skin depth, the latter term usually applying to metals in reference to the decay of electrical currents or we only use penetration depth to descrie the media

Highly Conducting Media

For highly conducting medium, σ/ωε >>1, the k constant can be simplify to

$$k \sim \omega\sqrt{\mu\varepsilon}\left(-j\,\frac{\sigma}{\omega\varepsilon}\right)^{1/2} = \sqrt{\omega\mu\left(\frac{\sigma}{2}\right)}(1-j)$$

The penetration depth $\delta_p = 1/\alpha = \sqrt{\left(\frac{2}{\omega\mu\sigma}\right)} = \delta$ (skin depth) only for highly conductive media.

higher the frequency higher the attenuation

$$\alpha = \omega\mu\sigma/2$$

For Slightly Conducting Media

For slightly conducting media, where σ/ωε <<1, the constant k can be approximated by

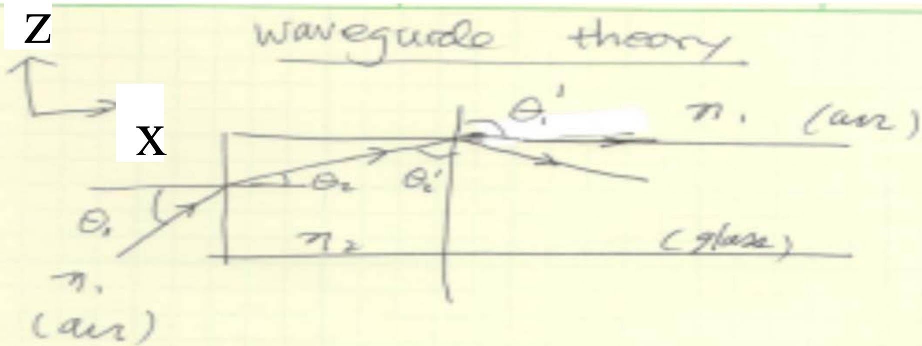
$$k = k - ja = \omega\sqrt{\mu\varepsilon}(1\text{-}j\,\frac{\sigma}{\omega\varepsilon})^{1/2} \approx \omega\sqrt{\mu\varepsilon}(1\text{-}j\,\frac{\sigma}{2\omega\varepsilon})$$

Penetration depth $\delta_p = 1/\alpha = \frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}}$ (here we don't have skin depth, skin depth only refers to metal)

Independent of wavelength

$$\alpha = \frac{2}{\sigma}\sqrt{\frac{\varepsilon}{\mu}}$$

waveguide theory



$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

$$n_2 \sin\theta_2' = n_1 \sin\theta_1'$$

$\theta_1' = 90°$ ∴ total reflection occurs

then $\theta_2' = \theta_{critical}$

no wave travel outside the glass substrate, therefore glass substrate becomes a wave guide.

However if $\theta_2' > \theta_{critical}$

Then $n_2 \sin\theta_2' = n_1 \underbrace{\sin\theta_1'}_{>1}$

$$1 = \sin^2\theta_1' + \cos^2\theta_1'$$

$\sin^2\theta_1'$: >1 $\quad$ $\cos^2\theta_1'$: <0

since $k_1^2 = k_{1z}^2 + k_{1x}^2$

$$k_{1z}^2 = k_1^2 - k_{1x}^2 = k_1^2 - (k_1 \sin\theta_1')^2 \Rightarrow k_{1z}^2 < 0$$

$$k_1^2 = k_1^2\sin^2\theta_1' + k_1^2\cos^2\theta_1'$$

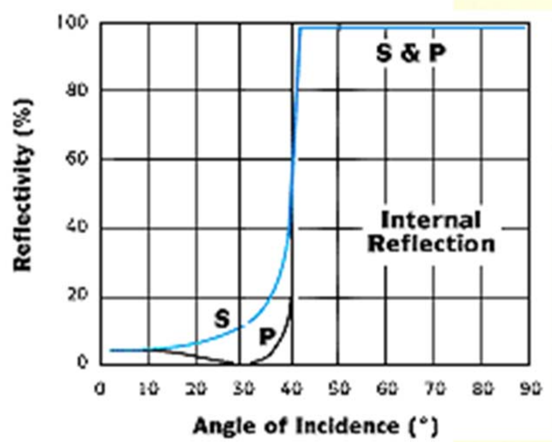
k_{1z} = imaginary

$$Ee^{jk_{1x}x + jk_{1z}z}$$

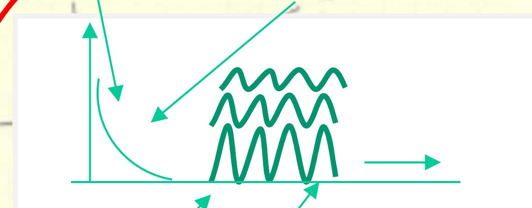
$$\mathrm{Re}\left\{ \left(Ee^{jk_{1x}x}\right) e^{j\overset{\text{imaginary}}{k_{1z}}z} \right\}$$

$$= E_0 \cos k_{1x}x \; e^{-j\alpha_z z}$$

where α_z = imaginary k_{1z} = imaginary



$e^{-j\alpha_z z}$



$\cos k_{1x}x$

Evanescent wave

use for sensing and wave coupling

$\cos k_{1x}x$ at different z

Typical Metal and Dielectric Waveguides

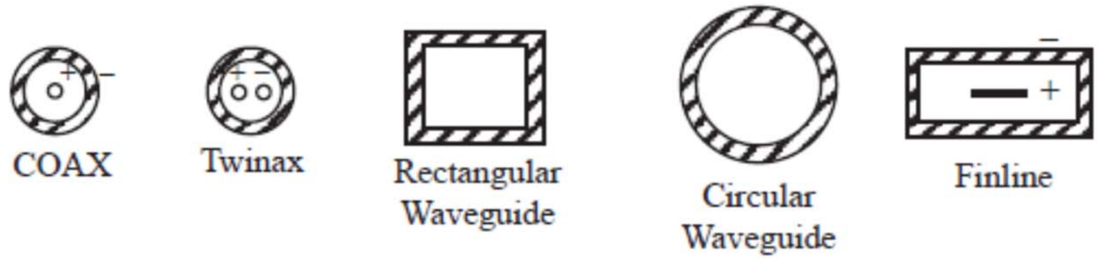


Figure 1.1: Examples of closed waveguides.

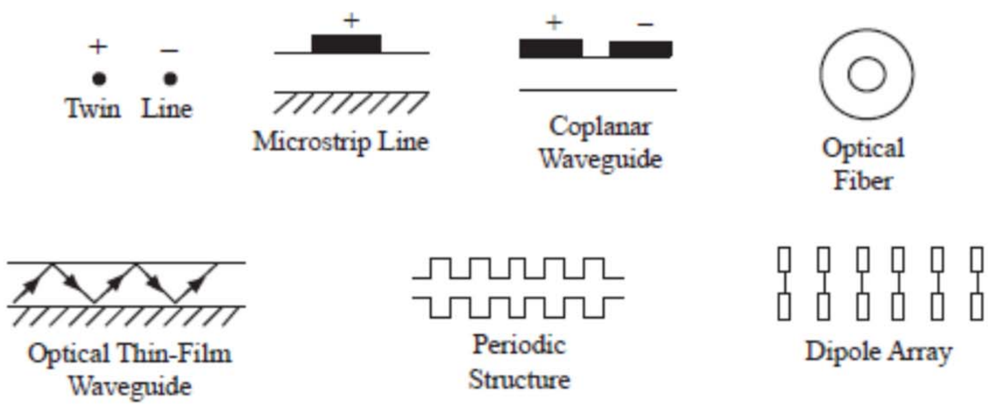
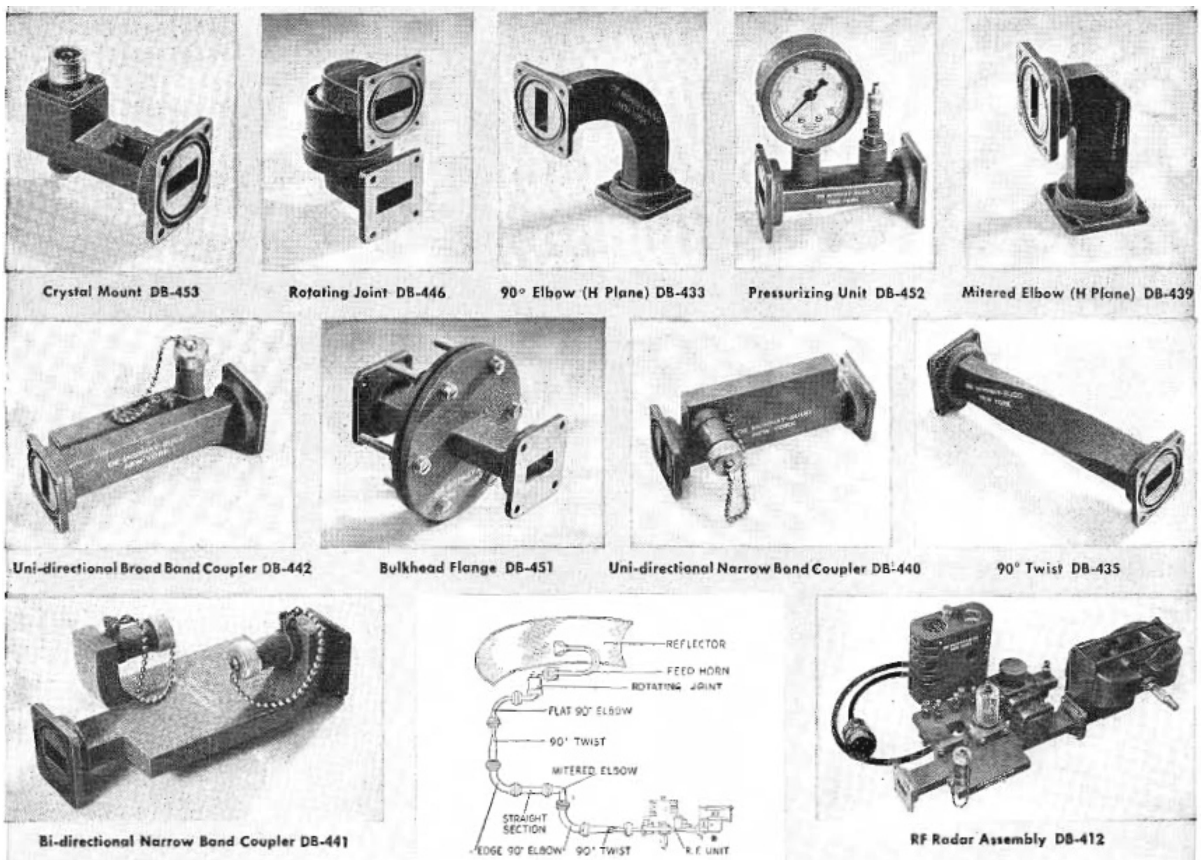
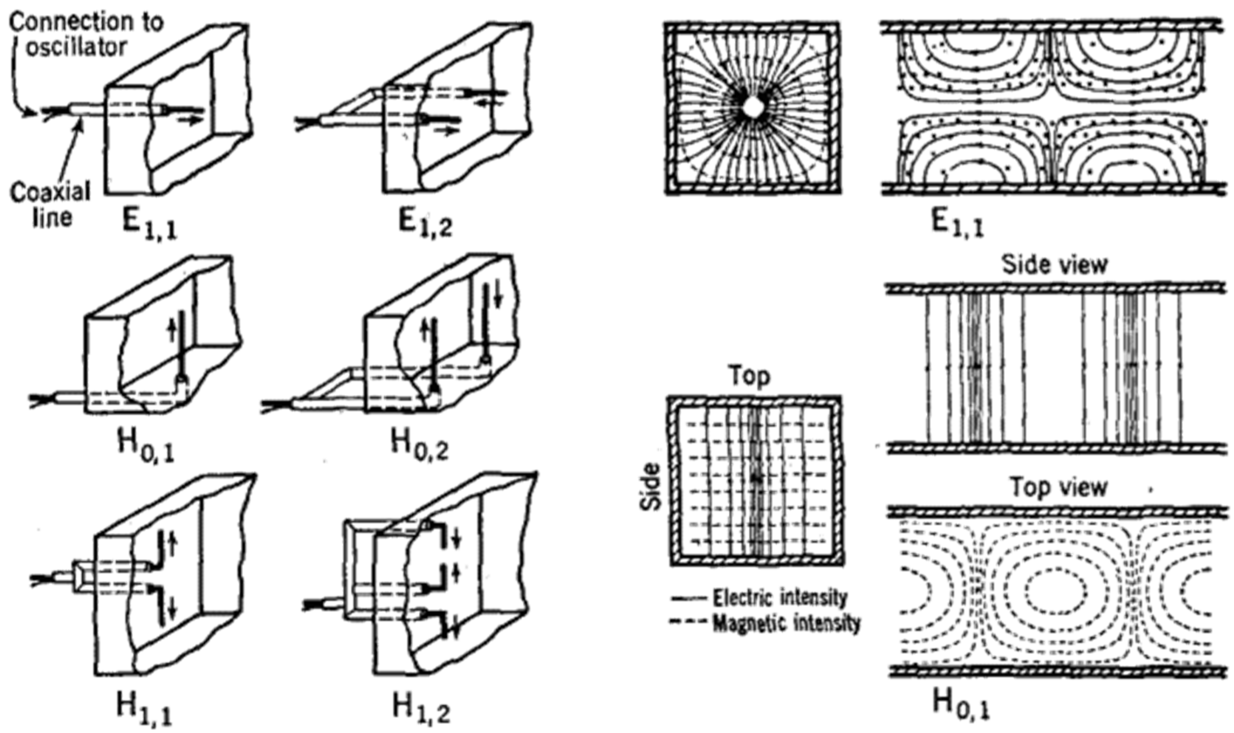


Figure 1.2: Examples of open waveguides.

RF metallic waveguide

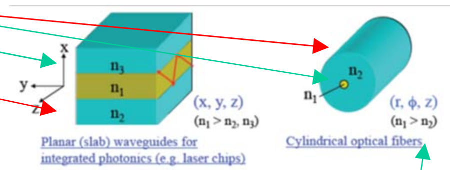


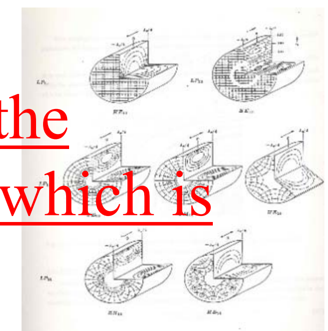
Field confinement



Dielectric Waveguide

• The basic structure of a dielectric waveguide consists of a longitudinally extended high-index optical medium, called the *core*, which is transversely surrounded by low-index media, called the *cladding*. A guided optical wave propagates in the waveguide along its *longitudinal* direction.





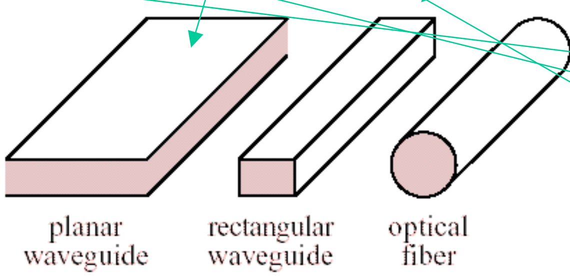
• The characteristics of a waveguide are determined by the **transverse profile of its dielectric constant $\varepsilon(x, y)/\varepsilon_o$**, which is ***independent of the longitudinal* (z) direction**.

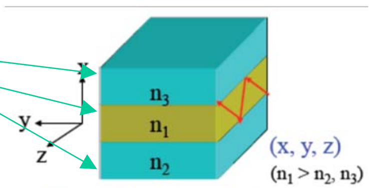
• For a waveguide made of **optically isotropic media**, we can characterize the waveguide **with a single spatially dependent transverse profile of the index of refraction n(x, y).**

Dielectric Waveguide

There are two basic types of waveguides:

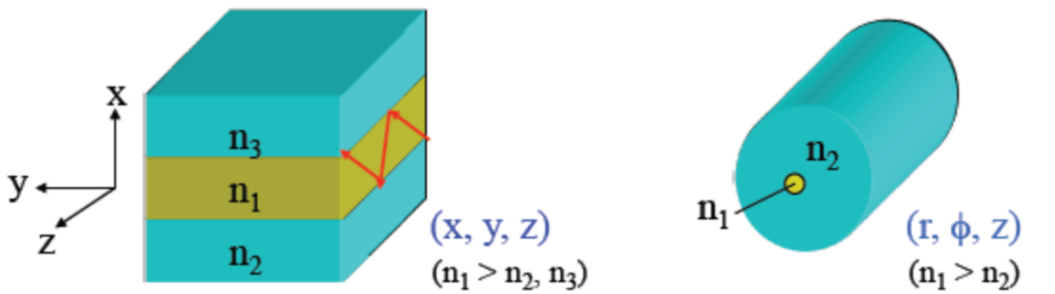
- In a ***nonplanar waveguide*** of two-dimensional transverse optical confinement, the core is surrounded by cladding in *all transverse directions*, and n(x, y) **is a function of both x and y coordinates** (e.g. *channel waveguides* and the *optical fiber)*
- In a ***planar waveguide*** that has **optical confinement in *only one transverse direction***, the core is sandwiched between cladding layers in only one direction, say the x direction, with an index profile n(x). The core of a planar waveguide is also called the *film*, while the upper and lower cladding layers are called the *cover* and the *substrate*.





Planar (slab) waveguides for integrated photonics (e.g. laser chips)

Dielectric Waveguide

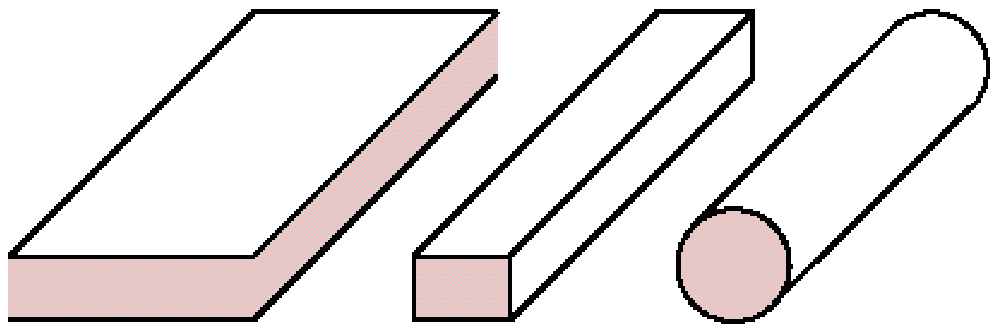


Planar (slab) waveguides for integrated photonics (e.g. laser chips)

Cylindrical optical fibers

Optical waveguides are the *basic elements for confinement and transmission of light* over various distances, **ranging from tens or hundreds of μm in integrated photonics to hundreds or thousands of km in long-distance fiber-optic transmission**. Optical waveguides *also form key structures in semiconductor lasers, and act as* ***passive and active devices*** such as waveguide couplers and modulators.

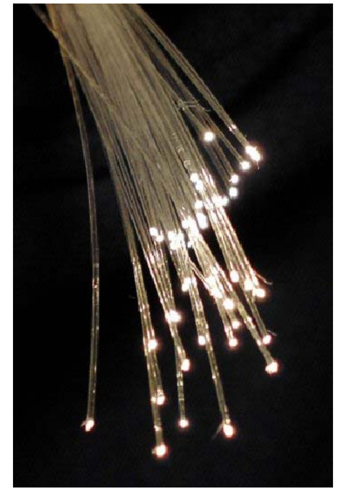
Light can be guided by planar or rectangular wave guides, or by optical fibers.

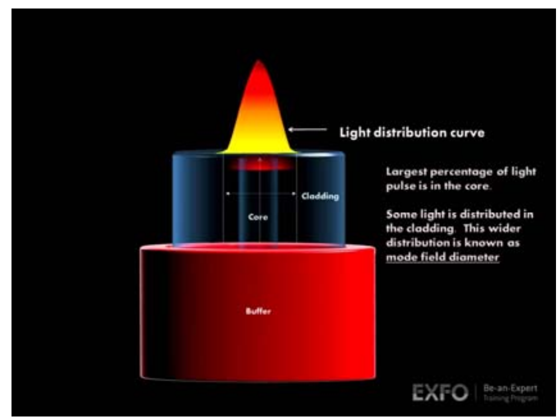


Optical waveguides are used as components in sensors, integrated optical circuits or as the transmission medium in local and long haul optical communication systems.

Optical Fiber

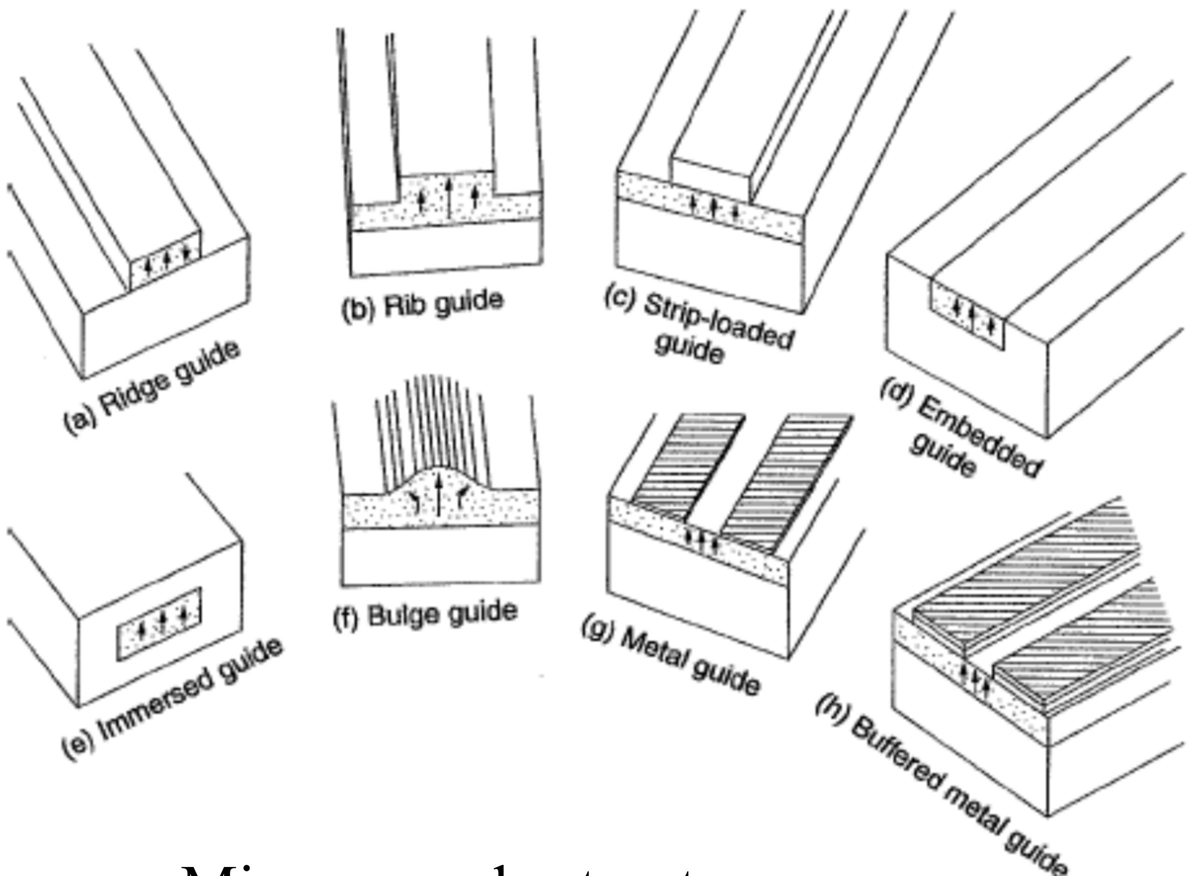
Optical fiber is typically a circular cross-section dielectric waveguide consisting of a dielectric material surrounded by another dielectric material with a lower refractive index. Optical fibers are most commonly made from silica glass, however other glass materials are used for certain applications and plastic optical fiber can be used for short-distance applications.





D fiber and Rectangular fiber

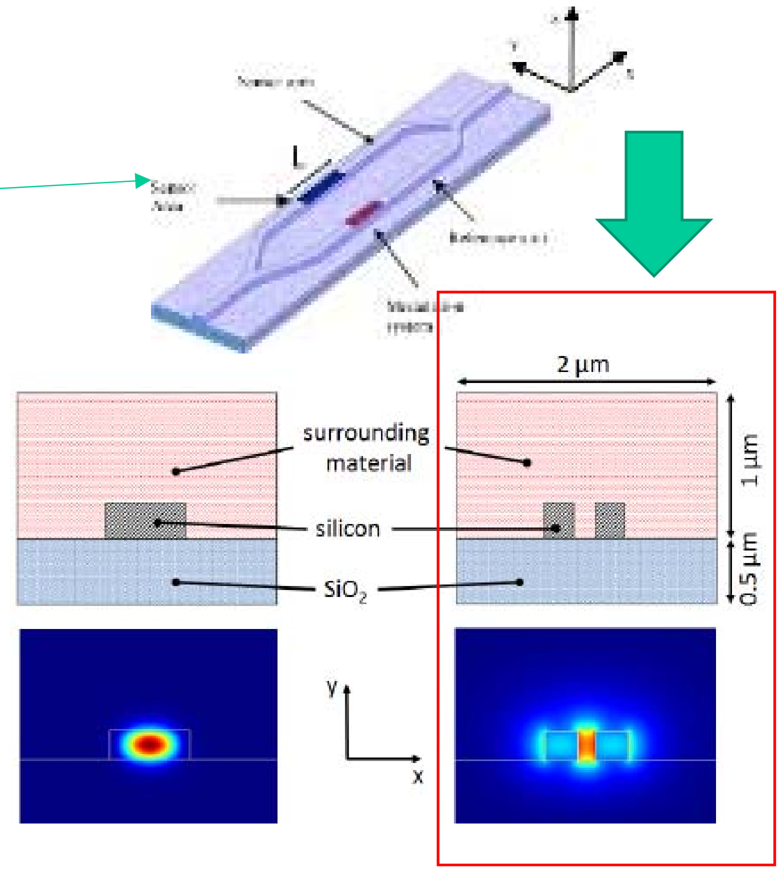
Rectangular Waveguide



Microns scale structure

Strip Waveguide

- Rectangular waveguide
- Use in integrated optical circuits and in laser diodes
- Mach-Zehnder interferometers and wavelength division multiplexers
- Produced by a variety of means, usually by a planar process
- The field distribution in a rectangular waveguide cannot be solved analytically, however approximate solution methods, such as Marcatili's method,[3] Extended Marcatili's method and Kumar's method, are known.
- Field distribution can be solved numerical using various FEM, FTDT or BPM method.

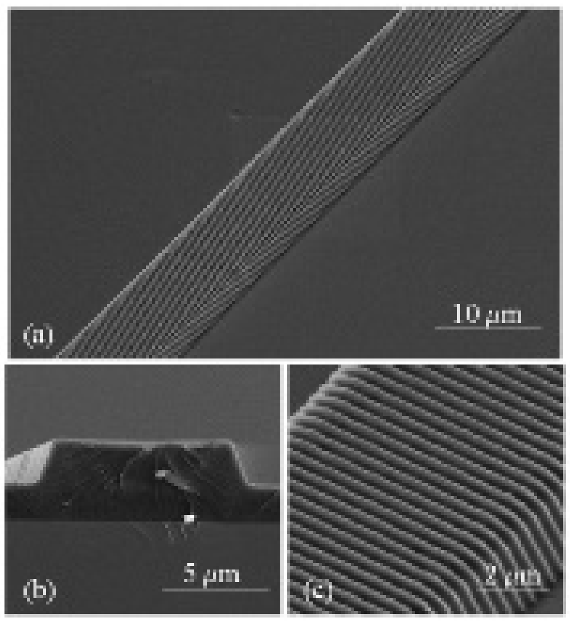


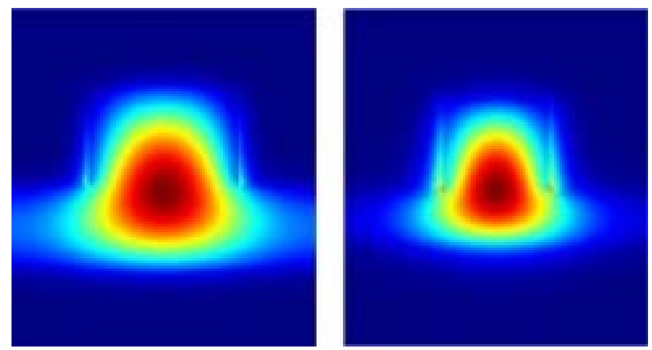
SOI-strip-waveguide and SOI-slot-waveguide

PHOTOPTICS, At Berlin, Volume: 3

Rib Waveguide

- Consists of a the slab with a strip (or several strips) superimposed onto it.

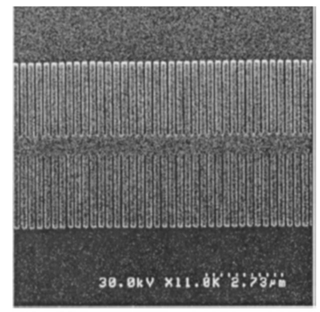




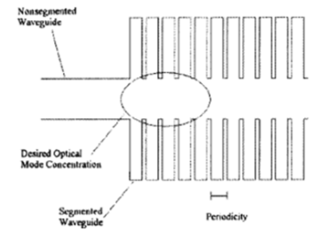
W . Wang, Journal of the Optical Society of America B, Vol. 26, Issue 6, pp. 1256-1262 (2009)

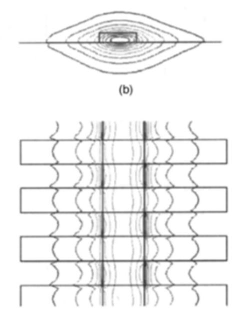
Segmented waveguides and photonic crystal waveguides

- have periodic changes in their cross-section while still allowing lossless transmission of light via so-called Bloch modes.
- segmented waveguides (with a 1D patterning along the direction of propagation) or as photonic crystal waveguides (with a 2D or 3D patterning).
- silicon-on-insulator waveguide designs for simultaneously achieving both low-loss optical confinement and electrical contacts



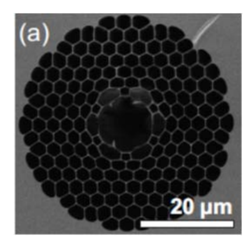
Segmented waveguide





Modal patterns of the Bloch mode

L. Gunn, Vol. 22, No. 7/July 2005/J. Opt. Soc. Am. B



hollow-core photonic crystal fiber

Vol. 16, No. 22 / OPTICS EXPRESS 17972

Waveguide Materials

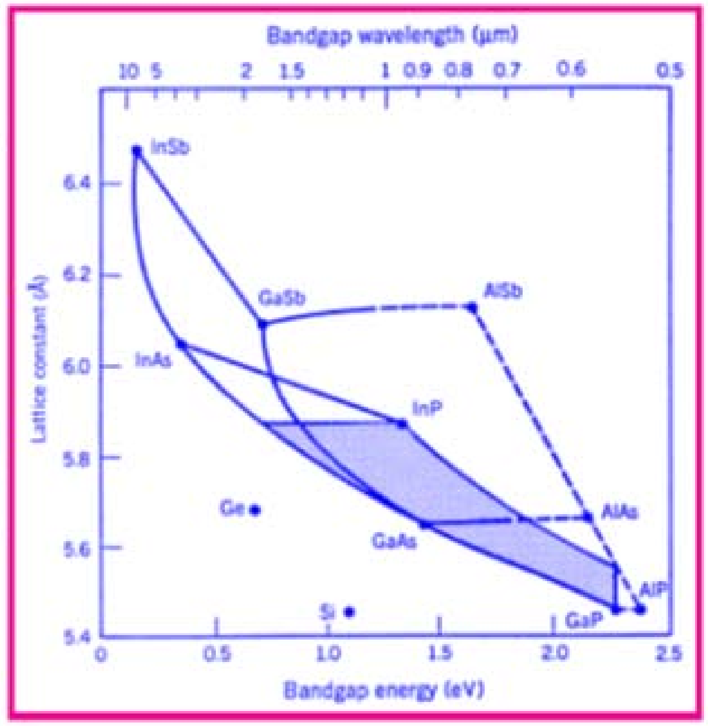
- Semiconductor waveguides: GasAs, InP, et.
- Electro-optic Waveguides: $LiNbO_3$, EO polymer
- Glass Waveguides: Silica (SiO_2), SiON
  - silica-on silicon technology
  - Laser-written waveguides
- Silicon-on-Insulator (SOI) Technology
- Polymer waveguides: Su8, PMMA, PDMS, PU, etc.

My research

Semiconductor Waveguides

Useful for semiconductor lasers, modulators, and photodetectors.

- Semiconductors allow fabrication of electrically active devices.
- Semiconductors belonging to III–V Group often used.
- Two semiconductors with different refractive indices needed.
- They must have different bandgaps but same lattice constant.
- Nature does not provide such semiconductors.



Ternary and Quaternary Compounds

- A fraction of the lattice sites in a binary semiconductor (GaAs, InP, etc.) is replaced by other elements.
- Ternary compound $Al_xGa_{1-x}As$ is made by replacing a fraction x of Ga atoms by Al atoms.
- Bandgap varies with x as

$$E_g(x) = 1.424 + 1.247x \qquad (0 < x < 0.45).$$

- Quaternary compound $In_{1-x}Ga_xAs_yP_{1-y}$ useful in the wavelength range 1.1 to 1.6 μm.
- For matching lattice constant to InP substrate, $x/y = 0.45$.
- Bandgap varies with y as $E_g(y) = 1.35 - 0.72y + 0.12y^2$.

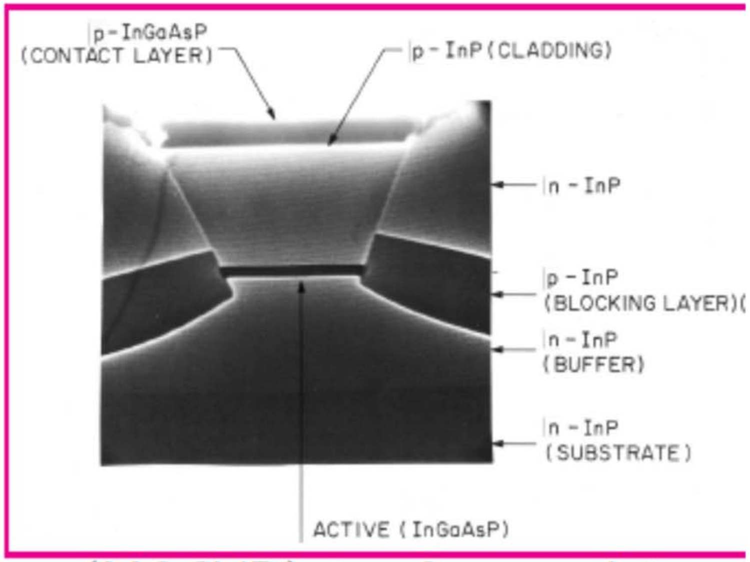
Fabrication Techniques

Epitaxial growth of multiple layers on a base substrate (GaAs or InP).

Three primary techniques:

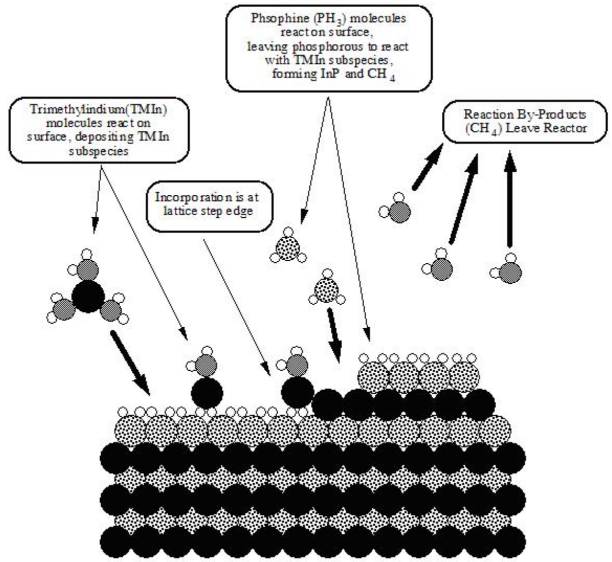
- Liquid-phase epitaxy (LPE)
- Vapor-phase epitaxy (VPE)
- Molecular-beam epitaxy (MBE)

VPE is also called chemical-vapor deposition (CVD).



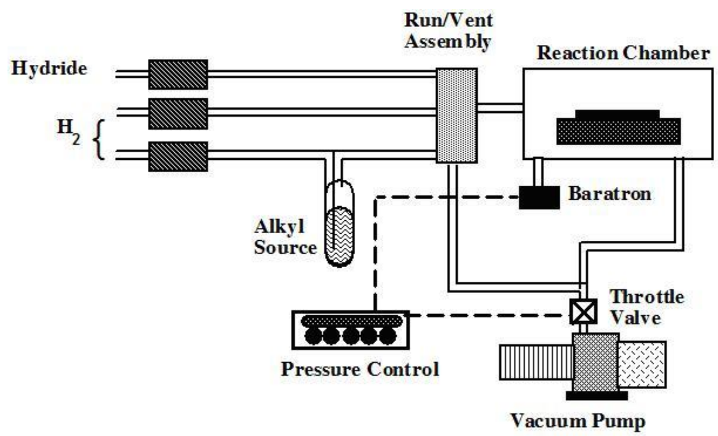
Metal-organic chemical-vapor deposition (MOCVD) is often used in practice.

Metalorganic vapor phase epitaxy



Metalorganic vapor phase epitaxy (MOVPE), also known as organometallic vapor phase epitaxy (OMVPE) or metalorganic chemical vapor deposition (MOCVD), is a chemical vapor deposition method used to **produce single or polycrystalline thin films**. It is a highly complex process for growing crystalline layers to create complex semiconductor multilayer structures**. In contrast to molecular beam epitaxy (MBE) the growth of crystals is by chemical reaction and not physical deposition. This takes place not in a vacuum, but from the gas phase at moderate pressures (10 to 760 Torr).** As such, this technique is preferred for **the formation of devices incorporating thermodynamically metastable alloys,** and it has become a major process in the manufacture of optoelectronics

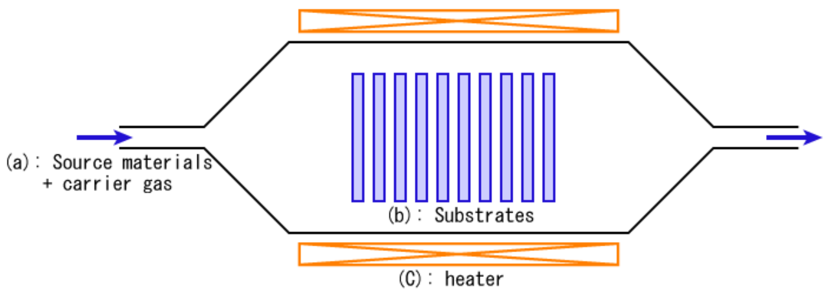
MOCVD



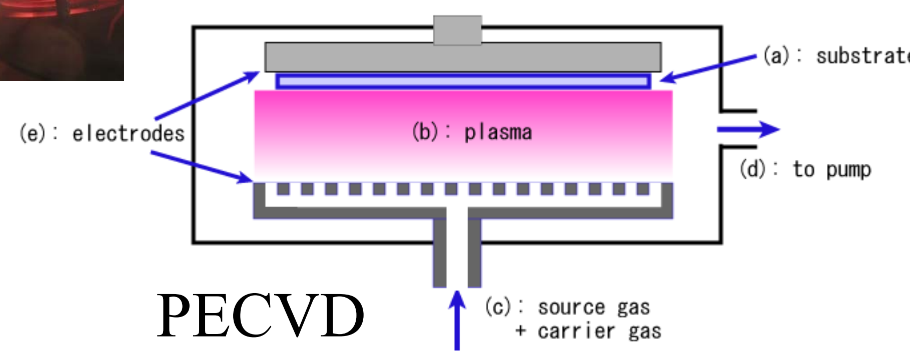
In the metal organic chemical vapor deposition (MOCVD) technique, **reactant gases are combined at elevated temperatures in the reactor to cause a chemical interaction, resulting in the deposition of materials on the substrate.**

A reactor is a chamber made of a material that does not react with the chemicals being used. It must also withstand high temperatures. This chamber is composed by reactor walls, liner, a susceptor, gas injection units, and temperature control units. Usually, the reactor walls are made from stainless steel or quartz. Ceramic or special glasses, such as quartz, are often used as the liner in the reactor chamber between the reactor wall and the susceptor. To prevent overheating, cooling water must be flowing through the channels within the reactor walls. A substrate sits on a susceptor which is at a controlled temperature. The susceptor is made from a material resistant to the metalorganic compounds used; graphite is sometimes used. For growing nitrides and related materials, a special coating on the graphite susceptor is necessary to prevent corrosion by ammonia (NH3) gas.

Chemical Vapor Deposition



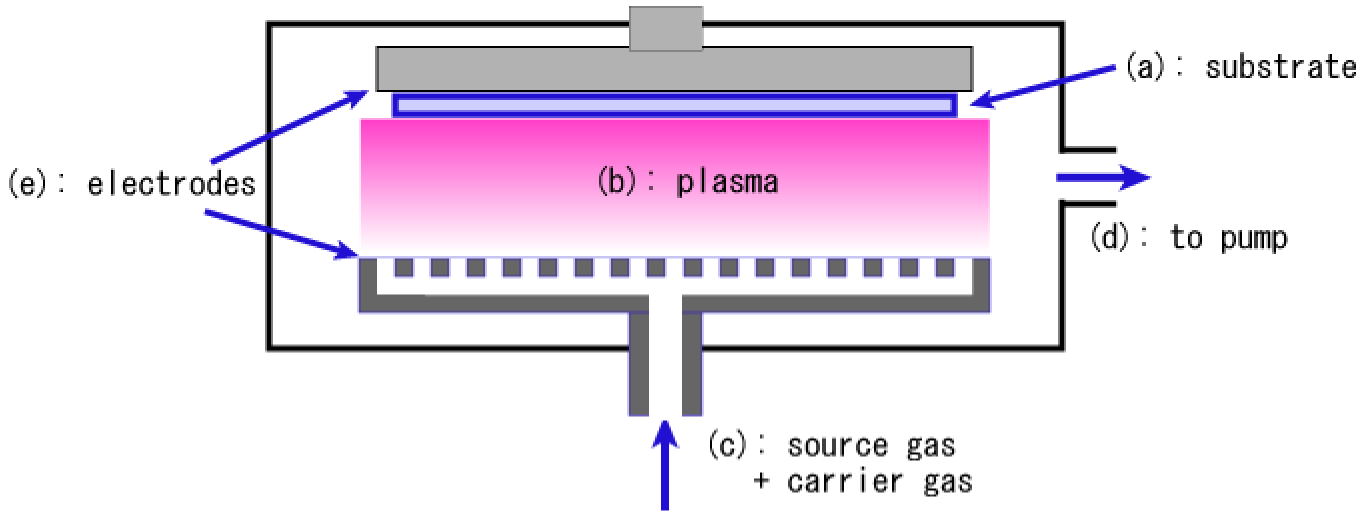
Thermal CVD



PECVD

Chemical vapor deposition (CVD) is a chemical process used to produce high quality, high-performance, solid materials. The process is often used in the semiconductor industry to produce thin films. In typical CVD, the wafer (substrate) is **<u>exposed to one or more volatile precursors, which react and/or decompose on the substrate surface to produce the desired deposit</u>**. Frequently, volatile by-products are also produced, which are removed by gas flow through the reaction chamber

PECVD



Plasma-Enhanced CVD (PECVD) – CVD **that utilizes plasma to enhance chemical reaction rates of the precursors**. PECVD processing allows deposition at lower temperatures, which is often critical in the manufacture of semiconductors. The lower temperatures also allow for the deposition of organic coatings, such as plasma polymers, that have been used for nanoparticle surface functionalization

Quantum-Well Technology

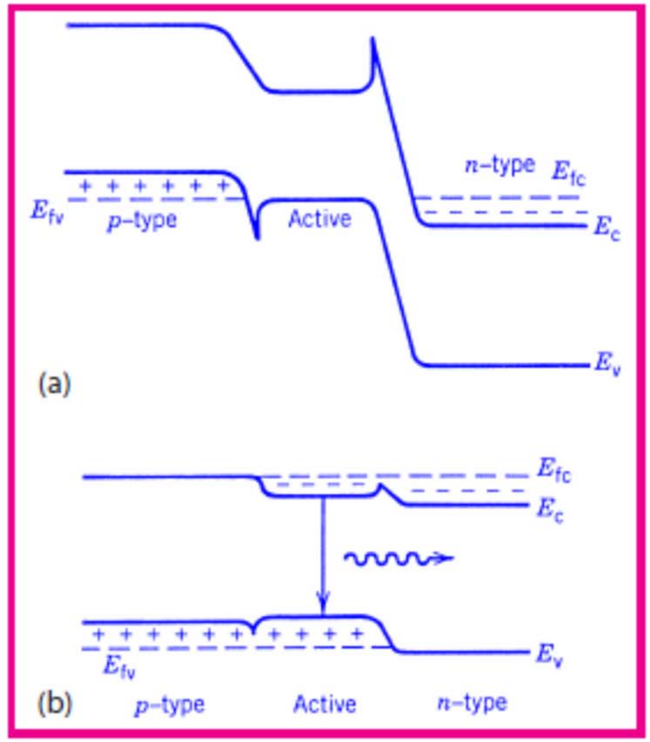
- Thickness of the core layer plays a central role.
- If it is small enough, electrons and holes act as if they are confined to a quantum well.
- Confinement leads to quantization of energy bands into subbands.
- Joint density of states acquires a staircase-like structure.
- Useful for making modern quantum-well, quantum wire, and quantum-dot lasers.
- in MQW lasers, multiple core layers (thickness 5–10 nm) are separated by transparent barrier layers.
- Use of intentional but controlled strain improves performance in *strained* quantum wells.

Doped Semiconductor Waveguides

- To build a laser, one needs to inject current into the core layer.
- This is accomplished through a p–n junction formed by making cladding layers p- and n-types.
- n-type material requires a dopant with an extra electron.
- p-type material requires a dopant with one less electron.
- Doping creates free electrons or holes within a semiconductor.
- Fermi level lies in the middle of bandgap for undoped (intrinsic) semiconductors.
- In a heavily doped semiconductor, Fermi level lies inside the conduction or valence band.

p–n junctions (basic light emitting diode)

- Fermi level continuous across the p–n junction in thermal equilibrium.
- A built-in electric field separates electrons and holes.
- Forward biasing reduces the built-in electric field.
- An electric current begins to flow: $I = I_s[\exp(qV/k_BT) - 1]$.
- Recombination of electrons and holes generates light.



Electro-Optic Waveguides

- Use Pockels effect to change refractive index of the core layer with an external voltage.
- Common electro-optic materials: $LiNbO_3$, $LiTaO_3$, $BaTiO_3$.
- $LiNbO_3$ used commonly for making optical modulators.
- For any anisotropic material $D_i = \varepsilon_0 \sum_{j=1}^{3} \varepsilon_{ij} E_j$.
- Matrix ε_{ij} can be diagonalized by rotating the coordinate system along the principal axes.
- Impermeability tensor $\eta_{ij} = 1/\varepsilon_{ij}$ describes changes induced by an external field as $\eta_{ij}(\mathbf{E}^a) = \eta_{ij}(0) + \sum_k r_{ijk} \mathbf{E}_k^a$.
- Tensor r_{ijk} is responsible for the electro-optic effect.

Lithium Niobate Waveguides

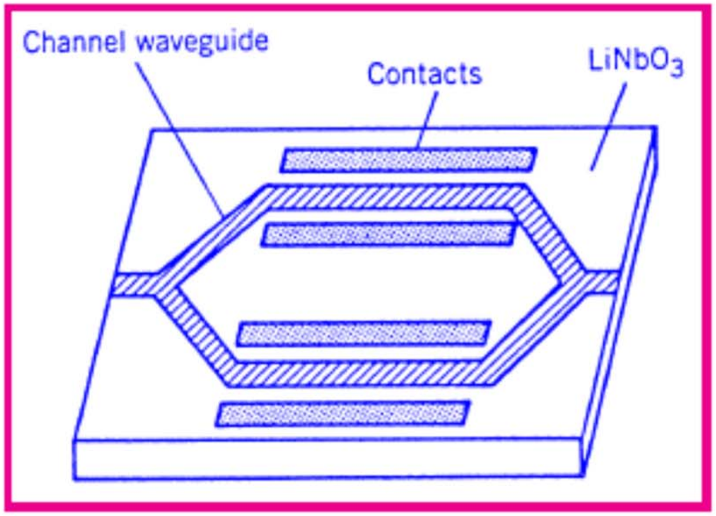
- $LiNbO_3$ waveguides do not require an epitaxial growth.
- A popular technique employs diffusion of metals into a $LiNbO_3$ substrate, resulting in a low-loss waveguide.
- The most commonly used element: Titanium (Ti).

- Diffusion of Ti atoms within $LiNbO_3$ crystal increases refractive index and forms the core region.
- Out-diffusion of Li atoms from substrate should be avoided.
- Surface flatness critical to ensure a uniform waveguide.

$LiNbO_3$ Waveguides

- A proton-exchange technique is also used for $LiNbO_3$ waveguides.
- A low-temperature process ($\sim 200°C$) in which Li ions are replaced with protons when the substrate is placed in an acid bath.
- Proton exchange increases the extraordinary part of refractive index but leaves the ordinary part unchanged.
- Such a waveguide supports only TM modes and is useful for some applications because of its polarization selectivity.
- High-temperature annealing used to stabilizes the index difference.
- Accelerated aging tests predict a lifetime of over 25 years at a temperature as high as 95°C.

$LiNbO_3$ Waveguides

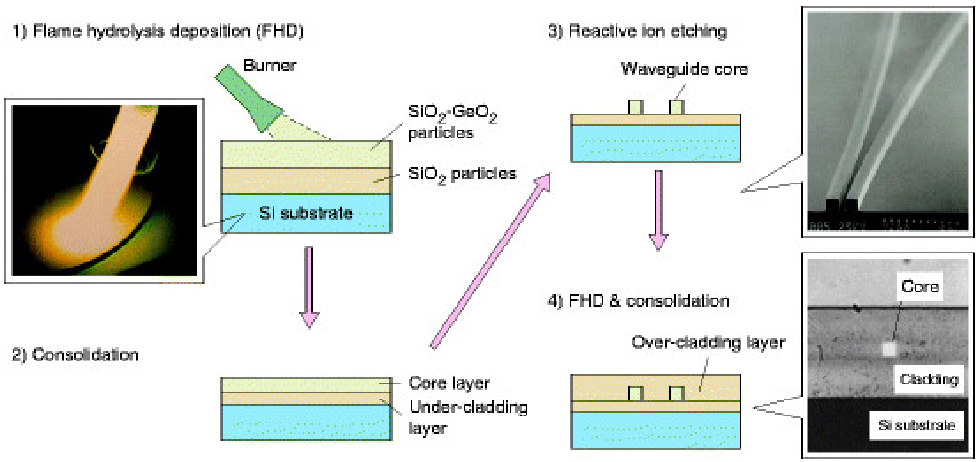


- Electrodes fabricated directly on the surface of wafer (or on an optically transparent buffer layer.
- An adhesion layer (typically Ti) first deposited to ensure that metal sticks to $LiNbO_3$.
- Photolithography used to define the electrode pattern.

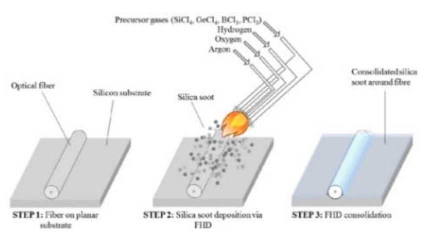
Silica Glass Waveguides

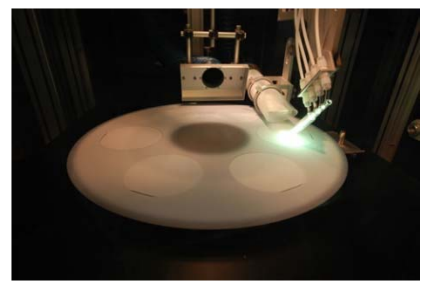
- Silica layers deposited on top of a Si substrate.
- Employs the technology developed for integrated circuits.
- Fabricated using flame hydrolysis with reactive ion etching.
- Two silica layers are first deposited using flame hydrolysis.
- Top layer converted to core by doping it with germania.
- Both layers solidified by heating at 1300°C (consolidation process).
- Photolithography used to etch patterns on the core layer.
- Entire structure covered with a cladding formed using flame hydrolysis. A thermo-optic phase shifter often formed on top.

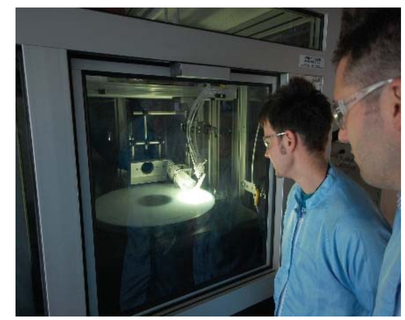
Flame Hydrolysis



control the flame temperature you can control the proportion of dopant deposited







The Flame Hydrolysis Deposition tool (FHD) is a state-of-the-art system for the growth of silica to form waveguide structures for integrated photonic circuits. The system allows the precise doping of germanium, phosphorous and boron within the silica films and is particularly optimised for the growth of films with high photosensitivity for direct UV laser writing of advanced photonic circuits. The tool can grow films ranging from 2 to 50 microns in thickness. The system is designed to deposit onto 150mm silicon wafers (other substrates can also be used) and has a high throughput of up to 30 wafers per day.

Vapor axial deposition

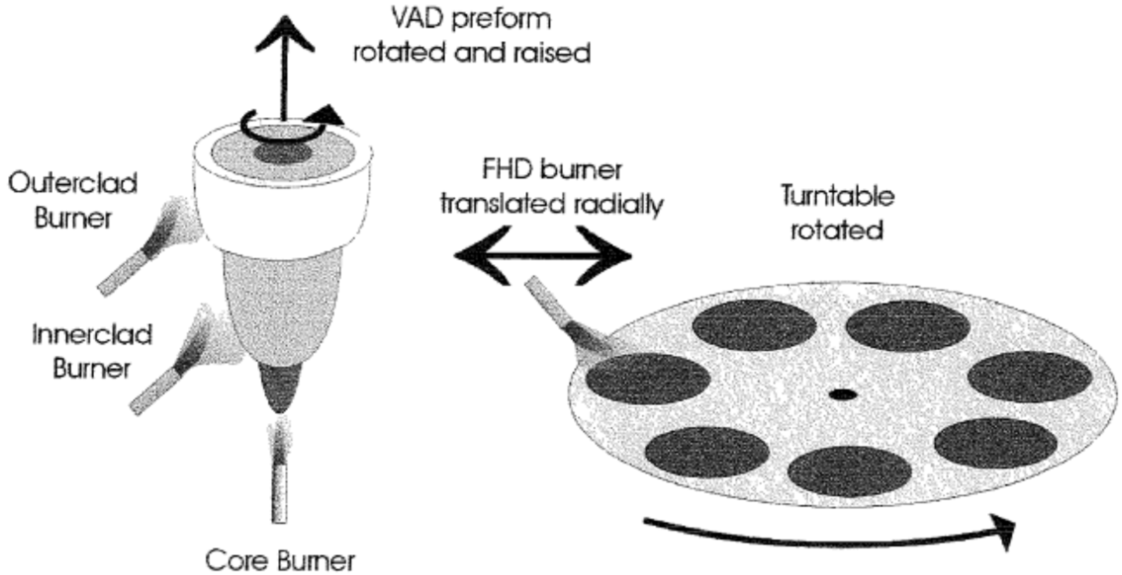
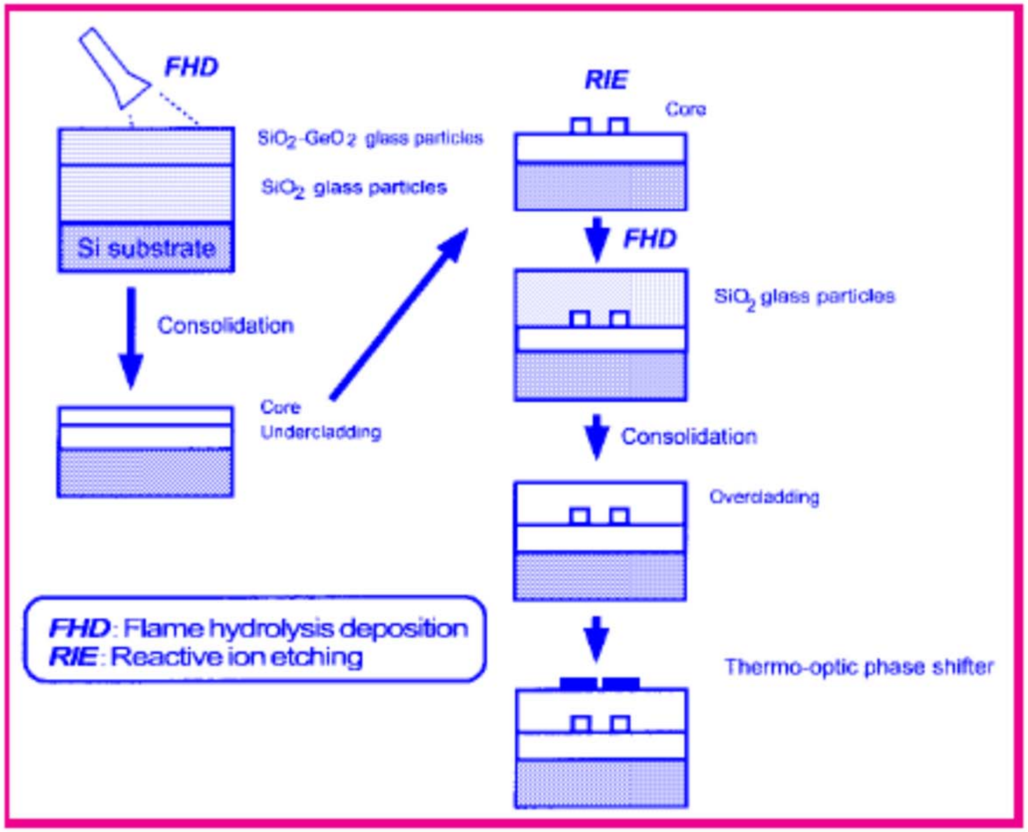


Figure 2.1, a) Schematic of Vapour Axial Deposition process,
b) Schematic of typical Flame Hydrolysis Deposition process

- Refractive indices can be carefully controlled using VAD
- VAD is a very important process and accounts for a large proportion of world fibre production. It was originally intended to be a continuous process which would have a lot lower cost than the batch processes.

Silica-on-Silicon Technique

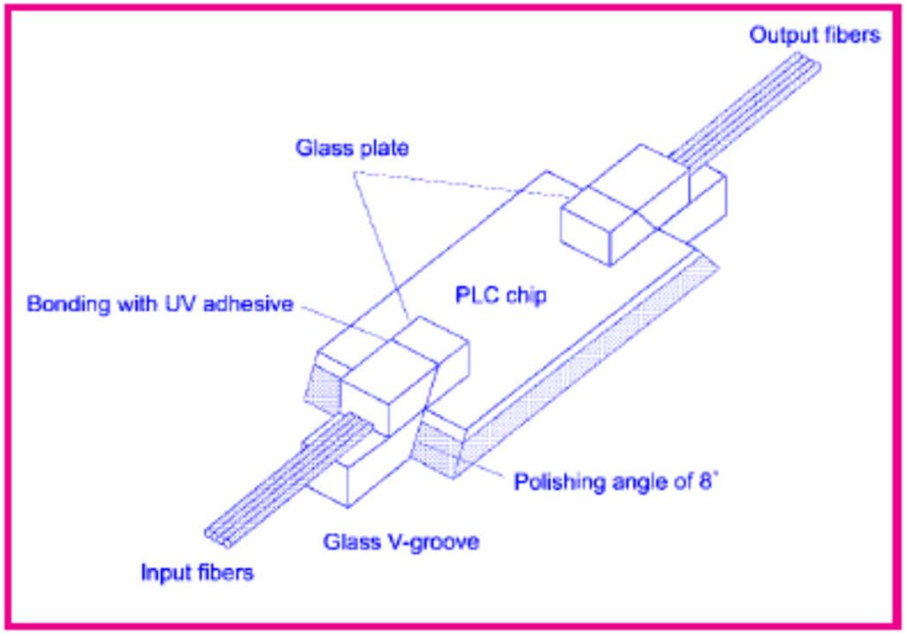


Steps used to form silica waveguides on top of a Si Substrate

Silica Waveguide properties

- Silica-on-silicon technology produces uniform waveguides.
- Losses depend on the core-cladding index difference $\Delta = (n_1 - n_2)/n_1$.
- Losses are low for small values of Δ (about 0.017 dB/cm for $\Delta = 0.45\%$).
- Higher values of Δ often used for reducing device length.
- Propagation losses ~0.1 dB/cm for $\Delta = 2\%$.
- Planar lightwave circuits: Multiple waveguides and optical components integrated over the same silicon substrate.
- Useful for making compact WDM devices ($\sim 5 \times 5$ cm^2).
- Large insertion losses when a PLC is connected to optical fibers.

Packaged PLCs

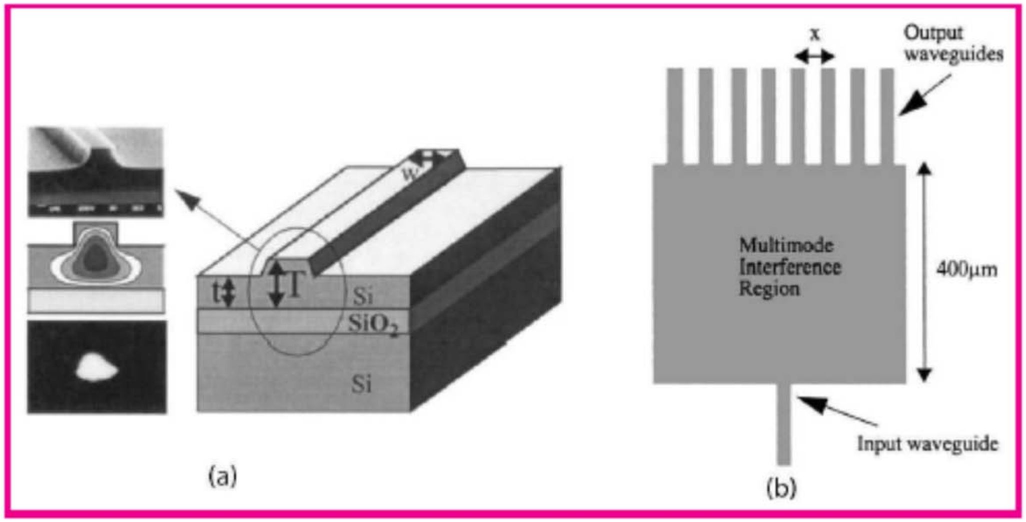


- Package design for minimizing insertion losses.
- Fibers inserted into V-shaped grooves formed on a glass substrate.
- Glass substrate connected to the PLC chip using an adhesive.
- A glass plate placed on top of V grooves is bonded to the PLC chip

Silicon Oxynitride Waveguides

- Employ Si substrate but use SiON for the core layer.
- SiON alloy is made by combining SiO_2 with Si_3N_4, two dielectrics with refractive indices of 1.45 and 2.01.
- Refractive index of SiON layer can vary from 1.45–2.01.
- SiON film deposited using plasma-enhanced chemical vapor deposition (SiH_4 combined with N_2O and NH_3).
- Low-pressure chemical vapor deposition also used (SiH_2Cl_2 combined with O_2 and NH_3).
- Photolithography pattern formed on a 200-nm-thick chromium layer.
- Propagation losses typically <0.2 dB/cm.

Silicon-on-Insulator

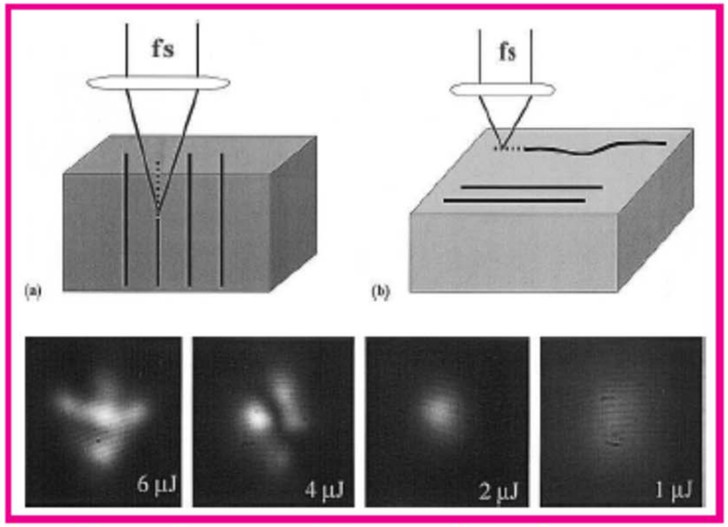


- Core waveguide layer is made of Si ($n_1 = 3.45$).
- A silica layer under the core layer is used for lower cladding.
- Air on top acts as the top cladding layer.
- Tightly confined waveguide mode because of large index difference.
- Silica layer formed by implanting oxygen, followed with annealing.

Laser Written Waveguides

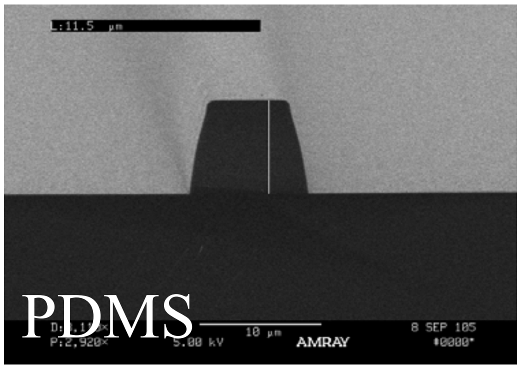
- CW or pulsed light from a laser used for writing waveguides in silica and other glasses
- Photosensitivity of Ge. Doped silica exploited to enhanced refractive index in the region exposed to a UV laser (245nm)
- Absorption of 244nm light from KrF laser changes refractive index by 10^{-4} only in the region exposed to UV light
- Index changes > 10^{-3} can be realized with a 193nm ArF laser
- A planar waveguide formed first through CVD, but core layer is doped with Ge.
- An UV beam focused to 1 micron scanned slowly to enhanced n selectively. UV written sample then annelid at 80^oC.

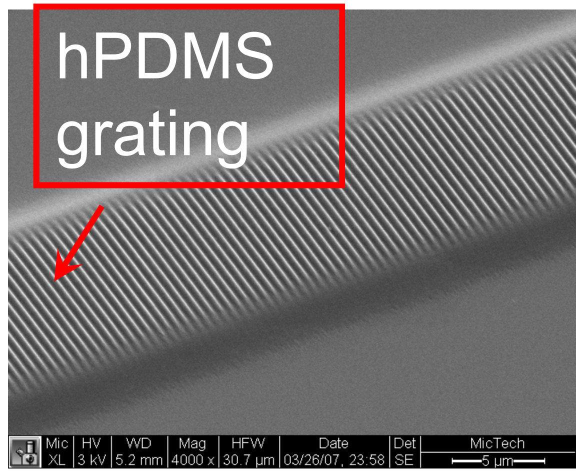
Laser Written Waveguides

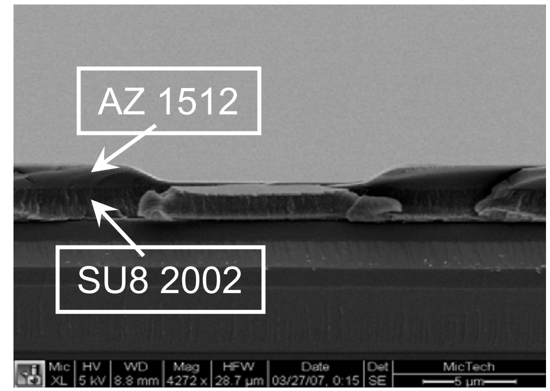


- Femtosecond pulses from a Ti:sapphire laser can be used to write waveguides in bulk glasses. (800nm)
- Intense pulses modify the structure of silica through multiphoton absorption.
- Refractive-index changes $\sim 10^{-2}$ are possible.

Polymer Waveguides







w. wang

- Polymers such as halogenated acrylate, fuorinated polyimide, and deuterated polymethylmethacrylate (PMMA) have been used.
- Polymer films can be fabricated on top of Si, glass, quartz,or plastic through spin coating.
- Photoresist layer on top used for reactive ion etching of the core layer through a photomask.

Additional Lecture on Polymer Optics Fabrication

- We will discuss fabrication of polymer optics later in the quarter mainly on latest development in using novel approach in polymer optics.

Week 13

- Course Website:
  http://courses.washington.edu/me557/sensors
- Reading Materials:

  - Week 13 reading materials can be found:

http://courses.washington.edu/me557/reading/

- Proposals due today
- Work on Lab 2 (arrange time to meet with TA, please finish it this week)
- HW 3 due today
- HW 4 assigned due week 16 if need more time send the HW electronically to abong@uw.edu after week 16
- Final presentation is on 12/23, final report due 1/7/20

Outline

- Waveguide structures and materials
- Field equations
- Wave equations in Waveguides
- Waveguide modes, $n_{eff,}$ dispersion equation
- Guided modes in symmetric and asymmetric slab waveguides
- General formalisms for step-index planar waveguides

Waveguide Theory

- Ray approach
- Wave approach

Wave Equation in free space and waveguide

→ Faraday's Law

$\Rightarrow ma + kx = 0$

equivalent to a mass spring mechanical system

$$\nabla \times E = -\frac{\partial B}{\partial t} \quad \text{— (1)}$$

→ take curl of Faraday's Law

$$\nabla \times (\nabla \times E) = -\frac{\partial (\nabla \times B)}{\partial t} \quad \text{— (2)}$$

→ Substitute Ampere's Law for a charge and current free region :

$$\text{Ampere's Law: } \nabla \times H = \cancel{J}^{\,0} + \frac{\partial D}{\partial t}$$

since

$B = \mu H + M$

$H = \frac{B}{\mu}$ & $\mu = \mu_0$ for non magnetic material

$M = 0$

$D = \epsilon E$ electric elasticity equation

$C = \frac{1}{\sqrt{\epsilon \mu}}$ Speed of light

$$\boxed{\nabla \times H = \frac{\partial D}{\partial t} \Rightarrow \nabla \times B = \mu_0 \epsilon \frac{\partial E}{\partial t}} \quad \text{— (3)}$$

$$= \frac{1}{C^2} \frac{\partial E}{\partial t}$$

$$\nabla \times (\nabla \times E) = -\frac{\partial(\nabla \times B)}{\partial t} \quad — \quad (2)$$

$$\nabla \times H = \frac{\partial D}{\partial t} \Rightarrow \nabla \times B = \mu_0 \epsilon \frac{\partial E}{\partial t} = \frac{1}{c^2}\frac{\partial E}{\partial t} \quad — \quad (3)$$

$$c = \frac{c_o}{n} = \frac{1}{\sqrt{\mu_r \mu_o \varepsilon_r \varepsilon_o}}$$

→ sub (3) back into 2 :

$$\nabla \times (\nabla \times E) = -\mu_0 \epsilon \frac{\partial^2 E}{\partial t^2} = -\frac{1}{c^2}\frac{\partial^2 E}{\partial t^2}$$

$$\nabla \times \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix} = -\frac{1}{c^2}\frac{\partial E}{\partial t^2} = -\frac{1}{c^2}\left[\frac{\partial E_x}{\partial t}\hat{x} + \frac{\partial E_y}{\partial t}\hat{y} + \frac{\partial E_z}{\partial t}\hat{z}\right.$$

Recall wave equation

$$\nabla \times (\nabla \times E) = -\mu_0 \epsilon \frac{\partial^2 E}{\partial t^2} = -\frac{1}{c^2}\frac{\partial^2 E}{\partial t^2}$$

use an identity

$$\nabla \times (\nabla \times E) = \nabla(\nabla \cdot E) - \nabla^2 E \quad \text{— (4)}$$

above equation becomes

$$\nabla(\nabla \cdot E) - \nabla^2 E = -\frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} \quad \text{— (5)}$$

$$\boxed{\nabla(\nabla\cdot E) - \nabla^2 E = -\frac{1}{c^2}\frac{\partial^2 E}{\partial t^2}} \quad \text{— (5)}$$

since

$\nabla \cdot E = 0$ for charge free region (Gauss's Law of electricity)

$\frac{\partial E}{\partial t} = j\omega E$ assume time harmonic function

equation (5) now becomes

$$0 - \nabla^2 E = \frac{\omega^2}{c^2} E$$

(wave equation) $$\boxed{\nabla^2 E + \frac{\omega^2}{c^2} E = 0} \quad \text{— (6)}$$

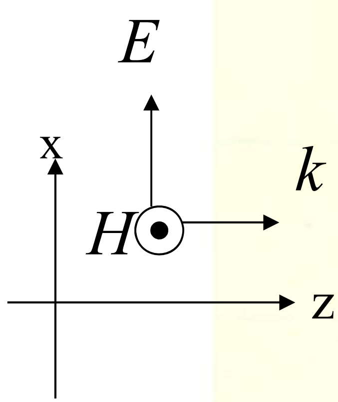
We consider a simple solution where E field is parallel to the x axis & is function of z coordinate only, the wave equation then becomes

$$\frac{\partial^2 E_x}{\partial z^2} + \frac{\omega^2}{c^2} E_x = 0 \quad \text{—— (7)}$$

A solution to the above differential equation is

$$E = \hat{x} E_o e^{-jkz} \quad \text{—— (8)}$$

(in phasor form because of time harmonic function)



Substitute above equation into wave equation (eq 7)

$$\left(-k^2 + \frac{\omega^2}{c^2}\right) E = 0 \Rightarrow k^2 = \frac{\omega^2}{c^2} \quad \text{(dispersion relation)}$$

$$k_o^2 n^2 = \omega^2/c^2$$

$$k = \frac{\omega}{c} = \frac{2\pi f}{c}$$ (k = wave number or propagation constant)

Let's transform the solution for the wave equation into real space & time (assume time harmonic field)

If looking at wave propagating in a slab waveguide, the wave equation from free space propagating in z direction:

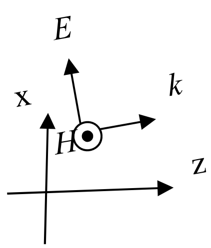
$$\frac{\partial^2 E(x,y)}{\partial z^2} + \omega^2 \mu_o \varepsilon_o E(x,y) = \frac{\partial^2 E(x,y)}{\partial z^2} + k_0^2\ n^2 E(x,y) = 0$$

is modified for wave propagating in the slab with confinement in x direction and plane wave propagating in z direction $E(r) = E(x,y)e^{-j\beta z}$ or $E(r) = (\hat{x}E_x + \hat{y}E_y)e^{-j\beta z}$, the wave equation becomes:

$$\nabla^2 E + \omega^2 \mu_o \varepsilon_o E = \frac{\partial^2 E(x,y)}{\partial x^2} + \frac{\partial^2 E(x,y)}{\partial y^2} + (k_0^2 n^2 - \beta^2)E(x,y) = 0$$

β is propagation constant

If only focusing on x variation:



$$\frac{\partial^2 E(x,y)}{\partial x^2} + (k_0^2 n^2 - \beta^2)E(x,y) = 0$$

E-M Field in a Planar Waveguide

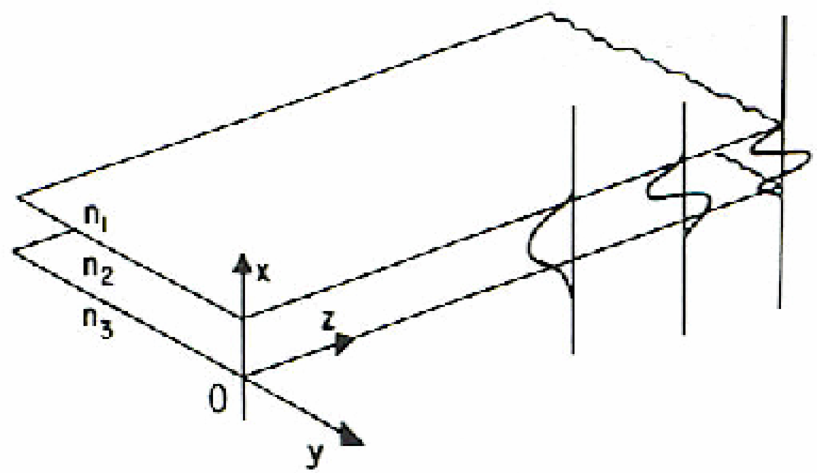


Fig. 2.1. Diagram of the basic three-layer planar waveguide structure. Three mode are shown, representing distributions of electric field in the x direction

Assuming a monochromatic wave propagating in z-direction

$$\mathbf{E}(\mathbf{r},t) = \mathbf{E}(\mathbf{r})e^{j\omega t} = \mathbf{E}(x,y)e^{-j\beta z}e^{j\omega t}$$

$$\nabla^2\mathbf{E}(\mathbf{r}) + k^2 n^2(\mathbf{r})\mathbf{E}(\mathbf{r}) = 0$$

Region I: $$\frac{\partial^2}{\partial x^2}E(x,y) + (k^2 n_1^2 - \beta^2)E(x,y) = 0$$

Region II: $$\frac{\partial^2}{\partial x^2}E(x,y) + (k^2 n_2^2 - \beta^2)E(x,y) = 0$$ (wave equations)

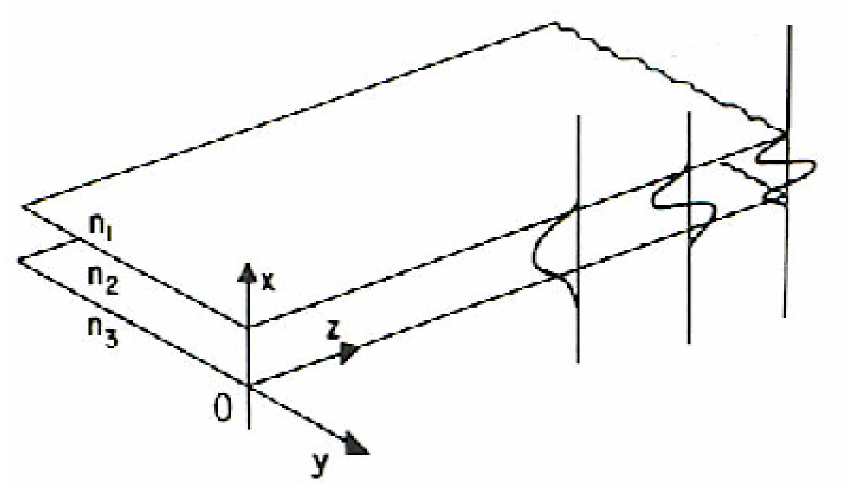
Region III: $$\frac{\partial^2}{\partial x^2}E(x,y) + (k^2 n_3^2 - \beta^2)E(x,y) = 0$$

Outline

- Waveguide structures and materials
- Field equations
- ➡ Waveguide modes, $n_{eff,}$ dispersion equation
- Guided modes in symmetric and asymmetric slab waveguides
- General formalisms for step-index planar waveguides

Waveguide modes

E-M Field in a Planar Waveguide



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Region III: $$\frac{\partial^2}{\partial x^2}E(x,y) + (k^2 n_3^2 - \beta^2)E(x,y) = 0$$

Things covered in this section

- k vector in propagating and confined direction (x, z)
- Mode field in confined direction (x)
- n_{eff}
- Dispersion equation in confine direction (x)

Waveguide modes

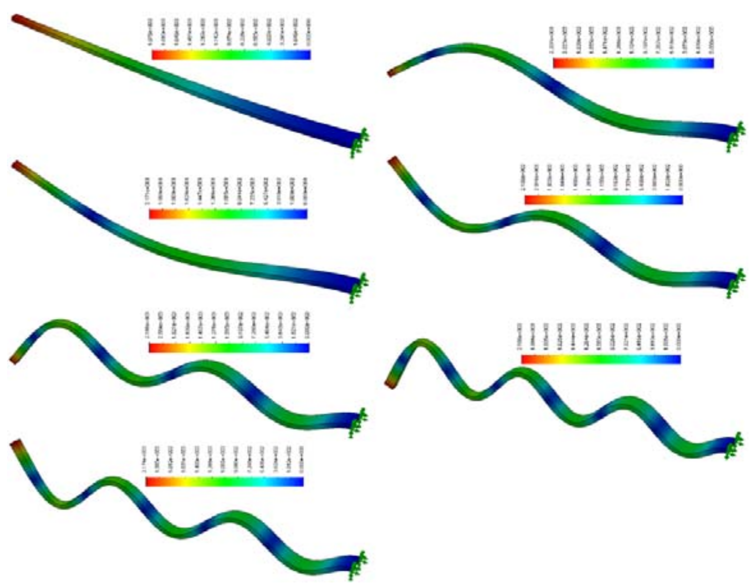
- ***Waveguide modes*** exist that are characteristic of a particular waveguide structure.
- A waveguide mode is a ***transverse field pattern*** whose *amplitude and polarization profiles* remain constant along the longitudinal z coordinate.
- Therefore, the electric and magnetic fields of a mode can be written as follows

$$E_\nu(r,t) = E_\nu(x,y)\exp i(\beta_\nu z - \omega t)$$

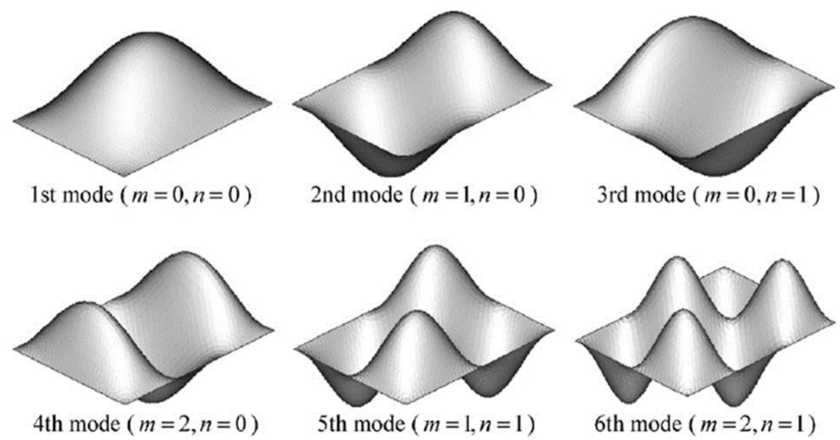
$$H_\nu(r,t) = H_\nu(x,y)\exp i(\beta_\nu z - \omega t)$$

where υ is the *mode index*, *$E_\nu(x, y)$ and $H_\nu(x, y)$* are the *mode field profiles*, and β_ν is the *propagation constant* of the mode

Mechanical Vibration Analogy Fix-Fix Boundary Mode Shape

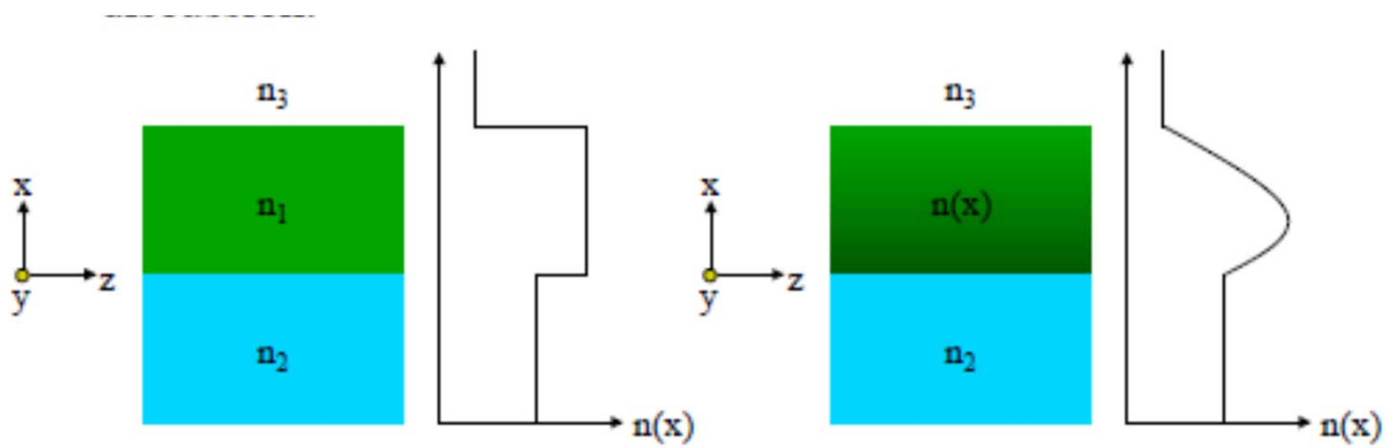


Fix-fix resonant mode in 1 D structure



Fix-fix resonant mode in 2 D structure

Index Profiles



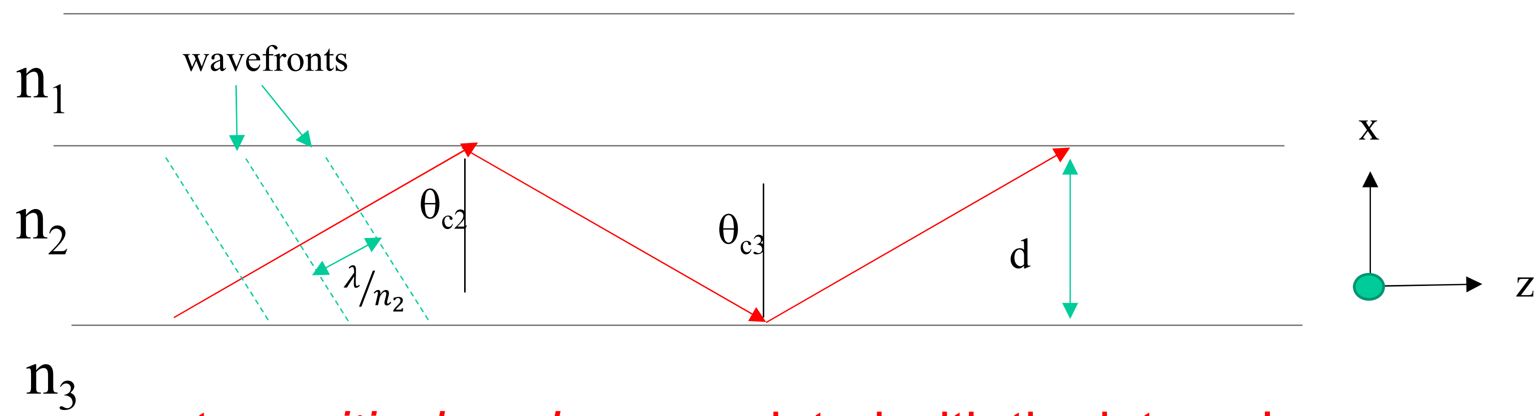
A waveguide in which the index profile has

1. abrupt changes between the core and the cladding is called a ***step-index waveguide***,

2. index profile varies gradually is called a ***graded-index waveguide***.

<u>We will focus only the step-index waveguide in the class.</u>

Ray Optics Approach to Optical Waveguide Theory



There are two *critical angles* associated with the internal reflections at the lower and upper interfaces:

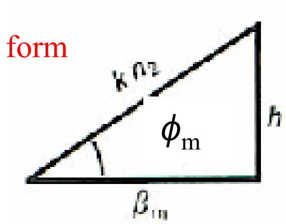
$$\theta_{c3} = sin^{-1}(n_3/n_2) \qquad \theta_{c2} = sin^{-1}(n_1/n_2)$$

$\theta_{c3} > \theta_{c2}$ because $n_3 > n_1$

If $\theta > \theta c_3 > \theta c_2$, the wave inside the core is totally reflected at both interfaces and is trapped by the core, resulting in *guided modes*.

Ray Patterns for Different Modes

This shows how ϕ needs to be to be able (in terms of refractive indices of each layer) to form either radiation or guided modes.



$$\phi_2 = \sin^{-1}\frac{\beta}{kn_2} < \sin^{-1}\frac{n_1}{n_2}$$

$\phi_{c3} > \phi_{c2}$ because $n_3 > n_1$

$$\frac{n_1}{n_2} < \sin\phi_2 < \frac{n_3}{n_2}$$

$$\phi_2 > \sin^{-1}\frac{n_3}{n_2}$$

Only n_2 sine $\phi_2 > n_3$sin 90o, greater than both sides are confined because $\phi_{c3} > \phi_{c2}$ because $n_3 > n_1$

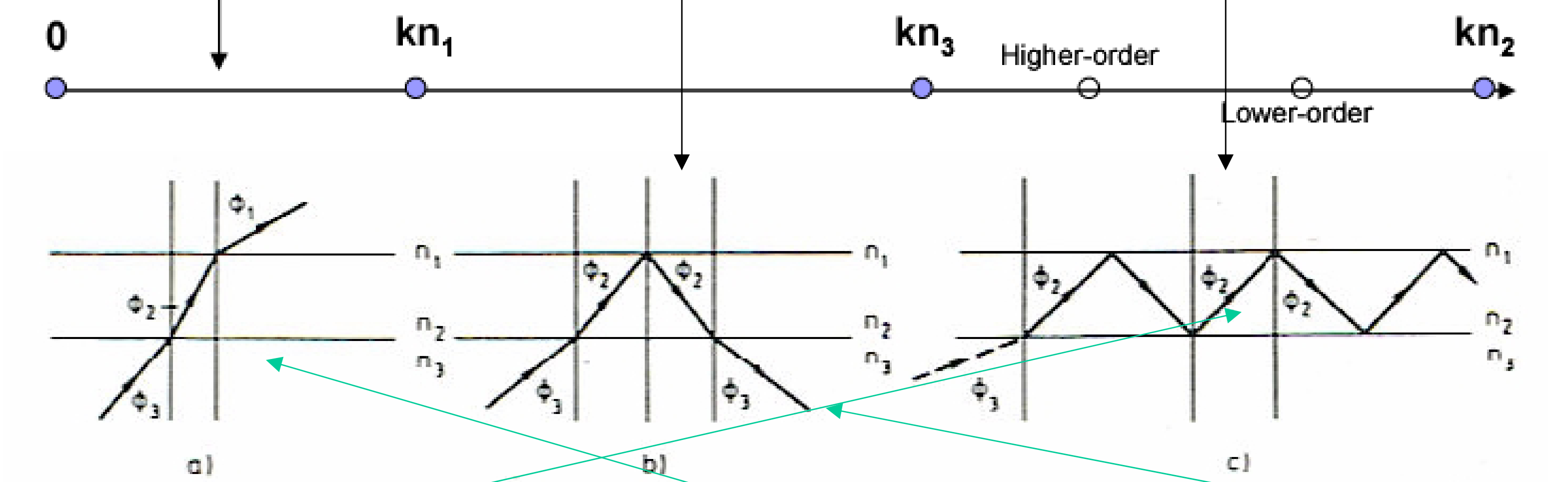
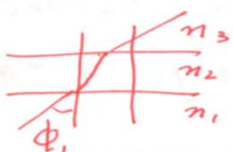


Fig. 2.10a–c. Optical ray patterns for a air radiation modes; **b** substrate radiation modes; **c** guided mode. In each case a portion of the incident light is reflected back into layer 3; however, that ray has been omitted from the diagrams

Snell's Law

$n_2 > n_1 > n_3$

(1)



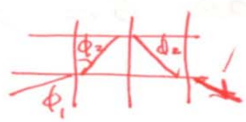
radiation mode
(non guide mode)

$$n_1 \sin\phi_1 = n_2 \sin\phi_2 = n_3 \sin\phi_3$$

(1) $\phi_2 < \sin^{-1}\frac{n_1}{n_2} = \sin^{-1}\frac{kn_1}{kn_2}$

$\rightarrow \beta \leq kn_1$

(2)



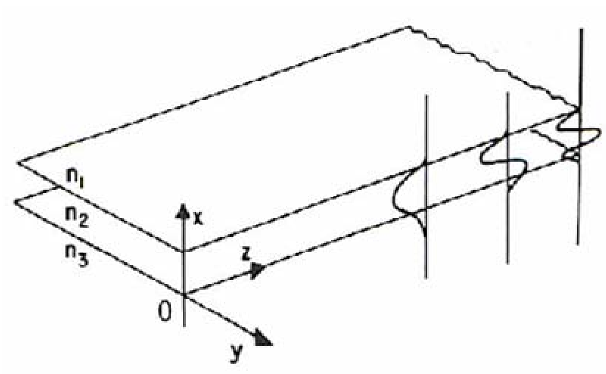
substrate radiation mode

(2) $\phi_2 > \sin^{-1}\frac{n_1}{n_2}$

$< \sin^{-1}\frac{n_3}{n_2}$

Waveguide modes

- Consider the qualitative behavior of an optical wave in an *asymmetric* planar step-index waveguide, where $n_2 > n_3 > n_1$.
- For an optical wave of angular frequency ω and free-space wavelength λ, the media in the three different regions of the waveguide define the following propagation constants:



$$k_1 = n_1\omega / c, k_2 = n_2\omega / c, k_3 = n_3\omega / c$$

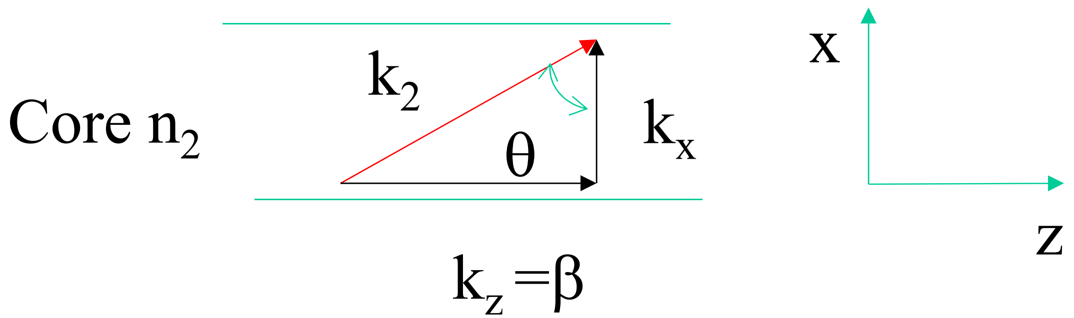
where $k_2 > k_3 > k_1$

- We can obtain useful intuitive picture from considering the path of an *optical ray*, or a *plane optical wave*, in the waveguide.

k-vector triangle

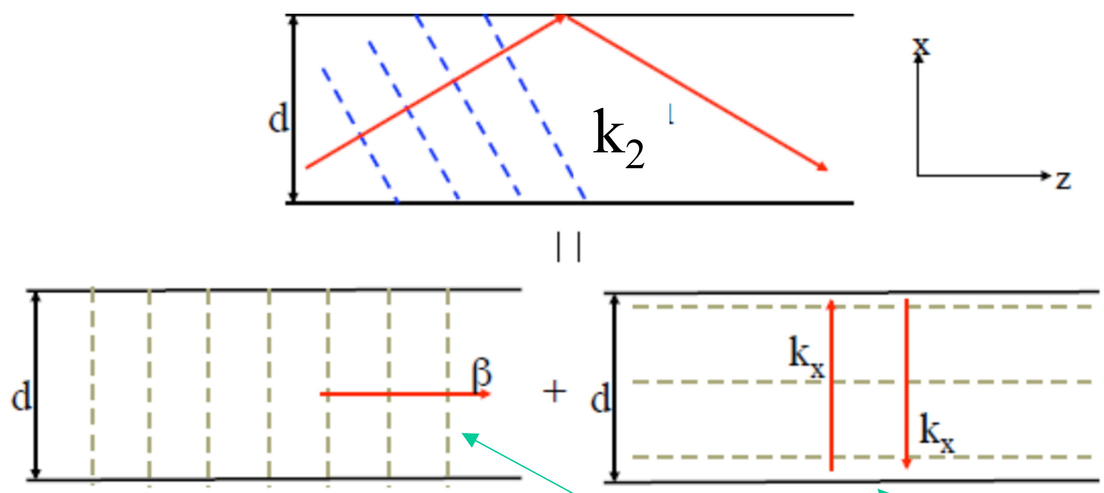
Remember only wave is propagating in Z direction

- The orthogonal components of the *propagation constant*, β and k_x, are related by the "**k**-vector triangle."



Transverse component $k_x = k_o n_2 \cos\theta = (n_2\omega/c) \cos\theta$
Longitudinal component $\beta = k_o n_2 \sin\theta = (n_2\omega/c) \sin\theta$
"k-vector triangle" $\beta^2 + k_x^2 = (n_2\omega/c)^2$

k_x and β components



- We can consider the "zig-zag" wave in the waveguide as two orthogonal components traveling in the longitudinal (z) and transverse (x) directions.
- The transverse component of the plane wave is reflected back and forth in the x direction, interfering with itself.

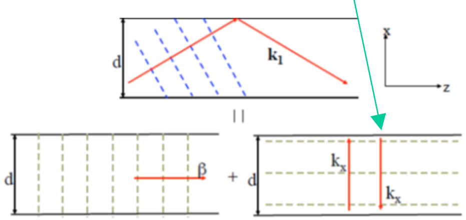
Things covered in this section

- k vector in propagating and confined direction (x, z)
- Mode field in confined direction (x)
- n_{eff}
- Dispersion equation in confine direction (x)

Guided modes

Ray approach

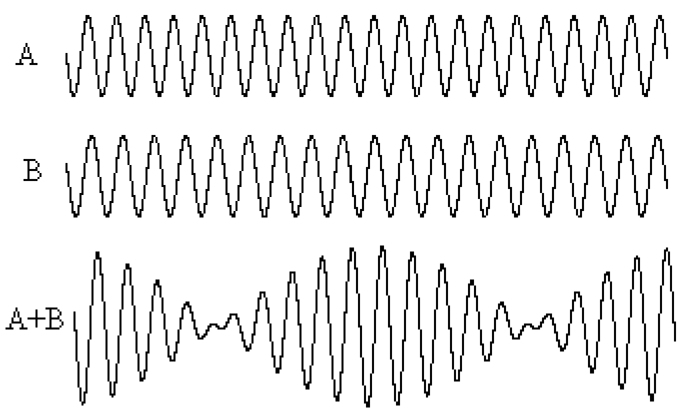
- As the wave is reflected back and forth between the two interfaces, it **interferes with itself.**
- **A guided mode can exist only when a transverse resonance (x direction) condition is satisfied (e.g. the repeatedly reflected wave has constructive interference with itself).**
- In the core region, the x component of the wave vector is $k_x = k_2 \cos\theta$ for a ray with an angle of incidence θ, while the z component is $\beta = k_2 \sin\theta$.
- The phase shift in the optical field due to a round-trip transverse passage in the core of thickness $\phi = 2k_2 d\cos\theta$.



Most important summary

Power of sin(a)+sin(b)

Everything can be expanded or explained in a series of sin function either a summation, multiplication or convolution. Many of our optical theory and sensor concept exploit this concept to allow us to study small physical changes using resolution of optical wavelength but observed at relatively lower frequency or longer wavelength or simplify the way we calculate them. e.g. interference or beats



sinA+sinB = 2sin(A+B)/2 ***cos(A-B)/2**

Let $A = k_1x+\omega_1t+\phi_1$ $\quad k_1 = 2\pi n_1/\lambda$

$B = k_2x+\omega_1t+\phi_2$ $\quad k_2 = 2\pi n_2/\lambda$

Light phenomena is just a superposition of waves with different wave lengths, phase, etc. (ambient light)

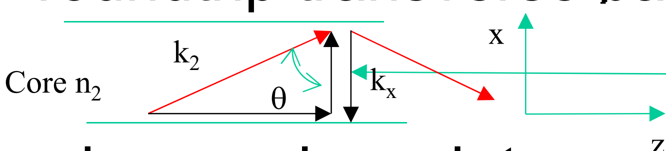
Ray approach

Transverse Resonance Condition

- There are phase shifts φ_2 and φ_3 associated with the **internal reflections in the lower and upper interfaces**.
- These phase shifts can be obtained from the phase angle of **r_s** (reflection coeff) for a TE wave (s wave) and that of **r_p** for a TM wave (p wave) for a given $\theta > \theta_{c3}, \theta_{c2}$. r is reflection coefficient.

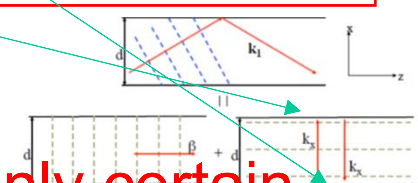
We will talk about it soon

- Because φ_2 and φ_3 are functions of θ, the ***transverse resonance condition*** for constructive interference in a roundtrip transverse passage is

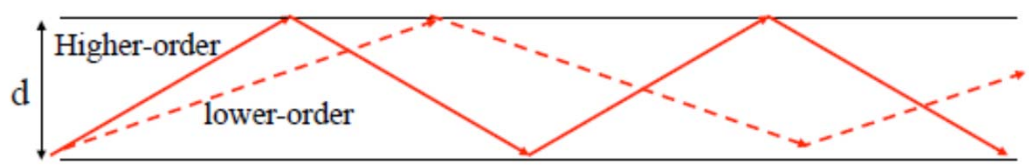


$$2k_2 d\cos\theta + \varphi_2(\theta) + \varphi_3(\theta) = 2m\pi$$

where m is an integer = 0, 1, 2, …

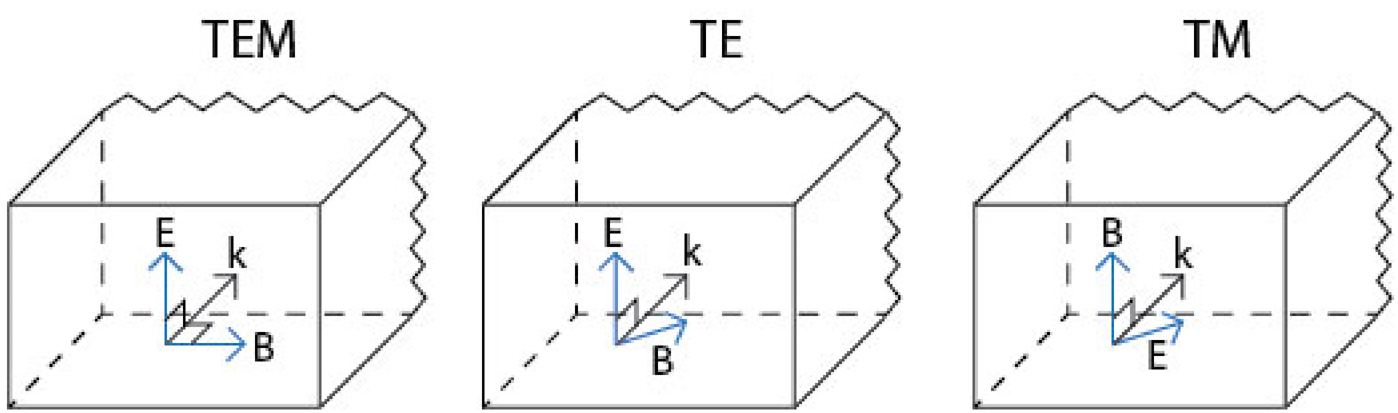


Because m can assume only integral values, only certain discrete values of θ can satisfy the transverse resonance condition.



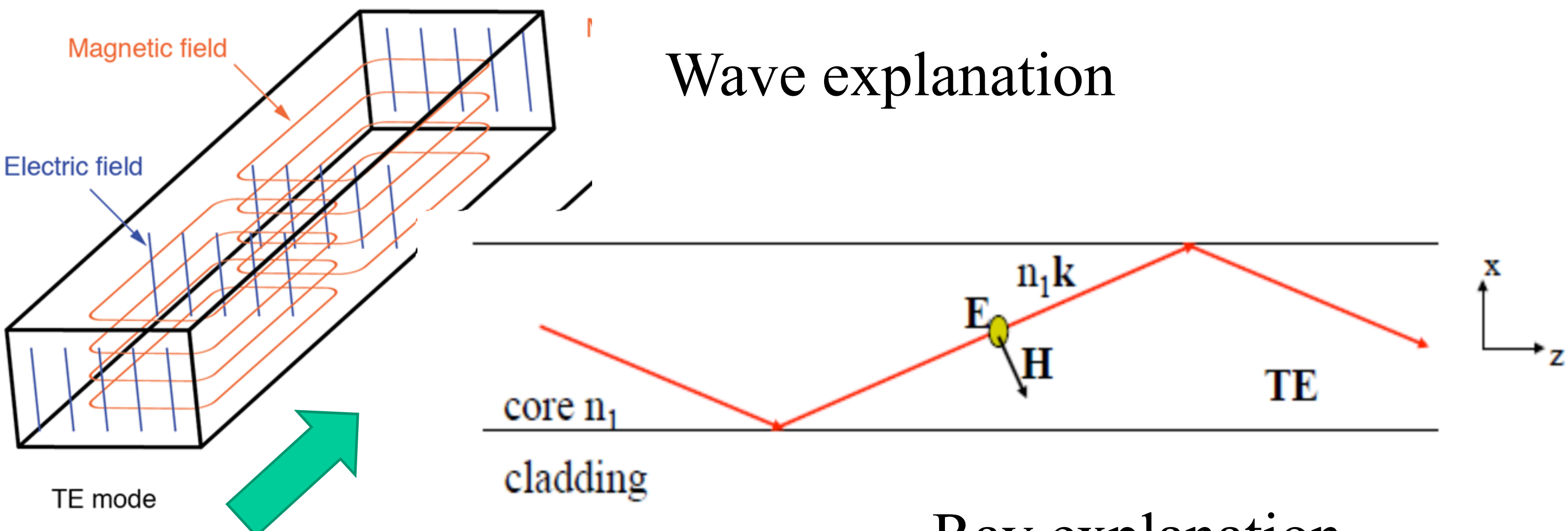
TEM Mode

Transverse electromagnetic (TEM) waves. In this case both Ez and Hz are zero. An example of this is a plane electromagnetic wave which has both electric and magnetic field perpendicular to the propagation direction. **There is no cutoff frequency for supporting TEM mode.** It can be shown that at least two separate conductors are needed for TEM waves. Examples of waveguides that allow TEM modes include **A coaxial cable, parallel waveguide, strip line and microstrip. Rectangular, circular, elliptical or any hallow waveguides cannot support TEM mode.**



Transverse Electric Polarization (TE mode)

- For slab waveguides, we define the x-z plane as the plane of incidence.
- An electric field pointing in the y direction corresponds to the perpendicular, or s, polarization.
- Waves with this polarization are labeled transverse electric (TE) fields because the electric field vector lies entirely in the x y plane (i.e. $E_z = 0$) that is *transverse* to the direction of net travel (the z direction).

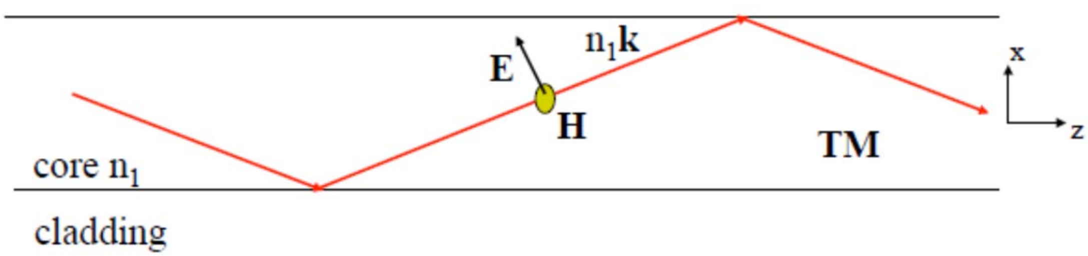


Ray explanation

W. Wang

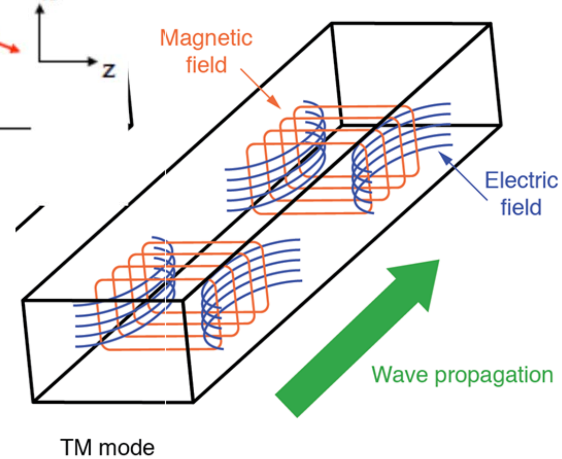
Transverse Magnetic Polarization (TM mode)

• For the parallel, or p, polarization, the electric field is no longer purely transverse. It has a component along the z direction.
• However, the magnetic field points in the y direction for this polarization is entirely transverse (i.e. Hz = 0).
The p polarization is labeled transverse magnetic (TM) in the slab.



Ray explanation

Wave explanation



Remember

If neither two are perfect conductors, J_s=0, then boundary conditions requires both the tangential electric-filed and magnetic-field components be continuous at **z =0** thus,

$$e^{-jk_x x} + R_l e^{-jk_{rx} x} = T_l e^{-jk_{tx} x} \qquad \text{(E component)}$$

$$\frac{-k_z}{\omega\mu_1} e^{-jk_x x} + \frac{k_{rz}}{\omega\mu_1} R_l e^{-jk_{rx} x} = \frac{-k_{tz}}{\omega\mu_2} T_l e^{-jk_{tx} x} \qquad \text{(B component)}$$

For the above equations to hold at all x, all components must be the same, thus we get the phase matching condition:

$$k_x = k \sin\theta_i = k_{rx} = k_r \sin\theta_r = k_{tx} = k_t \sin\theta_t$$

From this we obtain law of reflection:

$$\boxed{\theta_i = \theta_r}$$

Since $k = k_r$ *because* $k^2 = k_r^2 = \omega^2 \mu_1 \varepsilon_1 = k_1$

And Snell's Law:

$$\boxed{n_1 \sin\theta_1 = n_2 \sin\theta_2}$$

$$n_1 = c\sqrt{\mu_1 \varepsilon_1} = \frac{c}{\omega} k_1$$

$$n_2 = c\sqrt{\mu_2 \varepsilon_2} = \frac{c}{\omega} k_2$$

Remember

Reflection Coefficient

R is Complex (because n is and Function of θ

TE mode

TM mode

$$R_l = \frac{\mu_2 k_z - \mu_1 k_{tz}}{\mu_2 k_z + \mu_1 k_{tz}}$$

$$R_{ll} = \frac{\varepsilon_2 k_z - \varepsilon_1 k_{tz}}{\varepsilon_2 k_z + \varepsilon_1 k_{tz}}$$

$$T_l = \frac{2\mu_2 k_z}{\mu_2 k_z + \mu_1 k_{tz}}$$

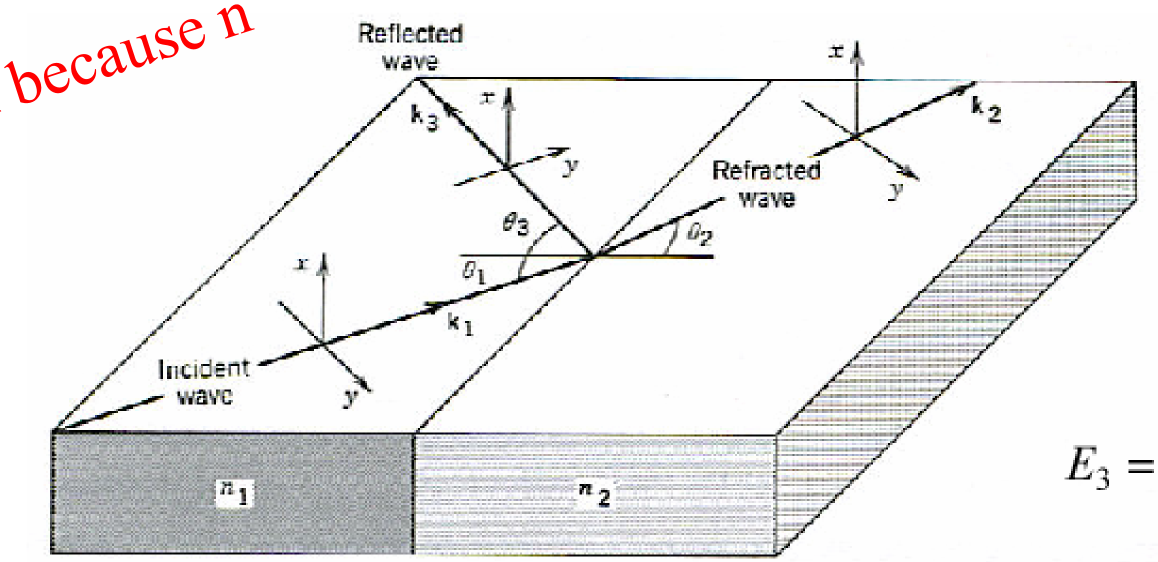
$$T_{ll} = \frac{2\varepsilon_2 k_z}{\varepsilon_2 k_z + \varepsilon_1 k_{tz}}$$

$$k_x^2 + k_z^2 = k_1^2 = k_{rx}^2 + k_{rz}^2$$

$$k_{tx}^2 + k_{tz}^2 = k_2^2$$

Reflection and Refraction

R is complex because n is complex



$$E_3 = rE_1, \quad E_2 = tE_1$$

For TE wave:

$$r_{TE} \quad \frac{n_1 \cos\theta_1 + n_2 \cos\theta_2}{n_1 \cos\theta_1 + n_2 \cos\theta_2} \qquad t_{TE} = 1 + r_{TE} \qquad r_s = r_{TE}$$

For TM wave:

$$r_{TM} \quad \frac{n_2 \cos\theta_1 - n_1 \cos\theta_2}{n_2 \cos\theta_1 + n_1 \cos\theta_2} \qquad t_{TM} = \frac{n_1}{n_2}(1 + r_{TM}) \qquad r_p = r_{TM}$$

Recall

$$r_{TE} = |r_{TE}| \exp(j\phi_{TE}), \quad r_{TM} = |r_{TM}| \exp(j\phi_{TM})$$

W. Wang

Discrete Guided Modes

Ray approach

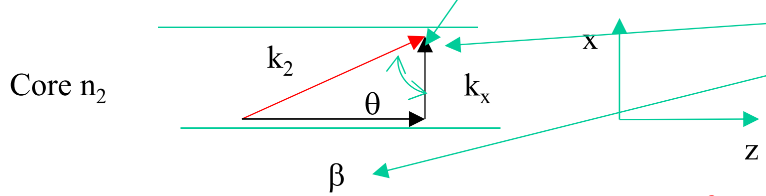
- The *transverse resonance condition* results in *discrete* values of the propagation constant β_m for guided modes identified by the mode number m.
- Although the critical angles, **θ_{c2} and θ_{c3}, do not depend on the polarization of the wave**, the phase shifts, **$\varphi_2(\theta)$ and $\varphi_3(\theta)$**, caused by the **internal reflection** at a given angle θ **depend on the *polarization***.
- Therefore, ***TE and TM waves have different solutions for the transverse resonance condition***, resulting in different β_m and different mode characteristics for a given mode number m.

Mode and incident angle

- For a given polarization, solution of the transverse resonance condition yields a smaller value of θ and a correspondingly smaller value of β for a larger value of m. Therefore, $\beta_0 > \beta_1 > \beta_2 > \ldots$
- The guided mode with m = 0 is called the ***fundamental mode*** and those with m ≠ 0 are ***higher-order modes***.

m=0, θ=90° kind of true but still depending on φ_2 and φ_3

$$2k_1 d\cos\theta + \varphi_2(\theta) + \varphi_3(\theta) = 2m\pi$$



Transverse component $k_x = (n_2\omega/c)\cos\theta$

Longitudinal component $\beta = (n_2\omega/c)\sin\theta$

"k-vector triangle" $\beta^2 + k_x^2 = (n_2\omega/c)^2$

k_x goes up with increasing m and β goes down because $k_1 d$ are fix so when m increases, cos θ has to increase because $2m\pi$ is increasing

Ray Patterns in the Three-Layer Planar Waveguide

In the guided region, $E \sim \sin(k_x + \gamma)$

$$\beta_m^2 + k_x^2 = k^2 n_2^2$$

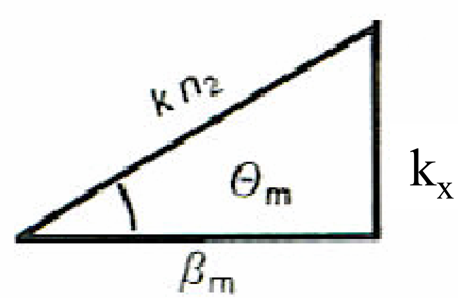


Fig. 2.9. Geometric (vectorial) relationship between the propagation constants of an optical waveguide

For the m-th mode, $\theta_m = \tan^{-1} \frac{k_x}{\beta_m}$

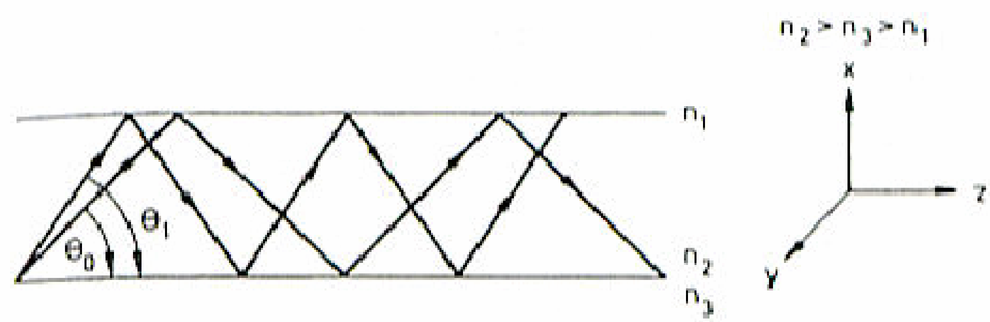
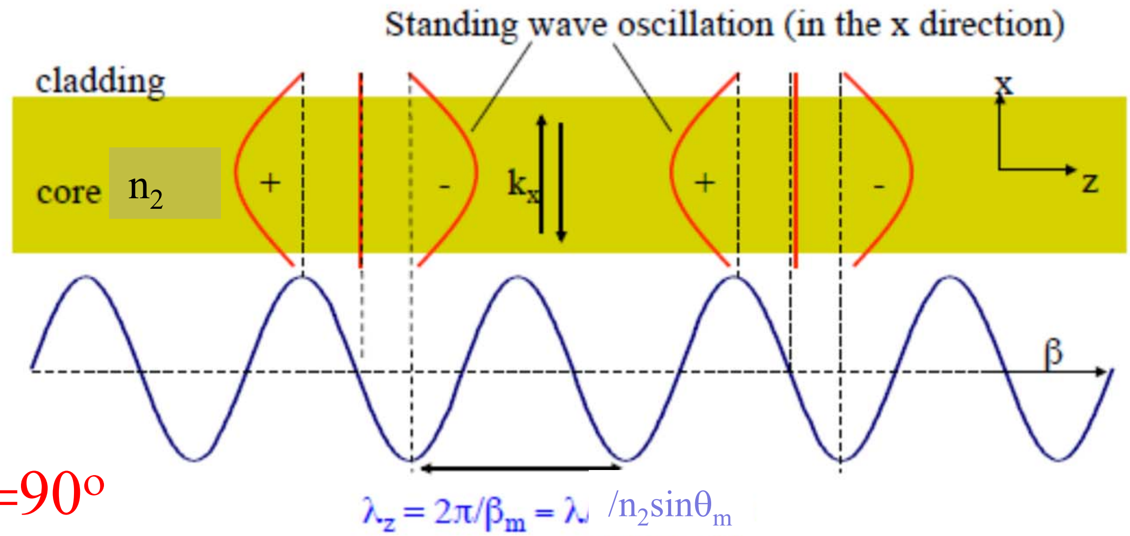


Fig. 2.8. Optical ray pattern within a multimode planar waveguide

Lower-order mode has smaller θ_m and larger β_m (propagating faster!) phasevelocity

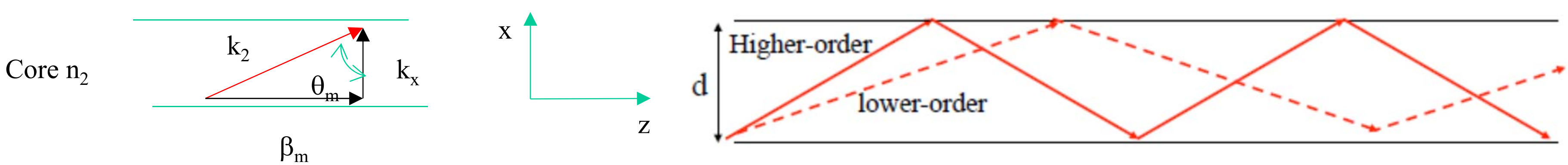
Qualitative Picture of A Waveguide Mode

Wave explanation



Look at how wave is propating inside waveguide in x and z direction over time

m=0, θ=90°



• The stable field distribution in the transverse direction with only a periodic longitudinal dependence is known as a waveguide mode.

Trigonometric Functions in Terms of Exponential Functions

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i} \qquad \csc x = \frac{2i}{e^{ix} - e^{-ix}}$$

$$\cos x = \frac{e^{ix} + e^{-ix}}{2} \qquad \sec x = \frac{2}{e^{ix} + e^{-ix}}$$

$$\tan x = \frac{e^{ix} - e^{-ix}}{i\left(e^{ix} + e^{-ix}\right)} \qquad \cot x = \frac{i\left(e^{ix} + e^{-ix}\right)}{e^{ix} - e^{-ix}}$$

Exponential Function vs. Trigonometric and Hyperbolic Functions

$$e^{ix} = \cos x + i \sin x \qquad e^{x} = \cosh x + \sinh x$$

Hyperbolic Functions in Terms of Exponential Functions

$$\sinh x = \frac{e^{x} - e^{-x}}{2} \qquad \operatorname{csch} x = \frac{2}{e^{x} - e^{-x}}$$

$$\cosh x = \frac{e^{x} + e^{-x}}{2} \qquad \operatorname{sech} x = \frac{2}{e^{x} + e^{-x}}$$

$$\tanh x = \frac{e^{x} - e^{-x}}{e^{x} + e^{-x}} \qquad \coth x = \frac{e^{x} + e^{-x}}{e^{x} - e^{-x}}$$

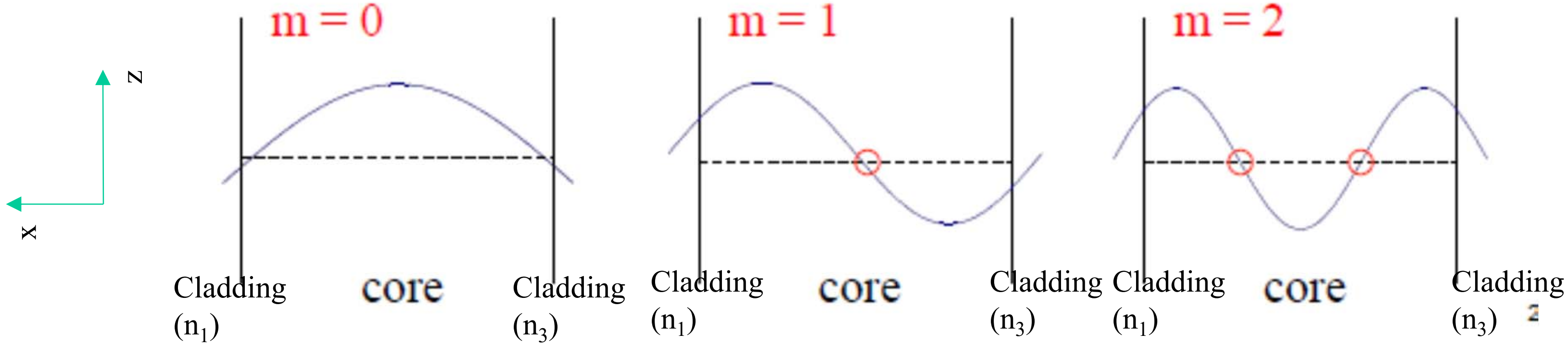
Wave explanation

Discrete Waveguide Mode

- Because m can assume only integral values, only certain *discrete* values of $\theta = \theta_m$ can satisfy the resonance condition.

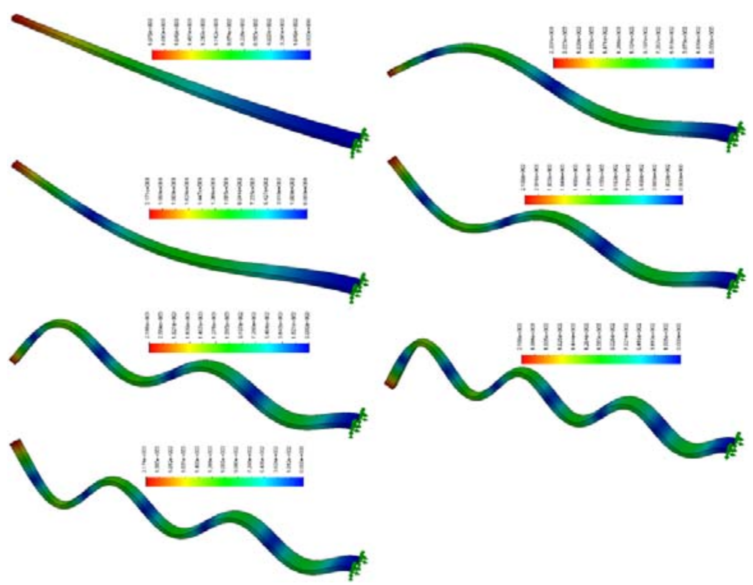
- This results in *discrete values of the propagation constant* β_m for guided modes identified by the mode number m.

- The guided mode with m = 0 is called the *fundamental mode* and those with m = 1, 2, … are *higher-order modes.*

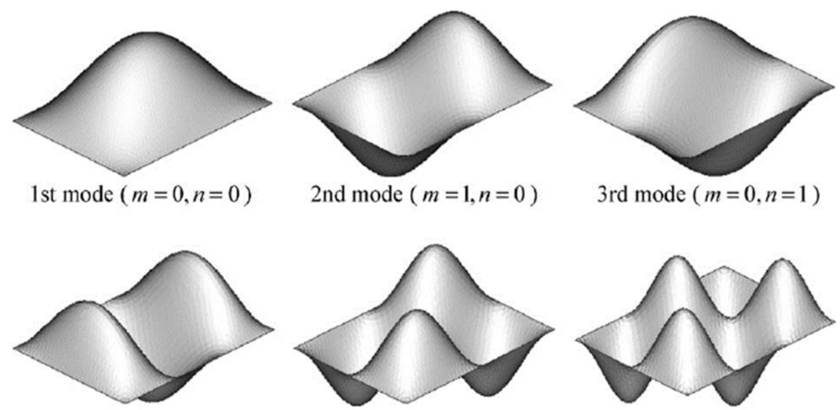


Mechanical Vibration Analogy
Fix-Fix Boundary Mode Shape

Recall



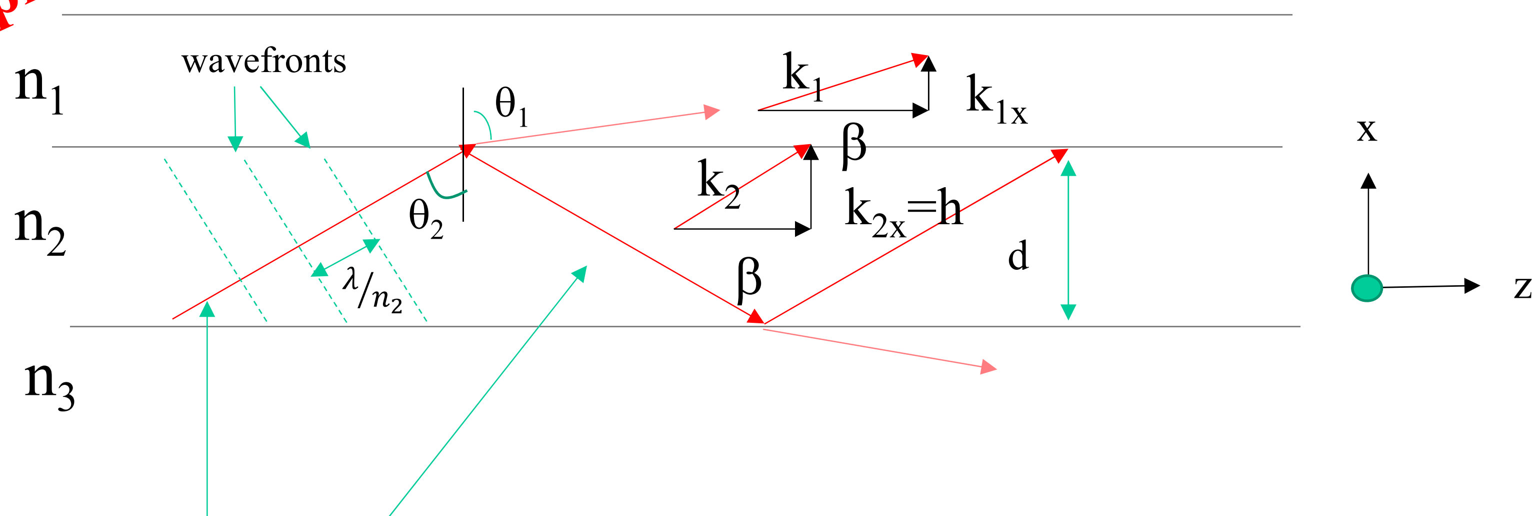
Fix-fix resonant mode in 1 D structure



Fix-fix resonant mode in 2 D structure

Plane Wave Propagating in a Planar Waveguide (TM mode)

Wave explanation



Upper field $E_u = E_o e^{-jk_u \bullet r} = E_o e^{-jh\,x - j\beta z}$

Down field $E_d = E_o e^{-jk_d \bullet r} = E_o e^{jh\,x - j\beta z}$

Where $r = x\hat{x} + z\hat{z}, k_u = -h\hat{x} - \beta\hat{z}, k_d = h\hat{x} - \beta\hat{z}$

Plane Wave Representation (TM mode)

Wave explanation

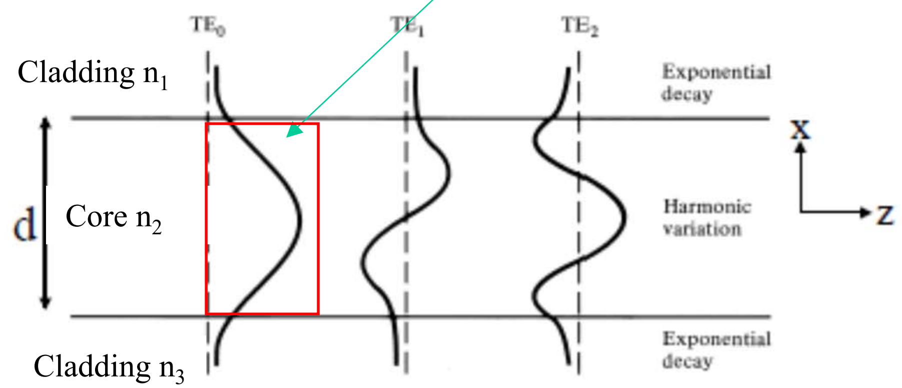
- The mode field is found through the superposition of the plane wave components. Assuming that these are in phase,

In core region

$$E_t = E_u + E_d = E_o e^{-jk_d \bullet r} = E_o \cos(hx) e^{-j\beta z}$$

- In *real instantaneous* form,

$$E_t(r,t) = E_u + E_d = E_o e^{-jk_d \bullet r} = 2E_o \cos(hx)\cos(\beta z - \omega t)$$



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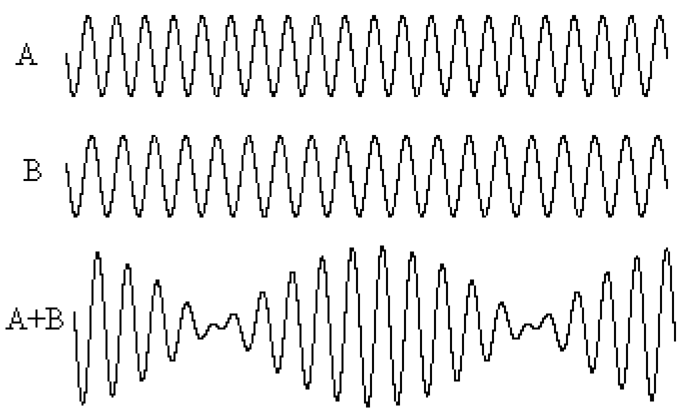


$$\sin A+\sin B = 2\sin(A+B)/2 * \cos(A-B)/2$$

Let $A = k_1x+\omega_1t+ \phi_1 \quad k_1 = 2\pi n_1/\lambda$

$B = k_2x+\omega_1t+ \phi_2 \quad k_2 = 2\pi n_2/\lambda$

Light phenomena is just a superposition of waves with different wave lengths, phase, etc. (ambient light)

Surface waves and the reflective phase shift

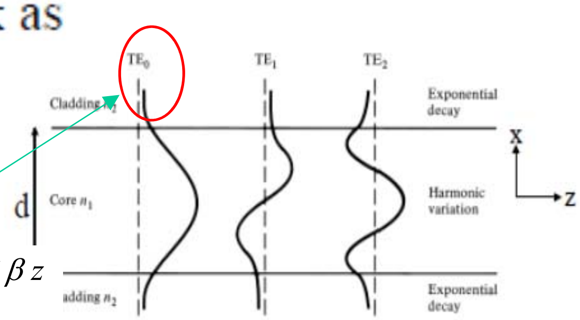
- An electric field component in the upper cladding region assumes the phasor form:

$$E_1 = E_{1o} e^{-jk_1 \bullet r} = E_{1o} e^{-jk_{1x}x} e^{-j\beta z}$$

- When $\theta_2 > \theta_{c2}, k_{1x}$ becomes *imaginary* and can be expressed in terms of a *real attenuation coefficient* κ as

$$\theta_2 > \theta_{c2} \Rightarrow k_{1x} = j\kappa$$

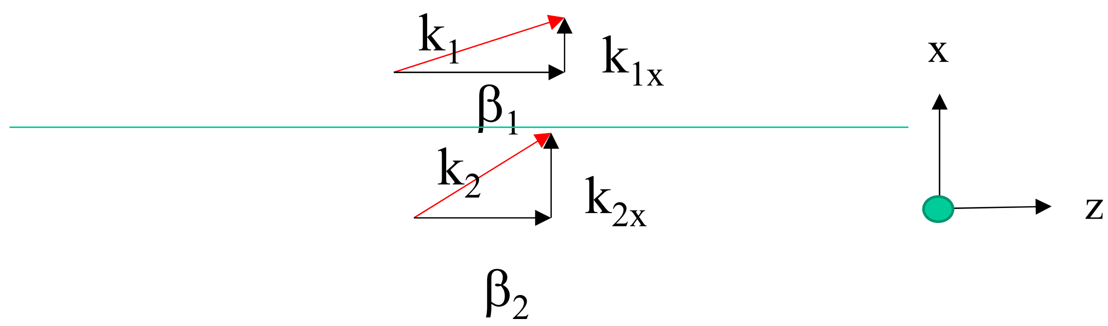
$$E_1 = E_{1o} e^{-jk_1 \bullet r} = E_{1o} e^{-j\kappa x} e^{-j\beta z}$$



- This is the phasor expression for a *surface* or *evanescent* wave – propagates only in the z direction, decays in the direction normal to the interface.

Wave optics

Phase-matching at an interface



As the spatial rate of change of phase at the boundary (or the projection of the wavefront propagation) on the n_2 side must match with that on the n_3 side, we have $\beta_2 = \beta_1 = \beta$. This condition is known as phase matching condition which allows coupling of oscillating field between the two media.

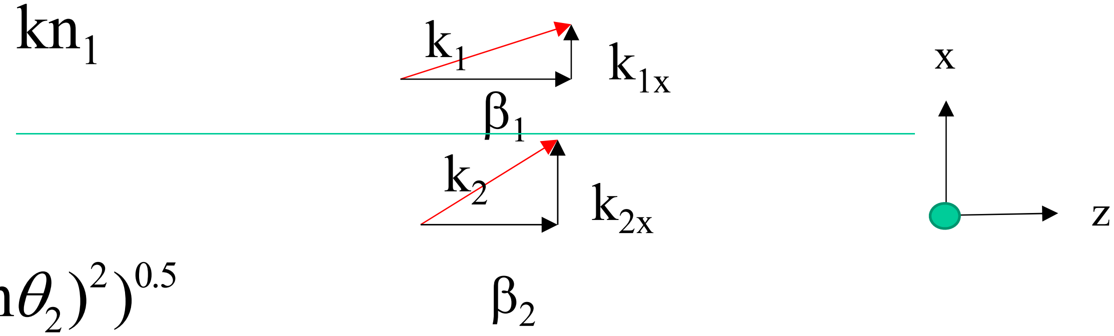
$$\beta_2 = \beta_1 \Rightarrow n_2 \sin\theta_2 = n_1 \sin\theta_1 \quad \text{(Snell's law)}$$

Wave optics

Evanescent field in total internal reflection

- Phase matching at TIR $\theta_2 > \theta_c$ (i.e. $\sin\theta_2 > n_1/n_2$)

$$\beta_2 = kn_2\sin\theta_2 = \beta_1 > kn_1$$



- k vector triangle in n_1

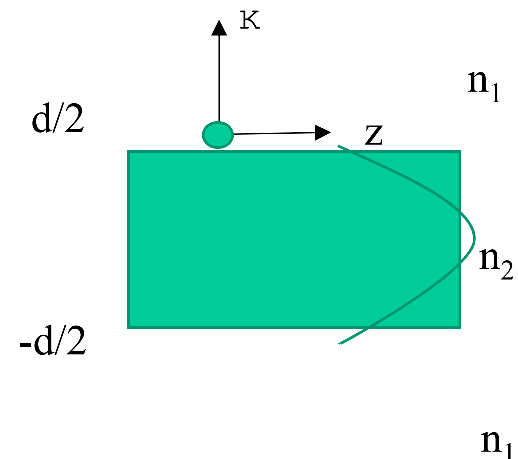
$$k_1 = ((n_1 k)^2 - (n_2 k \sin\theta_2)^2)^{0.5}$$

$$k_1 = j((n_2 k \sin\theta_2)^2 - (n_1 k)^2)^{0.5}$$

$$= j((n_2 k \sin\theta_2)^2 - (n_1 k)^2)^{0.5} = j\kappa$$

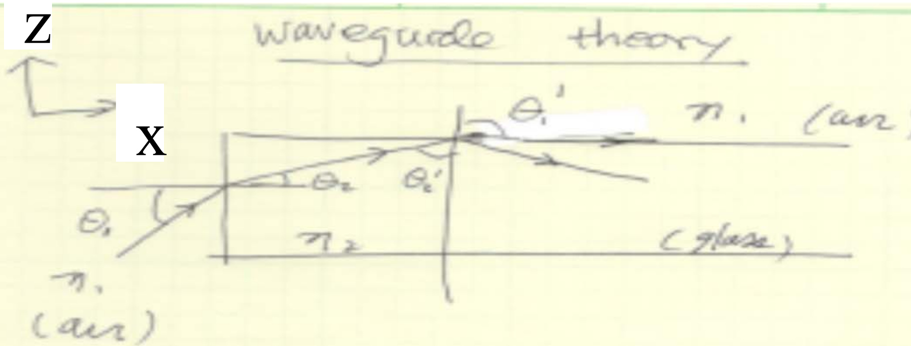
- Evanescent field in the transverse direction

$$E \propto e^{-\kappa x} e^{-j\beta z + \omega t}$$



κ is α like before

waveguide theory



$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

$$n_2 \sin\theta_2' = n_1 \sin\theta_1'$$

$\theta_1' = 90°$ ∴ total reflection occurs

then $\theta_2' = \theta_{critical}$

no wave travel outside the glass substrate, therefore glass substrate becomes a wave guide.

However if $\theta_2' > \theta_{critical}$

Then

$$n_2 \sin\theta_2' = n_1 \underbrace{\sin\theta_1'}_{>1}$$

$$1 = \underset{>1}{\sin^2\theta_1'} + \underset{<0}{\cos^2\theta_1'}$$

since $k_1^2 = k_{1z}^2 + k_{1x}^2$

$k_{1z}^2 = k_1^2 - k_{1x}^2$

$k_1^2 - (k_1 \sin\theta_1')^2$ $\Rightarrow k_{1z}^2 < 0$

$$k_1^2 = k_1^2\sin^2\theta_1' + k_1^2\cos^2\theta_1'$$

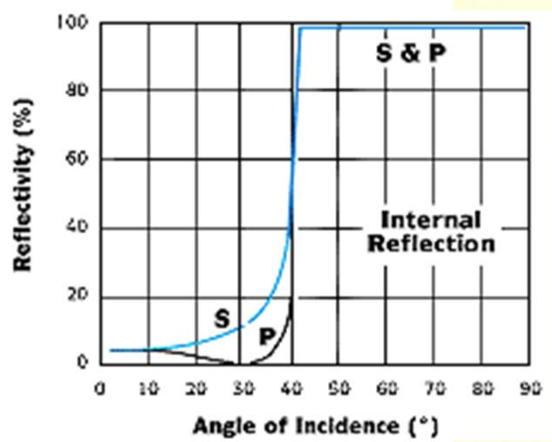
k_{1z} = imaginary

$$E e^{j k_{1x} x + j k_{1z} z}$$

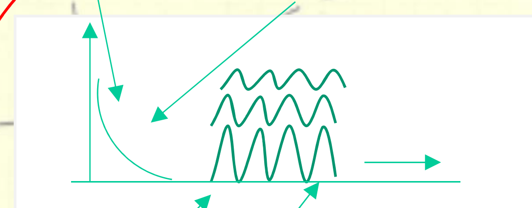
$$\mathrm{Re}\left\{ \left(E e^{j k_{1x} X}\right) e^{j \overset{\text{imaginary}}{k_{1z}} z} \right\}$$

$$= E_0 \cos k_{1x} X \; e^{-j \alpha_z z}$$

where α_z = imaginary k_{1z} = imaginary



$e^{-j \alpha_z z}$



$\cos k_{1x} X$

Evanescent wave

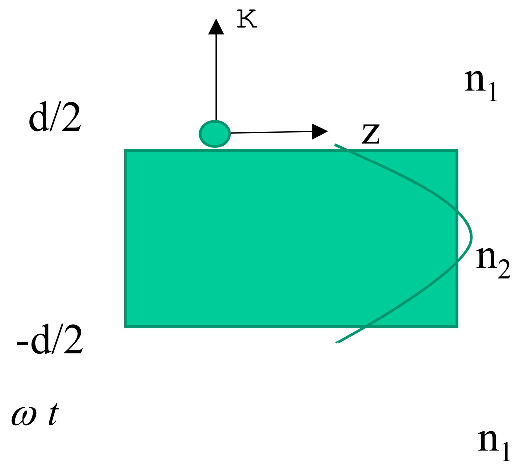
use for sensing and wave coupling

$\cos k_{1x} x$ at different z

Evanescent fields in the waveguide cladding

Evanescent wave outside the waveguide core decay exponentially
With an attenuation factor given by

$$\kappa = ((n_2 k \sin\theta_2)^2 - (n_1 k)^2)^{0.5}$$



In the upper layer ($x \geq d/2$)

$$E = E_1 e^{-\kappa(x-d/2)} e^{-j\beta z+\omega t}$$

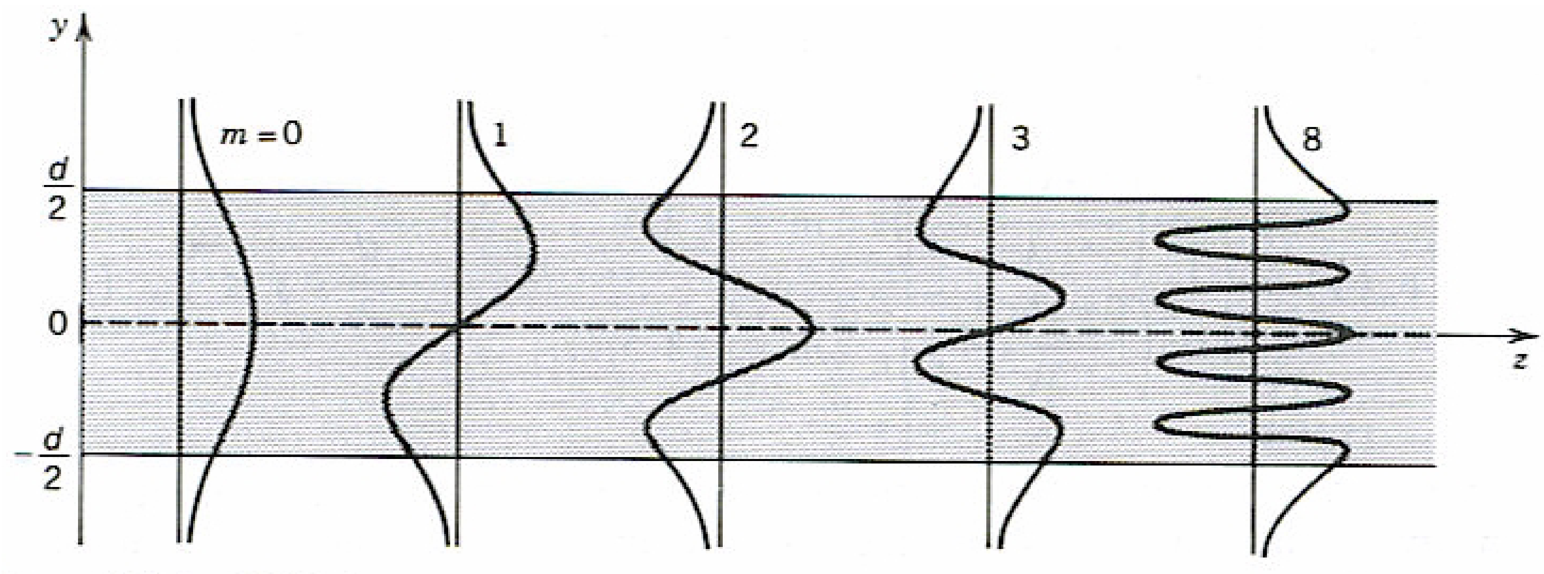
In the lower layer ($x \leq -d/2$)

κ is α like in hand derivatipon

$$E = E_1 e^{\kappa(x+d/2)} e^{-j\beta z+\omega t}$$

Where E_1 peak value of te electric field at lower (x=-d/2) and upper (x=d/2) boundaries.

Guided Modes in a Planar Waveguide



m: Mode order

Only discrete values of are allowed in a waveguide.

Waveguide Mode observed at the end of a Slab Waveguide

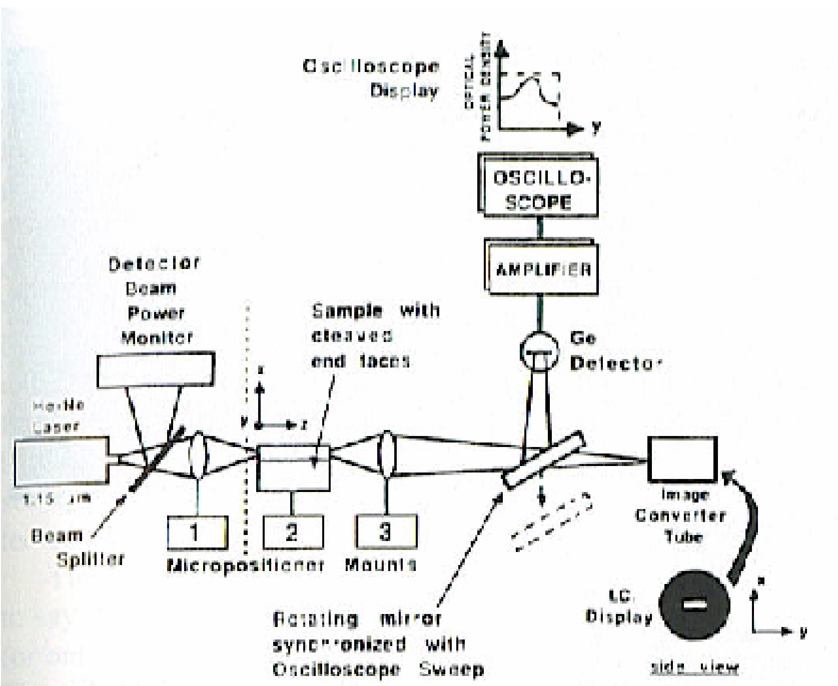
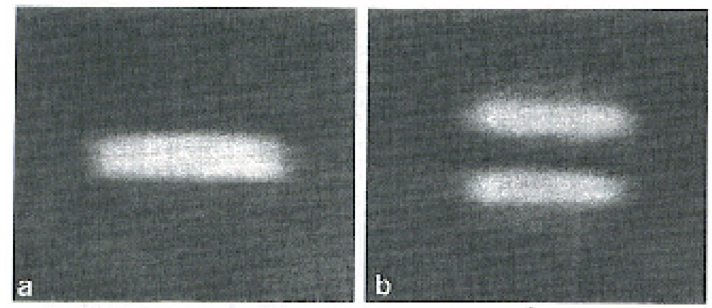


Fig. 2.3. Diagram of an experimental set-up than can be used to measure optical mode shapes [2.9]



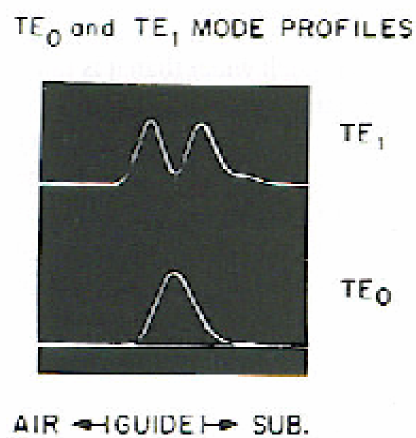


Fig. 2.5. Optical mode shapes are measured using the apparatus of Fig. 2.3. The waveguide in this case was formed by proton implantation into a gallium arsenide substrate to produce a 5 μm thick carrier-compensated layer [2.12]

Examples of Optical fiber modes

70 OPTICAL FIBERS WITH SINGLE-STEP INDEX PROFILES

3.7. LP MODES 71

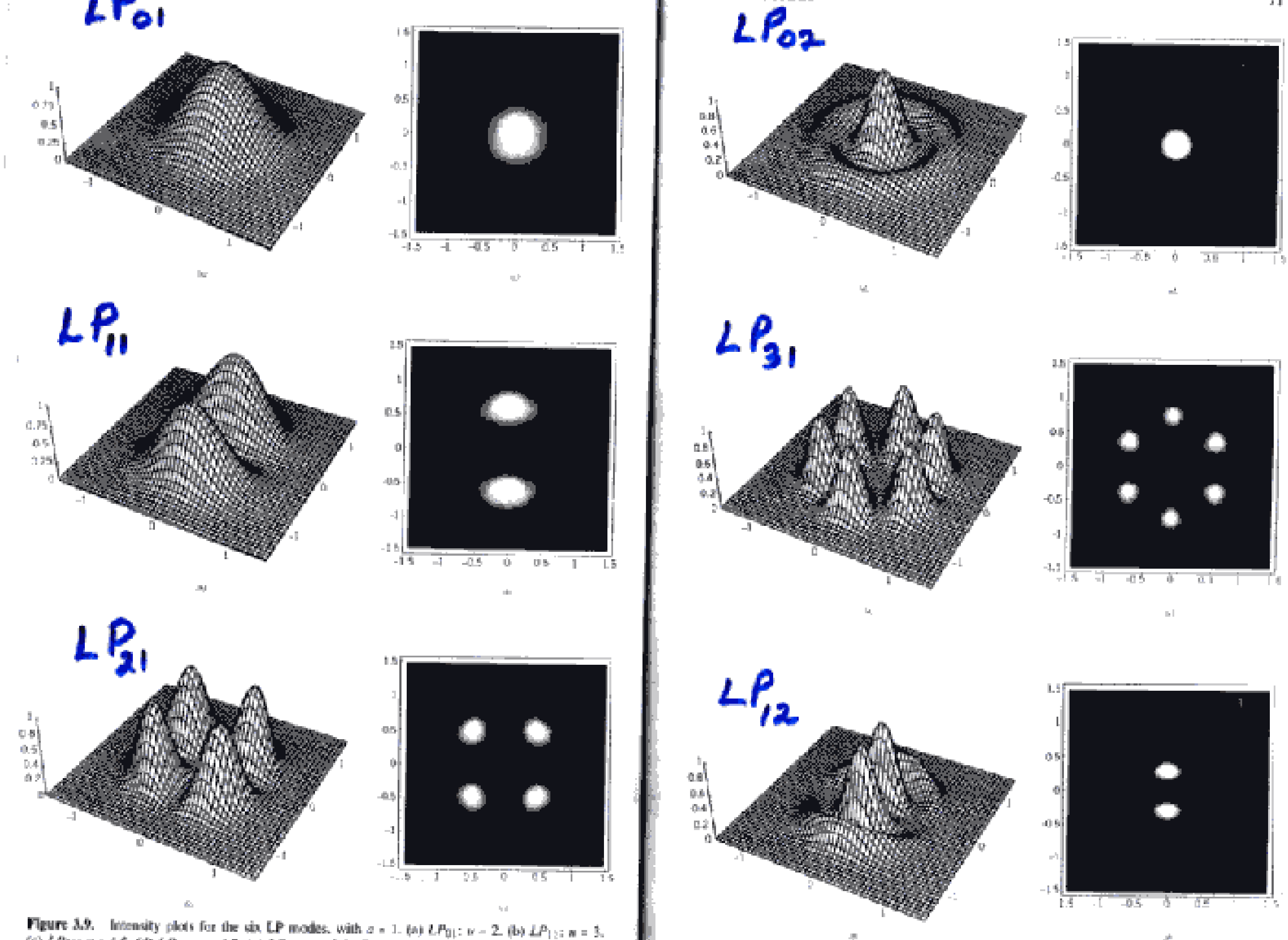


Figure 3.9. Intensity plots for the six LP modes, with $a = 1$. (a) LP_{01}: $v = 2$. (b) LP_{11}: $v = 3$. (c) LP_{21}: $v = 4.5$. (d) LP_{02}: $v = 4.5$. (e) LP_{31}: $v = 5.5$. (f) LP_{12}: $v = 6.3$.

Figure 3.9. (*Continued*)

Things covered in this section

- k vector in propagating and confined direction (x, z)
- Mode field in confined direction (x)
- n_{eff}
- Dispersion equation in confine direction (x)

Recall in free space

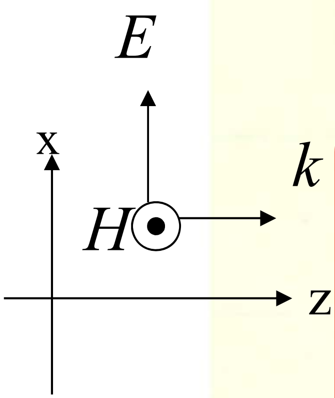
We consider a simple solution where E field is parallel to the X axis & is function of z coordinate only, the wave equation then becomes

$$\frac{\partial^2 E_x}{\partial z^2} + \frac{\omega^2}{c^2} E_x = 0 \quad \text{— (7)}$$

A solution to the above differential equation is

$$E = \hat{x} E_o e^{-jkz} \quad \text{— (8)}$$

(in phasor form because of time harmonic function)



Substitute above equation into wave equation (eq 7)

Dispersion equation

$$\left(-k^2 + \frac{\omega^2}{c^2}\right) E = 0 \Rightarrow k^2 = \frac{\omega^2}{c^2} \quad \text{(dispersion relation)}$$

$$k_o^2 n^2 = \omega^2 / c^2$$

$$k = \frac{\omega}{c} = \frac{2\pi f}{c} \quad (k = \text{wave number or propagation constant})$$

Let's transform the solution for the wave equation into real space & time (assume time harmonic field)

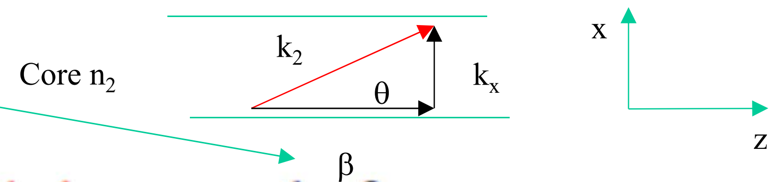
W W

Waveguide effective index

- We can define the waveguide phase velocity v_p as

Based on Dispersion equation, only Z direction is propagating so

$$v_p = \omega / \beta$$



- We now define an effective refractive index n_{eff} as the free-space velocity divided by the waveguide phase velocity.

$$n_{eff} = c / v_p$$

$$\text{Or} \quad n_{eff} = c\beta / \omega = \beta / k$$

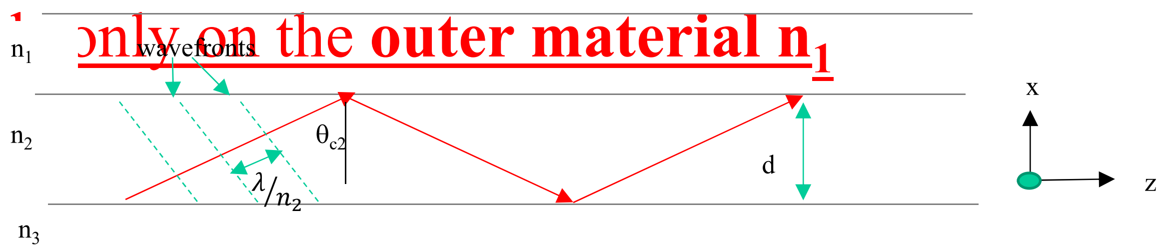
$$\Rightarrow n_{eff} = n_2 \sin\theta$$

- The effective refractive index is a key parameter in *guided propagation*, just as the refractive index is in unguided wave travel.

For wave guiding at n_2-n_1 interface, we see that **$n_1 < n_{eff} < n_2$ (n1=n3 or n1>n3)** Since $\beta_2 = \beta_1 \Rightarrow n_2 \sin\theta_2 = n_1 \sin\theta_1 = n_{eff}$

At $\theta_2 = 90^o$, $n_{eff} = n_2$ => **a ray traveling parallel to the slab** (core) has a effective index that depends on the guiding medium alone.

At $\theta_2 = \theta_c$, $\theta_1 = 0^o$, $n_{eff} = n_1$ => the effective index for critical angle rays depend only on the **outer material n_1**



The effective refractive index changes with the wavelength (i.e. dispersion) in a way related to that the bulk refractive index does.

The wavelength as measured in the waveguide is

$$\lambda_{waveguide} = \lambda / n_{eff}$$

Things covered in this section

- k vector in propagating and confined direction (x, z)
- Mode field in confined direction (x)
- n_{eff}
- Dispersion equation in confine direction (x)

Dispersion Equation in Waveguide

Dispersion Equation in Waveguide

- Looking at Phase term to figure out guided modes

Things covered in this section

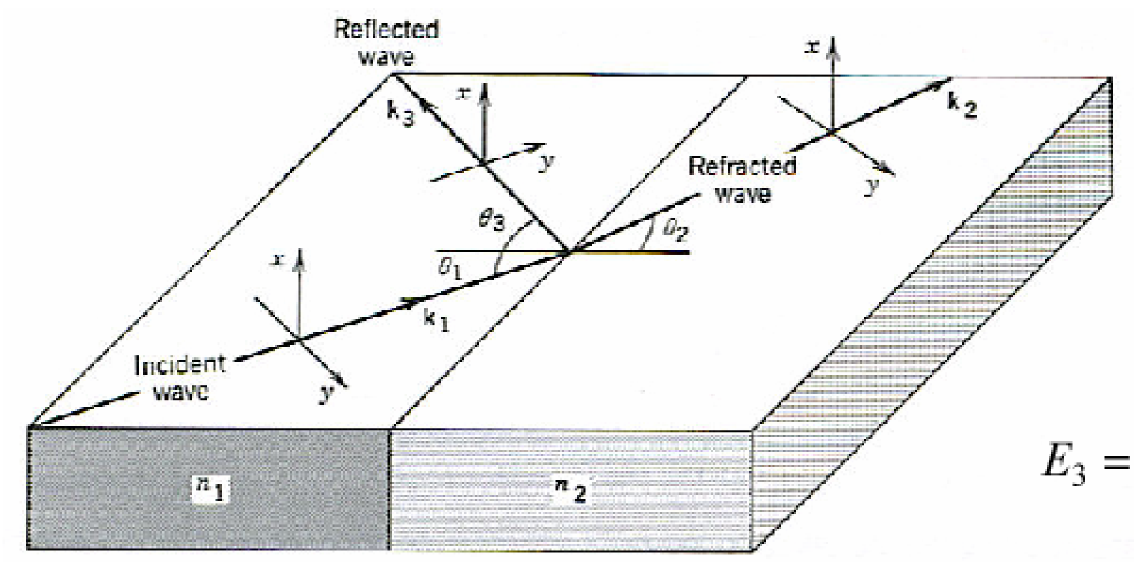
- Derive dispersion equation with TE TM reflection coefficient
- Dimensionless parameters: normalized frequency, normalized guided index, asymmetry, dispersion equation with above parameters, mode number, effective film thickness

Reflection coefficient at BC

Looking at :

- How phase shifts ϕ_2 and ϕ_3 associated with the internal reflections in the lower and upper interfaces.
- How these phase shifts can be obtained from the phase angle of r_s (reflection coefficient) for a TE wave (s wave) and that of r_p for a TM wave (p wave) for a given $\theta > \theta_{c3}$, θ_{c2} and θ_m.

Reflection and Refraction



$$E_3 = rE_1, \quad E_2 = tE_1$$

For TE wave: $$r_{TE} \quad \frac{n_1 \cos\theta_1 + n_2 \cos\theta_2}{n_1 \cos\theta_1 + n_2 \cos\theta_2} \qquad t_{TE} = 1 + r_{TE} \qquad r_s = r_{TE}$$

For TM wave: $$r_{TM} \quad \frac{n_2 \cos\theta_1 - n_1 \cos\theta_2}{n_2 \cos\theta_1 + n_1 \cos\theta_2} \qquad t_{TM} = \frac{n_1}{n_2}(1 + r_{TM}) \qquad r_p = r_{TM}$$

Recall

$$r_{TE} = |r_{TE}| \exp(j\phi_{TE}), \quad r_{TM} = |r_{TM}| \exp(j\phi_{TM})$$

Total Internal Reflection for TE Wave

$$\tan\frac{\phi_{TE}}{2} = \frac{\sqrt{\sin^2\theta_1 - \sin^2\theta_c}}{\cos\theta_1} = \frac{\sqrt{n_1^2\sin^2\theta_1 - n_2^2}}{n_1\cos\theta_1}$$

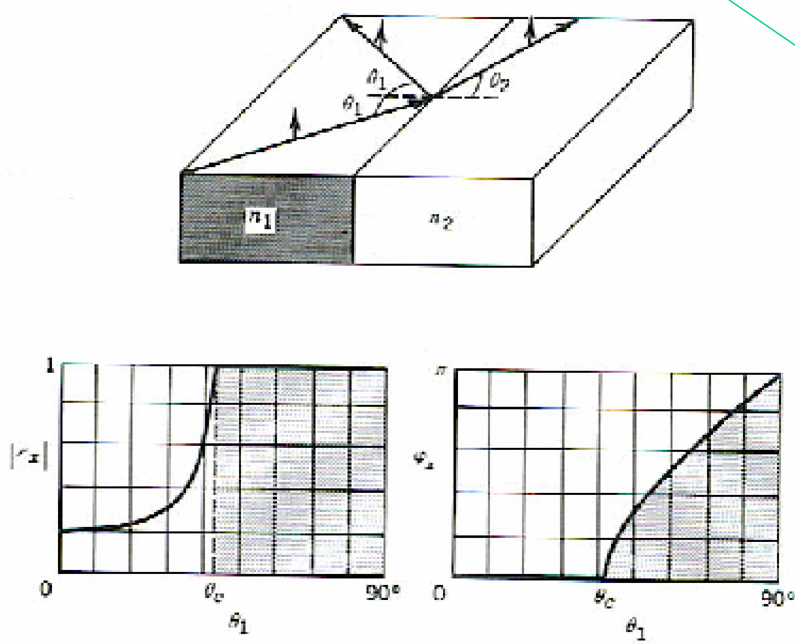


Figure 6.2-3 Magnitude and phase of the reflection coefficient for internal reflection of the TE wave ($n_1/n_2 = 1.5$).

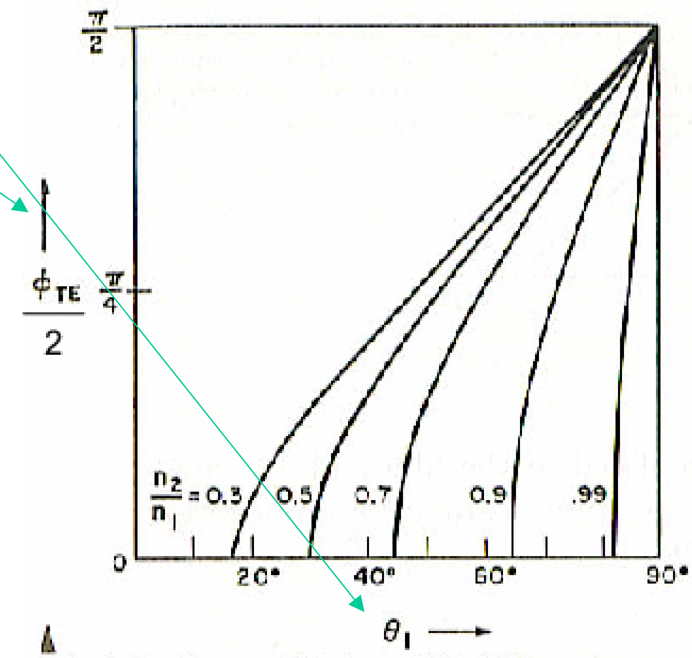


Fig. 2.3. Phase shift ϕ_{TE} of the TE mode as a function of the angle of incidence θ_1

Total Internal Reflection for TM Wave

$$\tan\frac{\phi_{TM}}{2} = \frac{\sqrt{\sin^2\theta_1 - \sin^2\theta_c}}{\cos\theta_1 \sin^2\theta_c} = \frac{n_1^2}{n_2^2}\frac{\sqrt{n_1^2\sin^2\theta_1 - n_2^2}}{n_1\cos\theta_1}$$

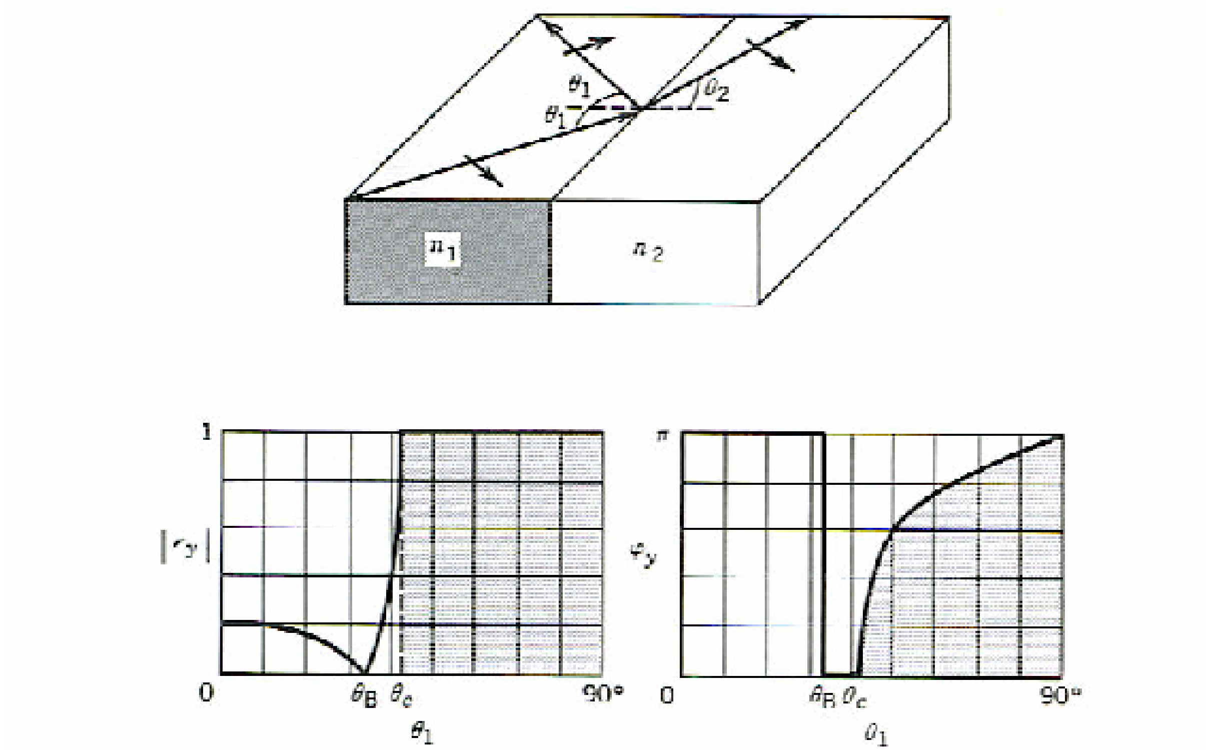


Figure 6.2-5 Magnitude and phase of the reflection coefficient for internal reflection of the TM wave ($n_1/n_2 = 1.5$).

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Derivation for r_{TE} and r_{TM}

→ Reflection & Refraction:

TE: $r_{TE} = \frac{n_1 \cos\theta_1 - n_2 \cos\theta_2}{n_1 \cos\theta_1 + n_2 \cos\theta_2}$ $t_{TE} = 1 + r_{TE}$

$r_{TM} = \frac{n_2 \cos\theta - n_1 \cos\theta_1}{n_1 \cos\theta_1 + n_2 \cos\theta_2}$ $t_{TM} = \frac{n_1}{n_2}(1 + r_{TM})$

complex $r_{TE} = |r_{TE}| e^{j\phi_{TE}}$ $r_{TM} = |r_{TM}| e^{j\phi_{TM}}$

magnitude

Phaseshift due to reflection

use $\cos\theta_2 = \sqrt{1 - \sin^2\theta_2} \Rightarrow [1 - (\frac{n_1}{n_2})^2 \sin^2\theta_2]^{1/2}$

can be > 1

* TE wave Total internal reflection

$n_1 > n_2, \quad \theta_1 > \theta_c$

$n_1 \;|\; n_2$

$$r_{TE} = \frac{n_1\cos\theta_1 - n_2\cos\theta_2}{n_1\cos\theta_1 + n_2\cos\theta_2}$$

$$= \frac{A + jB}{A - jB}$$

Remember $\theta_1 = \theta_c, \theta_2 = 90$ and $sin\theta_1 = \frac{n_2 sin\theta_2}{n_1} = \frac{n_2}{n_1}$

$$\cos\theta_2 = \sqrt{1 - \left(\frac{n_1}{n_2}\right)^2 \sin^2\theta_1} = -j\sqrt{\frac{\sin^2\theta_1}{\sin^2\theta_c} - 1}$$

$\left(\frac{n_1}{n_2}\right)^2 = \frac{1}{\sin^2\theta_c}$

Look at next page and see why $\cos\theta$ is imaginary

Remember $n_2^2\cos^2\theta_2 = n_2^2 - n_1^2\sin\theta_1$ because $n_1\sin\theta_1 = n_2\sin\theta_2$ (phase matching condition)

and $n_2^2(\sin^2\theta_2 + \cos^2\theta_2) = n_2^2$

$$A = n_1\cos\theta_1$$

$$B = \sqrt{\frac{\sin^2\theta_1}{\sin^2\theta_c} - 1} \cdot n_2$$

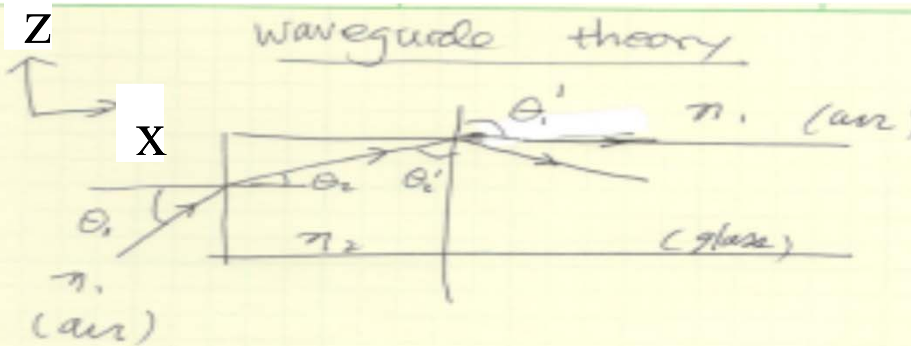
$$= \frac{1}{A^2 + B^2}\left(A^2 - B^2 + 2jAB\right)$$

$$= |r_{TE}|\, e^{j\phi_{TE}} = |r_{TE}|\left(\cos\phi_{TE} + j\sin\phi_{TE}\right)$$

$|r_{TE}| = 1$ (TIR)

Total internal reflection

waveguide theory



$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

$$n_2 \sin\theta_2' = n_1 \sin\theta_1'$$

$\theta_1' = 90°$ ∴ total reflection occurs

then $\theta_2' = \theta_{critical}$

no wave travel outside the glass substrate, therefore glass substrate becomes a wave guide.

However if $\theta_2' > \theta_{critical}$

Then

$$n_2 \sin\theta_2' = n_1 \underbrace{\sin\theta_1'}_{>1}$$

$$1 = \underbrace{\sin^2\theta_1'}_{>1} + \underbrace{\cos^2\theta_1'}_{<0}$$

since $k_1^2 = k_{1z}^2 + k_{1x}^2$

$$k_{1z}^2 = k_1^2 - k_{1x}^2 = k_1^2 - (k_1 \sin\theta_1')^2 \Rightarrow k_{1z}^2 < 0$$

$$k_1^2 = k_1^2\sin^2\theta_1' + k_1^2\cos^2\theta_1'$$

$\phi_{c \text{ or } sTE} = \tan\frac{\phi_{TE}}{2} = \frac{\sin\phi_{TE}}{1+\cos\phi_{TE}} = \frac{2AB}{2A^2} = \frac{B}{A}$

(10)

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phase shift @ interface TE mode

$$= \frac{\frac{n_2}{n_1}\sqrt{\frac{\sin^2\theta_1}{\sin^2\theta_c}-1}}{\cos\theta_1} = \frac{\sqrt{n_1^2\sin^2\theta_1 - n_2^2}}{n_1\cos\theta_1}$$

TM wave

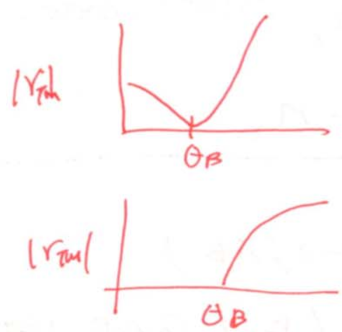
$$r_{TM} = \frac{n_2\cos\theta_1 - n_1\cos\theta_2}{n_2\cos\theta_1 + n_1\cos\theta_2}$$

$\phi_{c \text{ or } s\,TM} = \tan\frac{\phi_{TM}}{2} = \frac{\frac{n_1}{n_2}\sqrt{\frac{\sin^2\theta_1}{\sin\theta_c}-1}}{\cos\theta_1}$

($\frac{n_2}{n_1}$ = $\sin\theta_c$)

phase shift @ interface TM mode

$$= \frac{n_1^2}{n_2^2}\frac{\sqrt{n_1^2\sin^2\theta_1 - n_2^2}}{n_1\cos\theta}$$



Dispersion Equation

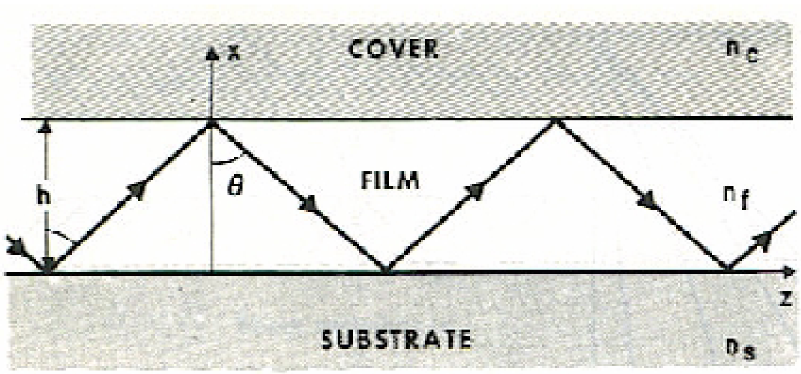
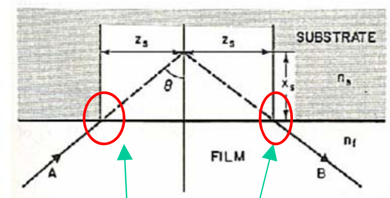


Fig. 2.5. Side-view of a slab waveguide showing wave normals of the zig-zag waves corresponding to a guided mode



Transverse resonance condition:

$$2kn_f h\cos\theta - 2\phi_c - 2\phi_s = 2m\pi$$

m : mode number

$kn_f h\cos$: phase shift for the transverse passage through the film

$2\phi_c = \phi_{TE,TM}$: phase shift due to total internal reflection from film/cover interface

$2\phi_s = \phi_{TE,TM}$: phase shift due to total internal reflection from film/substrate interface

Dispersion equation (β vs. ω):

$$kn_f h\cos\theta - \phi_c - \phi_s = m\pi$$

The phase shift can be representing the zig-zag ray at a certain depth into the confining layers 1 ad 3 before it is reflected (goos-Hanchen shifts- lateral shift)

Effective guide index $N = \beta/k = n_f \sin\theta$ $n_s < N < n_f$

2-3 Guided modes

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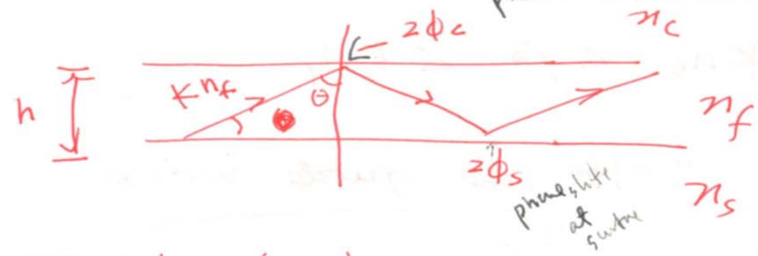


$\uparrow x$

Field of the wave $\propto \exp(-j\vec{k}\cdot\vec{r})$

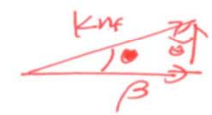
$$= \exp[-jkn_f\cdot(\pm x\cos\theta + z\sin\theta)]$$

Transverse resonance condition
(self consistency)

$$2kn_f h\cos\theta - 2\phi_c - 2\phi_s = 2m\pi$$

$2\phi_c = \phi_{TE,TM}$
$2\phi_s = \phi_{TE,TM}$

Dispersion equation



$$kn_f h\cos\theta - \phi_c - \phi_s = m\pi$$

(dispersion equation)

$\beta = kn_f\sin\theta$

$k = k(\omega)$ or $k(\lambda)$

$\beta\ (N_{effective})$ to ω, λ

↑ effective index light will see

Graphical Solution of the Dispersion Equation

Find θ with different h and λ **m=0**

Plotting left and right side of $kn_f h\cos\theta = \varphi_c + \varphi_s$

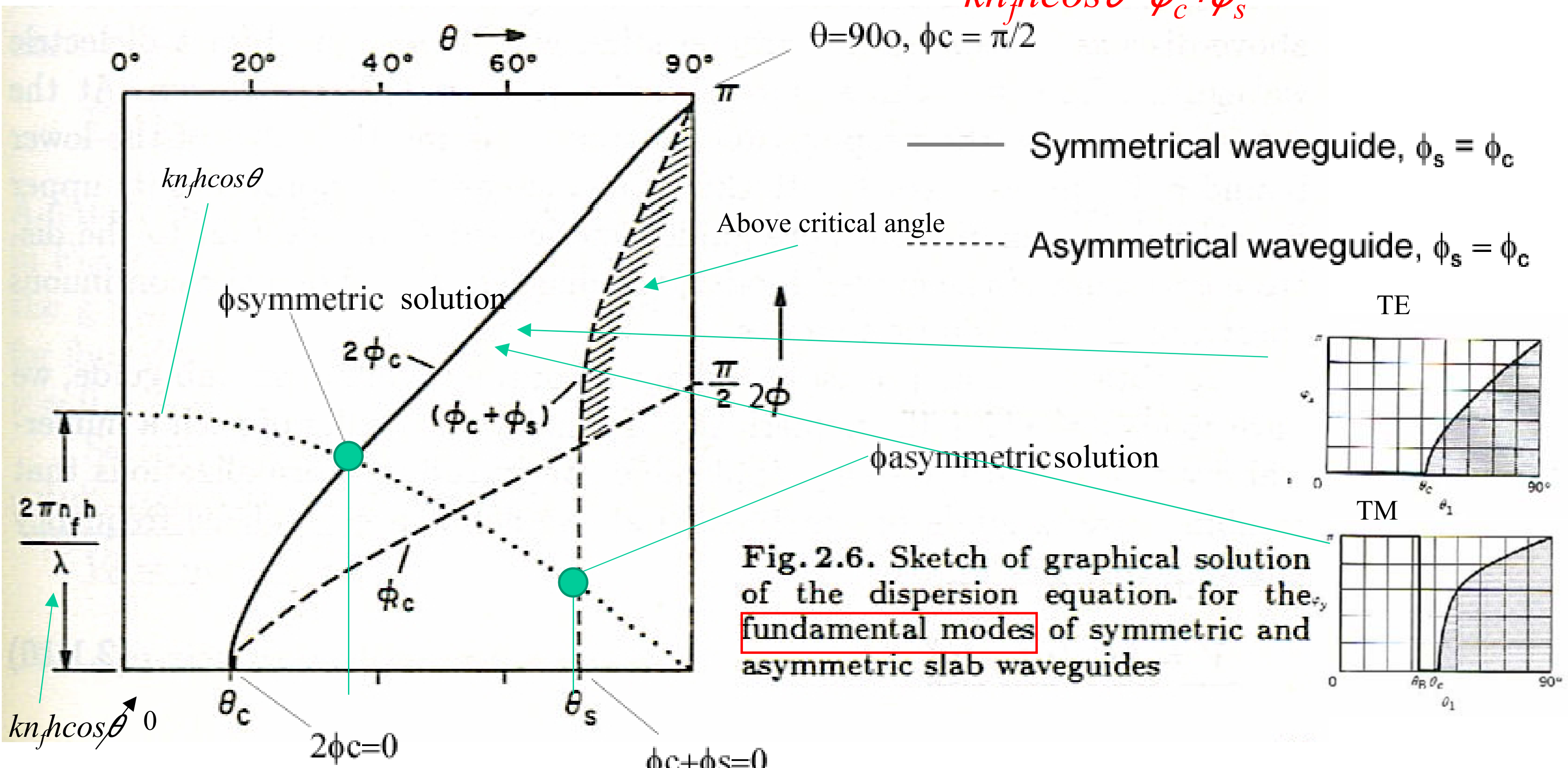


Fig. 2.6. Sketch of graphical solution of the dispersion equation for the fundamental modes of symmetric and asymmetric slab waveguides

For fundamental mode (m = 0), there is always a solution (no cut-off) for symmetrical waveguide.

Increasing h (and/or decreasing λ) will support more modes. (will show again in VB curve)

For guided modes :

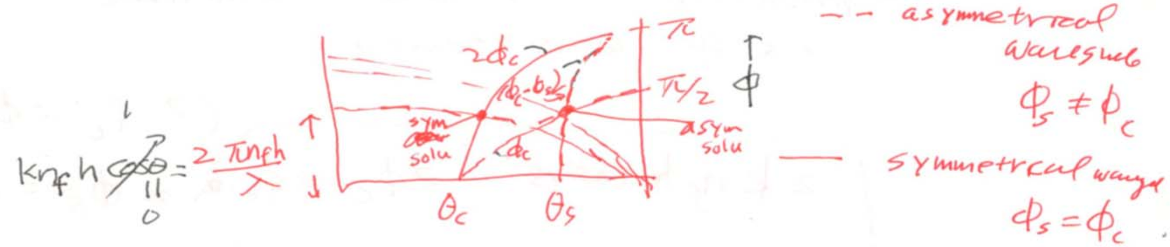
$k n_c < k n_s < \beta < k n_f$

Define "effective guide index" $\beta = k n_f \sin\theta$

$$\star \quad N = \beta / k = n_f \sin\theta$$

$\Rightarrow n_s < N < n_f$

graphical solution :



m=0

$$k_0 n_f h \cos\theta - \phi_c - \phi_s = m\pi$$

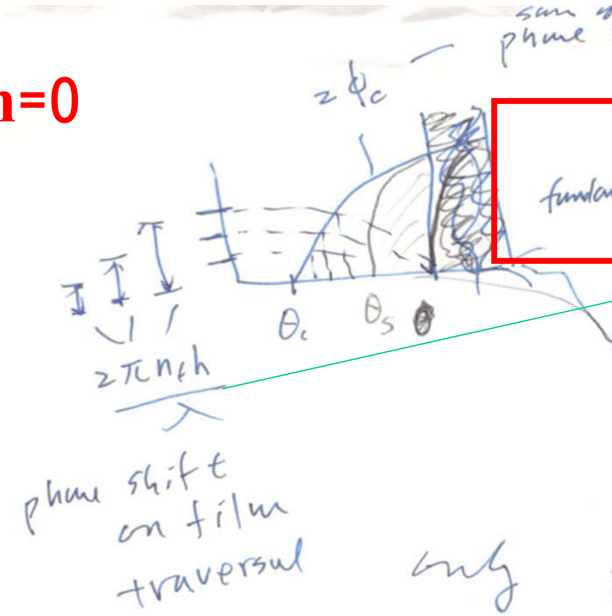
Assume $m = 0 \rightarrow$ $k n_f h \cos\theta = \phi_c + \phi_s$

$\star$ symmetrical waveguide there is always a solution (no cut off) for fundamental mode → Any ω will work TEM

Assume $m = 0 \rightarrow k n_f h \cos\theta = \phi_c + \phi_s$

$m \neq 0 \rightarrow k n_f h \cos - m\pi = \phi_c + \phi_s$

m=0



fundamental θ gets small as h/λ gets smaller

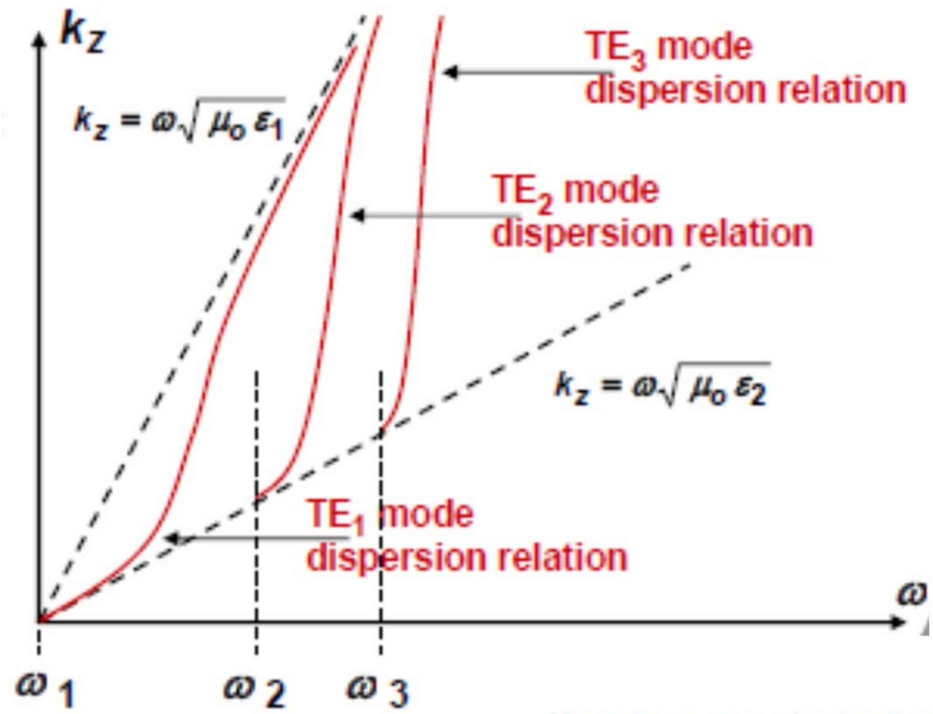
only portion of $\phi_s + \phi_c$ curve emphasized by shading is above critical angle θ_s of film substrate interferes

ϕ_s, ϕ_c are function of θ

as describe by $\tan \phi_{TE} = \dfrac{\sqrt{n_1^2 \sin^2 \theta_1 - n_2^2}}{n_1 \cos \theta_1}$

$\phi_{TM} = \ldots$

See how β_m relates to frequency



$$2kn_f h\cos\theta - 2\phi_c - 2\phi_s = 2m\pi$$

$$\beta^2 + k_x^2 = (n_2\omega/c)^2$$

How does one obtain dispersion curves?

(1) For a given frequency ω find k_x using:

$$\left\{\begin{matrix} \tan(k_x h) \\ -\cot(k_x h) \end{matrix}\right\} = \sqrt{\frac{\omega^2 \mu_o(\varepsilon_1 - \varepsilon_2)\, h^2}{(k_x h)^2} - 1}$$

(2) Then find k_z using:

$$k_z = \sqrt{\omega^2 \mu_o \varepsilon_1 - k_x^2}$$

Outline

- Waveguide structures and materials
- Field equations
- Waveguide modes, n_{eff}
- Wave equations in Waveguides
- ➡ Guided modes in symmetric and asymmetric slab waveguides
- General formalisms for step-index planar waveguides

Numerical Solution for Dispersion Relation (I)

Another common way to do this is to introduce the normalized parameters so that you don't have to worry about the dimension

Define:

Normalized frequency and film thickness

$$V = kh\sqrt{n_f^2 - n_s^2}$$

Normalized guide index

$$b = \frac{N^2 - n_s^2}{n_f^2 - n_s^2}$$

Know V find b and From b you can find N and from N your incident angle
Or modes

b = 0 at cut-ooff (N = n_s), and approaches 1 as N -> n_f.
(substrate radiation mode, m>=0) (guide mode, m=0)

Measure for the asymmetry

$$a = \frac{n_s^2 - n_c^2}{n_f^2 - n_s^2} \text{ for TE}, \quad a = \frac{n_f^4}{n_c^4}\frac{n_s^2 - n_c^2}{n_f^2 - n_s^2} \text{ for TM}$$

a = 0 for perfect symmetry ($n_s = n_c$), and a approaches infinity for strong asymmetry ($n_s \neq n_c$, $n_s \sim n_f$).

Table 2.2. Asymmetry measures for the TE modes (a_E) and the TM modes (a_M) of slab waveguides

Waveguide	n_s	n_f	n_c	a_E	a_M
GaAlAs, double heterostructure	3.55	3.6	3.55	0	0
Sputtered glass	1.515	1.62	1	3.9	27.1
Ti-diffused $LiNbO_3$	2.214	2.234	1	43.9	1093
Outdiffused $LiNbO_3$	2.214	2.215	1	881	21206

* Numerical solution (Kogelnick & Ramaswamy)

Textbook edited by Tamir

Define :

normalized frequency and film thickness

The **normalized frequency**, also known as the **V number**, of a step-index planar waveguide is defined as

$$V \equiv kh\sqrt{n_f^2 - n_s^2} \;\alpha\; (\omega)$$

normalized guided index b (dimensionless)

The propagation constant β can be represented by the normalized guide index:

$$b = \frac{N^2 - n_s^2}{n_f^2 - n_s^2} \;\alpha\; (\beta)$$

(dimensionless)

$$b=(\beta^2-k_2^2)/(k_1^2-k_2^2)$$

We know
$b = 0$ at cut off $(N = n_s)$ **(radiation mode)**

$\rightarrow 1$ as $N \rightarrow n_f$ **(Lowest order mode m=0)**

measure for the asymmetry

$$a \equiv \frac{n_s^2 - n_c^2}{n_f^2 - n_s^2} \text{ for TE}$$

$a = 0$ for perfect symmetry $(n_s = n_c)$

$a \rightarrow \infty$ for strong asymmetry

$$a = \frac{n_s^4}{n_c^4} \frac{n_s^2 - n_c^2}{n_f^2 - n_s^2} \text{ for TM}$$

Numerical Solution for Dispersion Relation (II)

For TE modes, dispersion relation

$$k n_f h\cos\theta - \phi_c - \phi_s = m\pi \Rightarrow V\sqrt{1-b} = m\pi + \tan^{-1}\sqrt{\frac{b}{1-b}} + \tan^{-1}\sqrt{\frac{b+a}{1-b}}$$

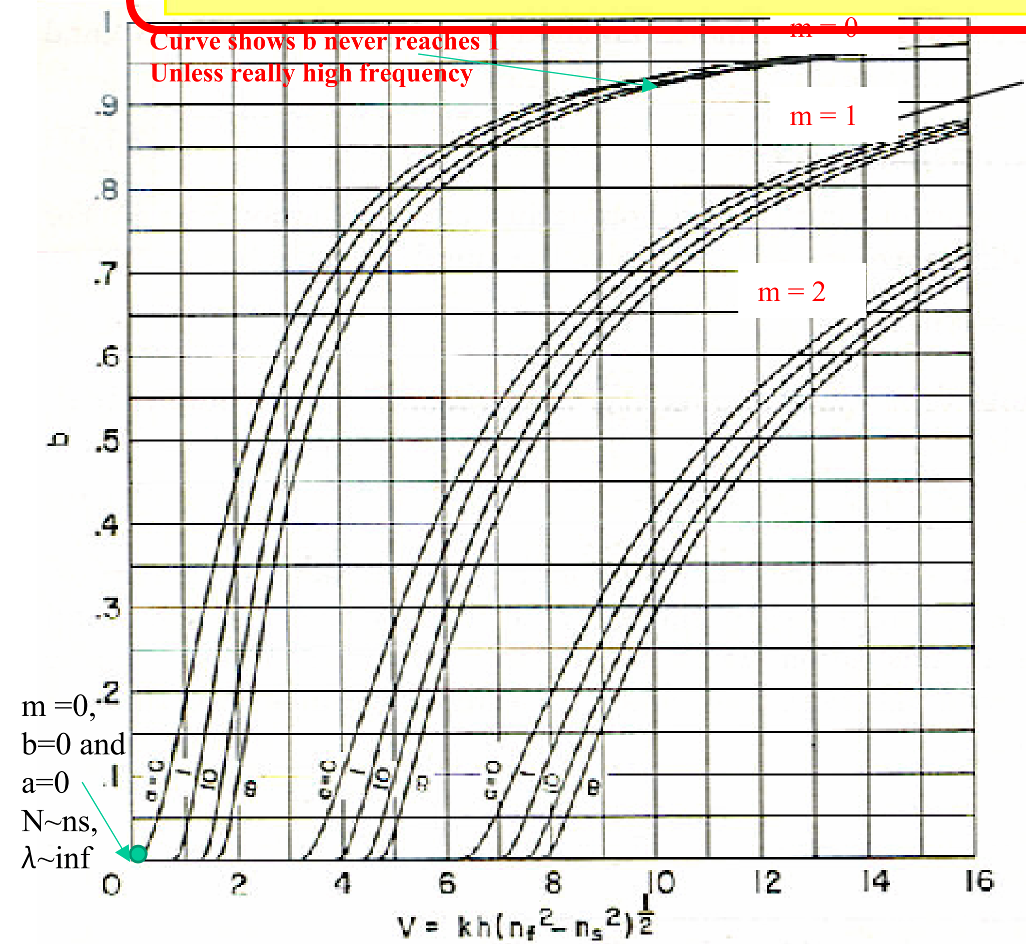


Fig. 2.8 Normalized ω-β diagram of a planar slab waveguide showing the guide index b as a function of the normalized thickness V for various degrees of asymmetry [2.20]

W. Wang

m : Mode number

(Normalized) cut-off frequency:

$$V_0 = \tan^{-1}\sqrt{a}$$

$$V_m = V_0 + m\pi$$

# of guided modes allowed:

$$m = \frac{2h}{\lambda}\sqrt{n_f^2 - n_s^2}$$

<Example>
AlGaAs/GaAs/AlGaAs double heterostructure
n = 3.55/3.6/3.55

Total Internal Reflection for TE Wave

recall

$$\tan\frac{\phi_{TE}}{2} = \frac{\sqrt{\sin^2\theta_1 - \sin^2\theta_c}}{\cos\theta_1} = \frac{\sqrt{n_1^2\sin^2\theta_1 - n_2^2}}{n_1\cos\theta_1}$$

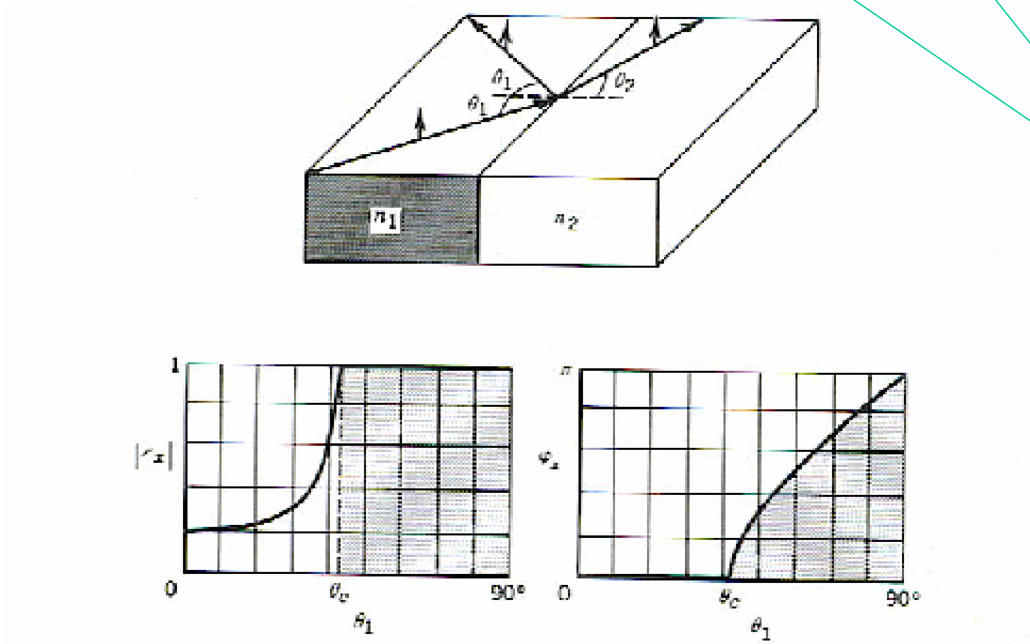


Figure 6.2-3 Magnitude and phase of the reflection coefficient for internal reflection of the TE wave ($n_1/n_2 = 1.5$).

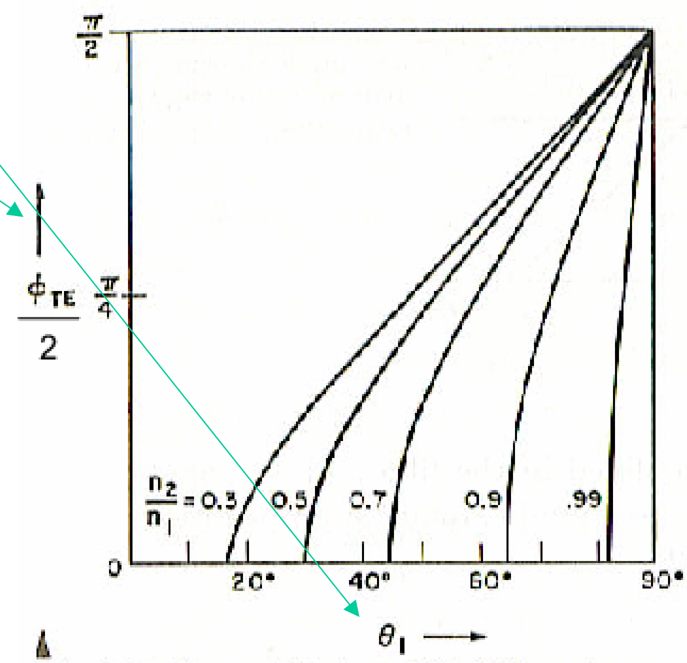


Fig. 2.3. Phase shift ϕ_{TE} of the TE mode as a function of the angle of incidence θ_1

Total Internal Reflection for TM Wave

recall

$$\tan\frac{\phi_{TM}}{2} = \frac{\sqrt{\sin^2\theta_1 - \sin^2\theta_c}}{\cos\theta_1 \sin^2\theta_c} = \frac{n_1^2}{n_2^2}\frac{\sqrt{n_1^2\sin^2\theta_1 - n_2^2}}{n_1\cos\theta_1}$$

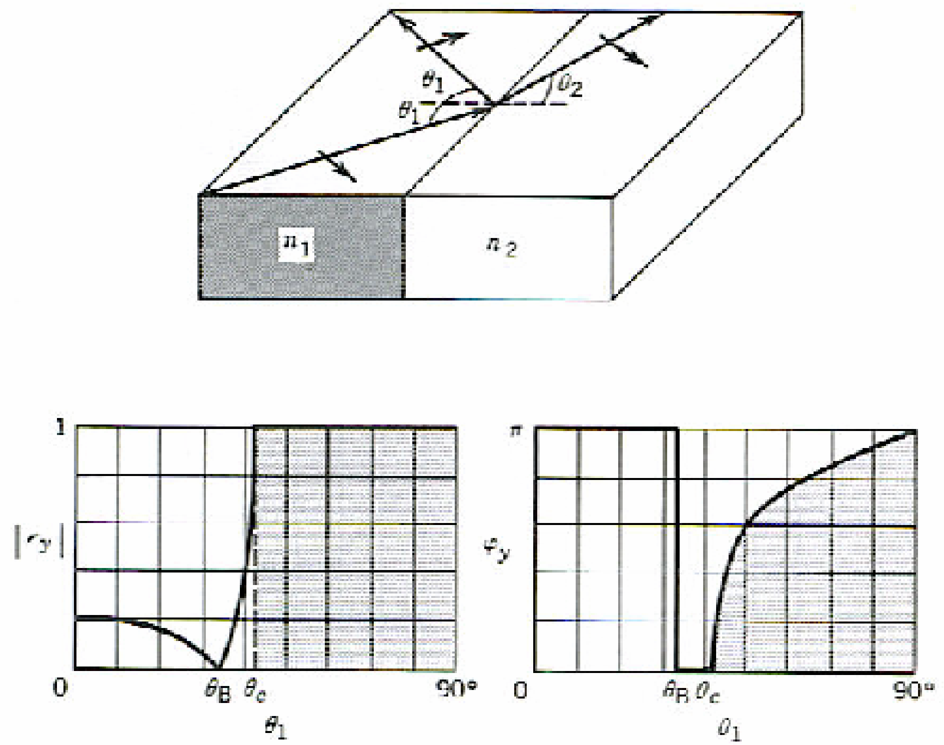


Figure 6.2-5 Magnitude and phase of the reflection coefficient for internal reflection of the TM wave ($n_1/n_2 = 1.5$).

Deriving cos θ, ϕ_s and ϕ_c

Using Snell's law and $1 = \cos\theta^2 + \sin\theta^2$

TE mode

$$k n_f h \cos\theta = m\pi + \phi_s + \phi_c$$

$$\tan\frac{\phi_{TE}}{2} = \frac{\sqrt{n_1^2 \sin^2\theta_1 - n_2^2}}{n_1 \cos\theta_1}$$

$$n_f \sin\theta = N$$

$$1 - \cos^2\theta = \sin^2\theta$$

$$n_f^2 - n_f^2 \cos^2\theta = n_f^2 \sin^2\theta = N^2$$

$$n_f^2 (1 - \cos^2\theta) = N^2$$

$$n_f^2 - N^2 = n_f^2 \cos^2\theta$$

$$n_f \sin\theta = N \rightarrow n_f \cos\theta = \sqrt{n_f^2 - N^2}$$

$\cos\theta = (n_f^2 - N^2)^{0.5}/n_f$

$$\phi_s = \tan^{-1}\frac{\sqrt{n_f^2 \sin^2\theta - n_s^2}}{n_f \cos\theta} = \tan^{-1}\frac{\sqrt{N^2 - n_s^2}}{\sqrt{n_f^2 - N^2}}$$

$$\phi_c = \tan^{-1}\frac{\sqrt{N^2 - n_c^2}}{\sqrt{n_f^2 - N^2}}$$

Numerical Solution for Dispersion Relation (II)

For TE modes, dispersion relation

$$k n_f h \cos\theta - \phi_c - \phi_s = m\pi \;\Rightarrow\; V\sqrt{1-b} = m\pi + \tan^{-1}\sqrt{\frac{b}{1-b}} + \tan^{-1}\sqrt{\frac{b+a}{1-b}}$$

b=1,N~n_f

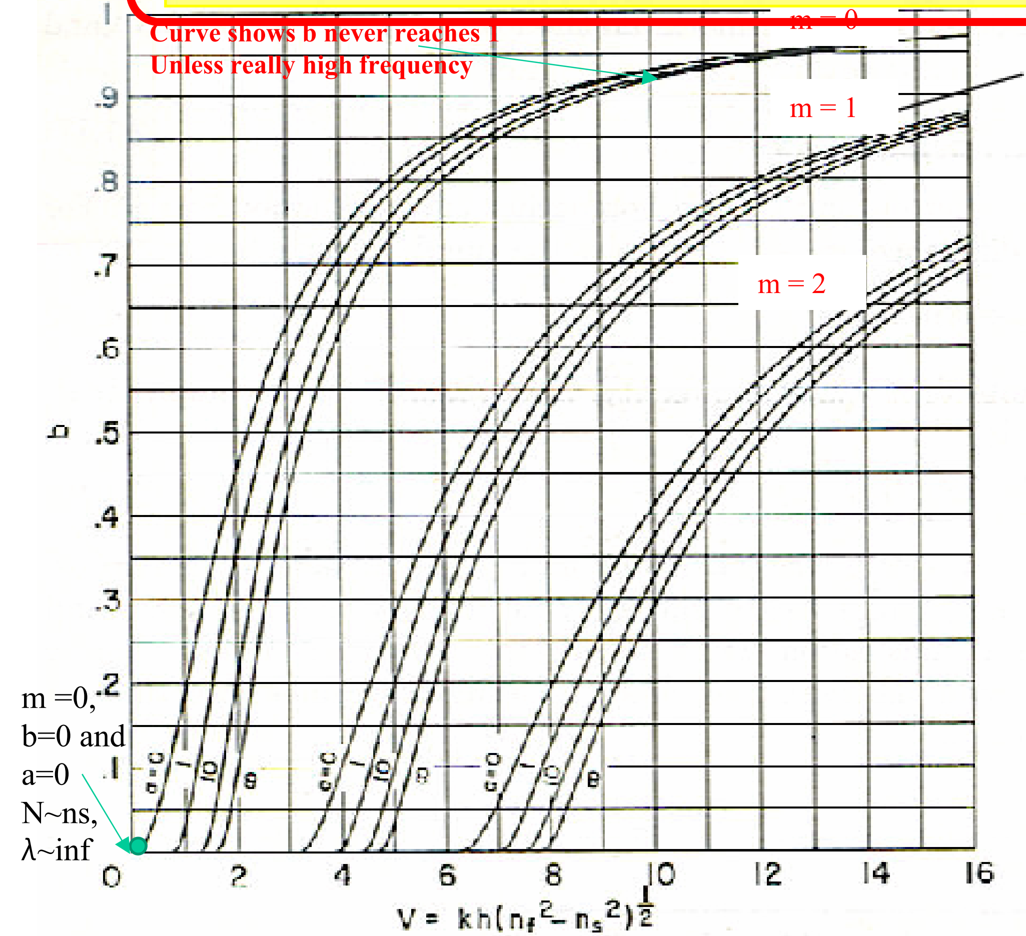


Fig. 2.8 Normalized ω-β diagram of a planar slab waveguide showing the guide index b as a function of the normalized thickness V for various degrees of asymmetry [2.20]

W. Wang

m : Mode number

(Normalized) cut-off frequency:

$$V_0 = \tan^{-1}\sqrt{a}$$

$$V_m = V_0 + m\pi$$

# of guided modes allowed:

$$m = \frac{2h}{\lambda}\sqrt{n_f^2 - n_s^2}$$

<Example>

AlGaAs/GaAs/AlGaAs double heterostructure

n = 3.55/3.6/3.55

(15)

(dispersion eq) $$kh\sqrt{n_f^2 - N^2} = m\pi + \tan^{-1}\frac{\sqrt{N^2 - n_s^2}}{\sqrt{n_f^2 - N^2}} + \tan^{-1}\frac{\sqrt{N^2 - n_c^2}}{\sqrt{n_f^2 - N^2}}$$

$$b = \frac{N^2 - n_s^2}{n_f^2 - n_s^2} \Rightarrow 1 - b = \frac{n_f^2 - N^2}{n_f^2 - n_s^2}$$

$$a = \frac{n_s^2 - n_c^2}{n_f^2 - n_s^2} \Rightarrow b + a = \frac{N^2 - n_c^2}{n_f^2 - n_s^2}$$

remember

$$V = kh\sqrt{n_f^2 - n_s^2}$$

$$V\sqrt{1-b} = m\pi + \tan^{-1}\sqrt{\frac{b}{1-b}} + \tan^{-1}\sqrt{\frac{b+a}{1-b}}$$

3.6
3.6
216
108
12.96

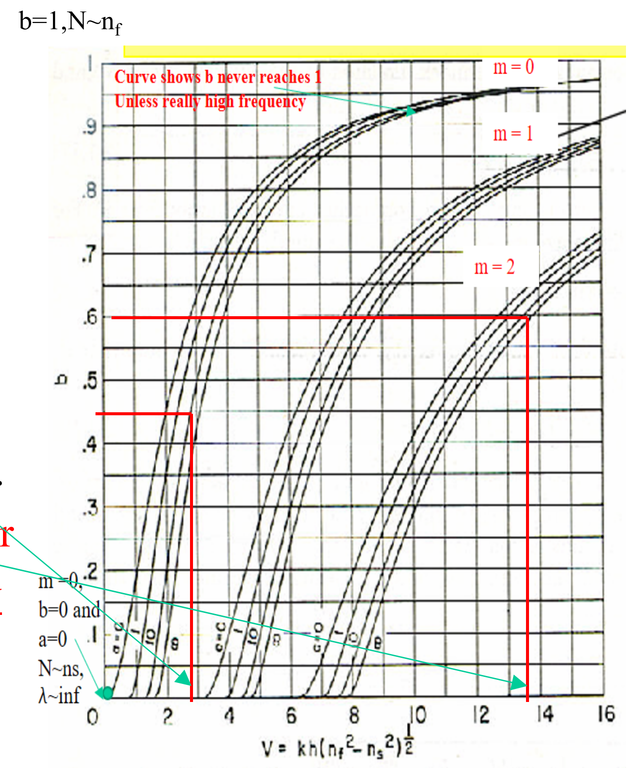
Normalized guide index vs. V number

• When the V number is very small (e.g. $h/\lambda << 1$) and the guided ray travels close to the critical angle ($b << 1$), the effective index is close to that of the cladding layer n_1 or n_s.

=>The wave penetrates deeply into the cladding layers, because the rays are near the critical angle. The evanescent decay is slow.

• As the V number increases, the ray travels more nearly parallel to the waveguide axis, and the effective refractive index lies between n_2 and n_1.

• For a very large V number (e.g. $h/\lambda >> 1$) and the effective index is near that of the core index n_2 or n_f, the wave in the cladding layer decays very rapidly for evanescent waves traveling at angles far above the critical angle.



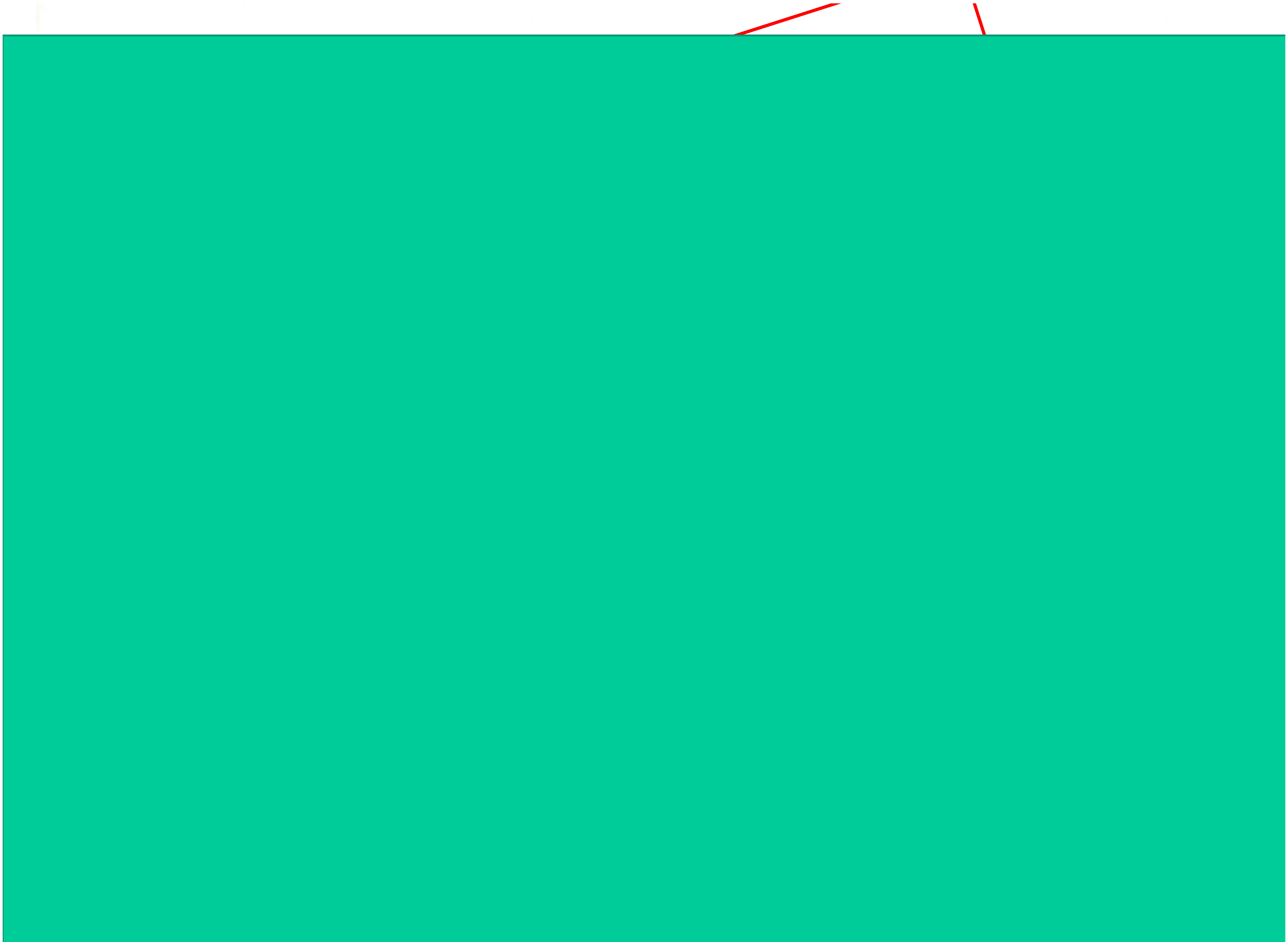
Intersections showing what mode that dimension and wavelengths can excite

Normalized frequency and film thickness

$$V = kh\sqrt{n_f^2 - n_s^2}$$

Normalized guide index

$$b = \frac{N^2 - n_s^2}{n_f^2 - n_s^2}$$

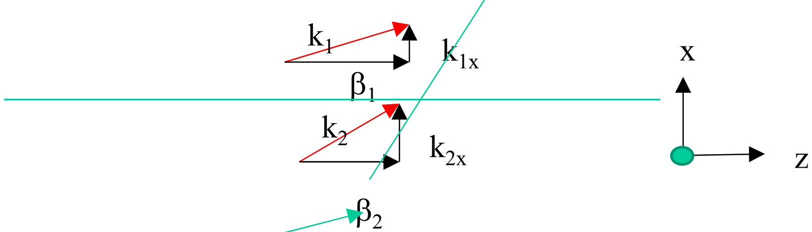


Cutoff Conditions

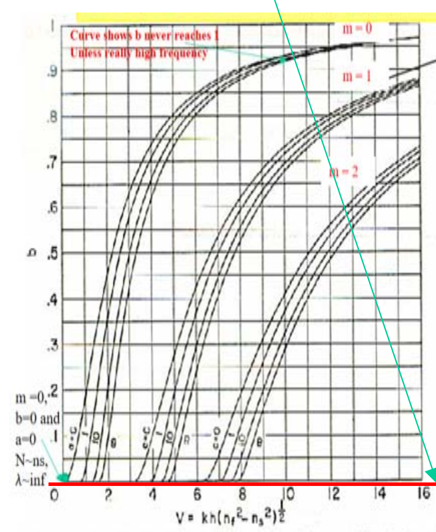
- *Cutoff* occurs when the propagation angle for a **given mode (different mode has different critical angle)** just equals the *critical angle* θ_c --- a guided mode transits to an *unguided* radiation mode.
- This corresponds to the condition that **$\beta_2 = k_2$ (b = 0,N=n_s) and $k_{1x} = 0 \Rightarrow \theta_1 = 90^o$.**
- The fields *extend to infinity* for $k_{1x} = 0$ (i.e. the fields become unguided!).

  This defines the *cutoff condition* for guided modes.

The size of waveguide determines its operating frequency, is determined by the dimension of the waveguide (~λ/2N) and at cutoff frequency and below, the waveguide energy will attenuate rapidly.



2 is layer 2 or film layer not m sorry!!!

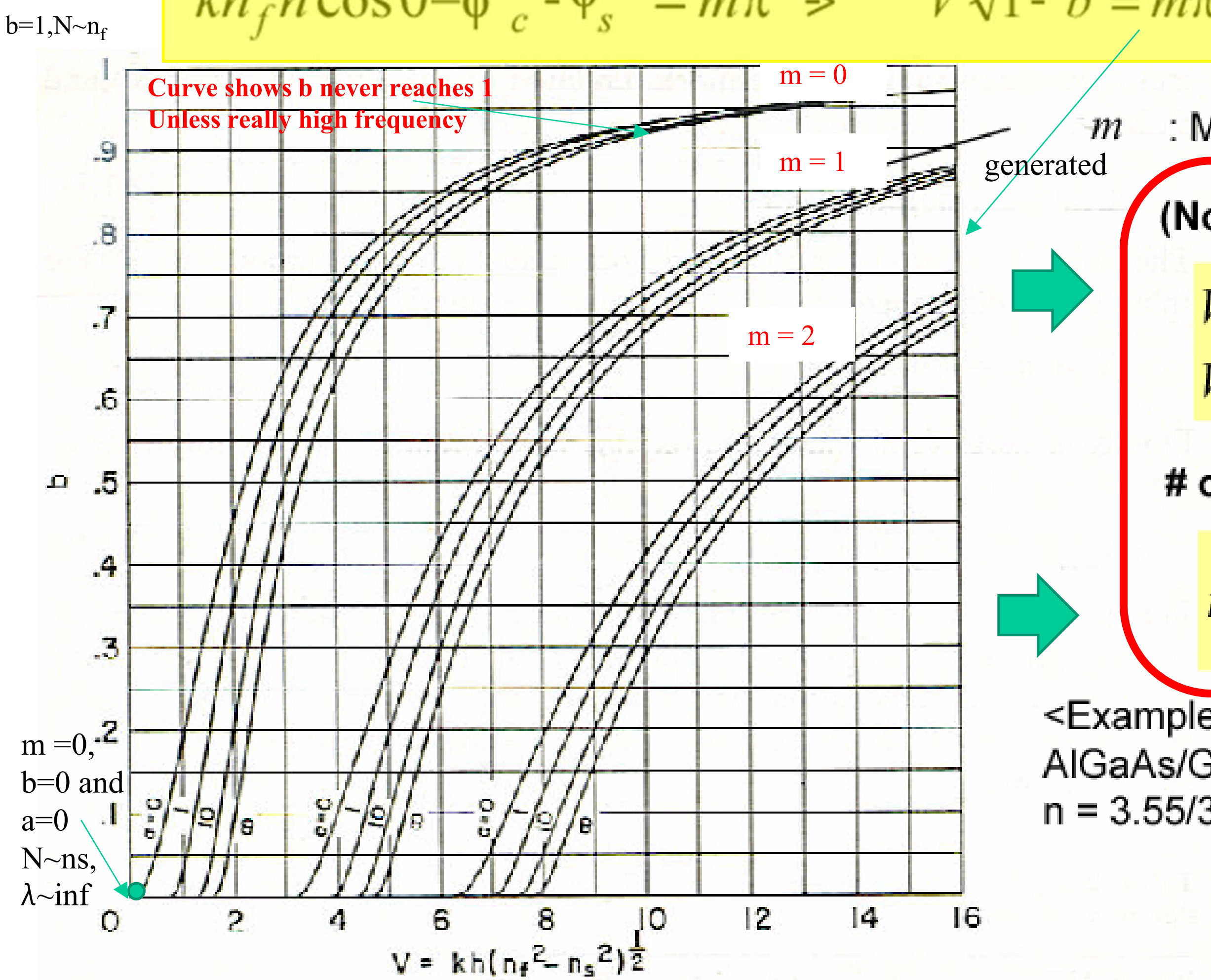


Intersections points between curves and x axis

Numerical Solution for Dispersion Relation (II)

For TE modes, dispersion relation

$$kn_f h\cos\theta - \phi_c - \phi_s = m\pi \;\Rightarrow\; V\sqrt{1-b} = m\pi + \tan^{-1}\sqrt{\frac{b}{1-b}} + \tan^{-1}\sqrt{\frac{b+a}{1-b}}$$



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W. Wang

m : Mode number

(Normalized) cut-off frequency:

$$V_0 = \tan^{-1}\sqrt{a}$$

$$V_m = V_0 + m\pi$$

# of guided modes allowed:

$$m = \frac{2h}{\lambda}\sqrt{n_f^2 - n_s^2}$$

<Example>

AlGaAs/GaAs/AlGaAs double heterostructure

n = 3.55/3.6/3.55

For m-th order mode Cutoff b=0

$$V_m = V_0 + m\pi$$

# of guided modes allowed in the waveguide
Assuming the cutoff for 0th mode is → 0 ($V_0 \to 0$)

so $m = \frac{V_m}{\pi} = \frac{2h}{\lambda}\sqrt{n_f^2 - n_s^2}$

① # of guide mode

$$m = \frac{2h}{\lambda}\sqrt{n_f^2 - n_s^2}$$

m= 0 when h is 0 or $n_f = n_s$ like TEM

② $$V\sqrt{1-b} = m\pi + \tan^{-1}\sqrt{\frac{b}{1-b}} + \tan^{-1}\sqrt{\frac{b+a}{1-b}}$$

b = 0 @ cutoff or N ≈ n_s

Obtain N from b

$$N = \sqrt{b(n_f^2 - n_s^2) + n_s^2}$$

$$b = \frac{N^2 - n_s^2}{n_f^2 - n_s^2}$$

$$V = kh\sqrt{n_f^2 - n_s^2}$$

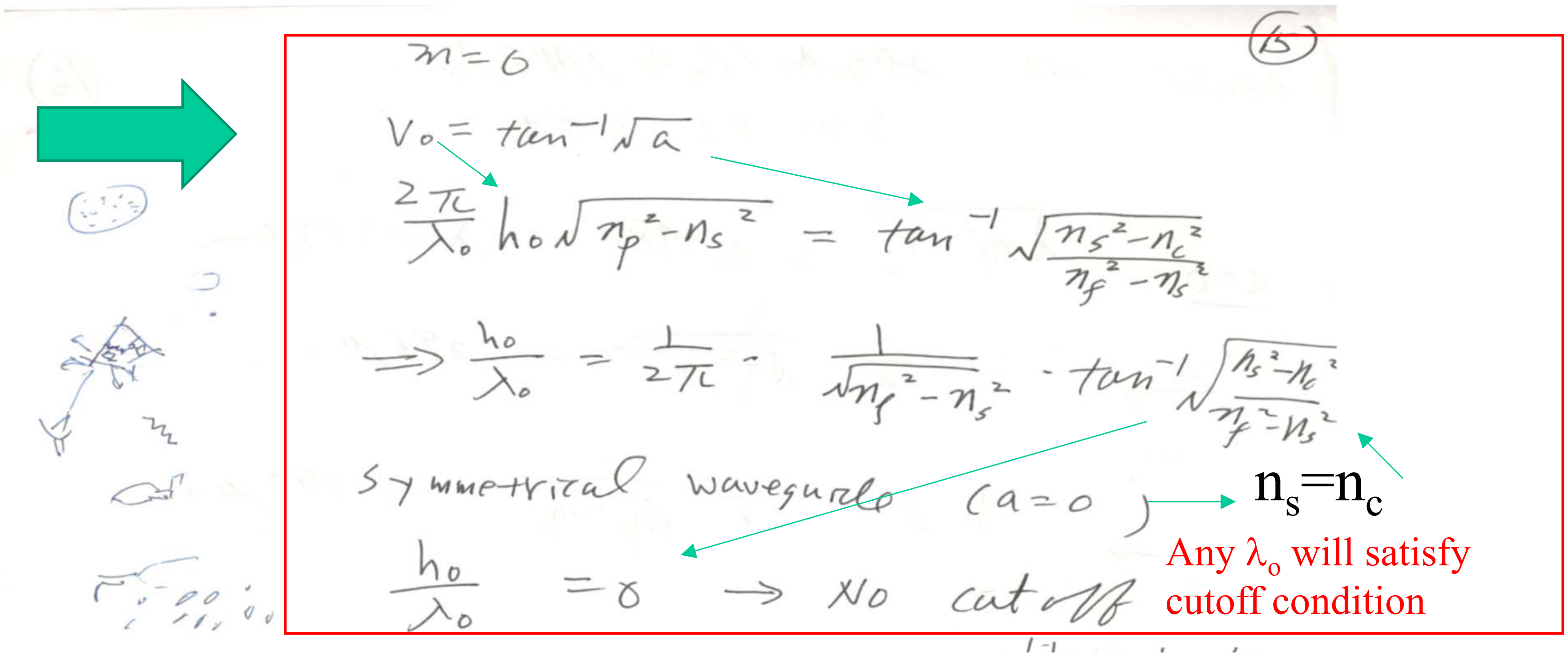
$$a_{TE} = \frac{n_s^2 - n_c^2}{n_f^2 - n_s^2}$$

$$a_{TM} = \frac{n_s^2 - n_c^2}{n_f^2 - n_s^2}\,\frac{n_f^4}{n_c^4}$$

b=0 and m=0 as N-> n_s

$$V_0 = \tan^{-1}\sqrt{a}$$

Normalized Cut off frequency @ m=0 and b =0, when wave Becomes radiation mode

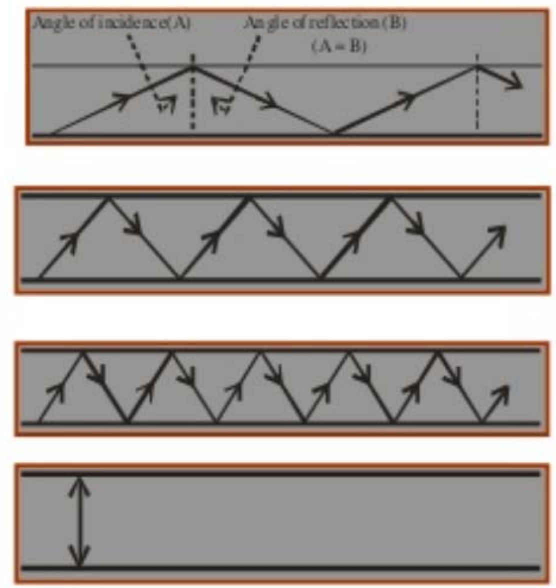


Look at the b v curve

Different modes will have different cutoff frequencies. In the Example is for m=0 and a=0

Wave path in metal waveguide

- High frequency
- Medium Frequency
- Low Frequency
- Cut off Frequency



Opposite of dielectric waveguide
Because index is lower in core!!!

Numerical Solution for Dispersion Relation (I)

Another common way to do this is to introduce the normalized parameters so that you don't have to worry about the dimension

Define:

Normalized frequency and film thickness

$$V = kh\sqrt{n_f^2 - n_s^2}$$

Normalized guide index

$$b = \frac{N^2 - n_s^2}{n_f^2 - n_s^2}$$

Know V find b and From b you can find N and from N your incident angle And modes

b = 0 at cut-ooff (N = n_s), and approaches 1 as N -> n_f.
(substrate radiation mode, m>=0) (guide mode, m=0)

Measure for the asymmetry

$$a = \frac{n_s^2 - n_c^2}{n_f^2 - n_s^2} \text{ for TE}, \quad a = \frac{n_f^4}{n_c^4}\frac{n_s^2 - n_c^2}{n_f^2 - n_s^2} \text{ for TM}$$

a = 0 for perfect symmetry ($n_s = n_c$), and a approaches infinity for strong asymmetry ($n_s \neq n_c$, $n_s \sim n_f$).

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Waveguide	n_s	n_f	n_c	a_E	a_M
GaAlAs, double heterostructure	3.55	3.6	3.55	0	0
Sputtered glass	1.515	1.62	1	3.9	27.1
Ti-diffused $LiNbO_3$	2.214	2.234	1	43.9	1093
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For TE modes, dispersion relation

$$k n_f h \cos\theta - \phi_c - \phi_s = m\pi \;\Rightarrow\; V\sqrt{1-b} = m\pi + \tan^{-1}\sqrt{\frac{b}{1-b}} + \tan^{-1}\sqrt{\frac{b+a}{1-b}}$$

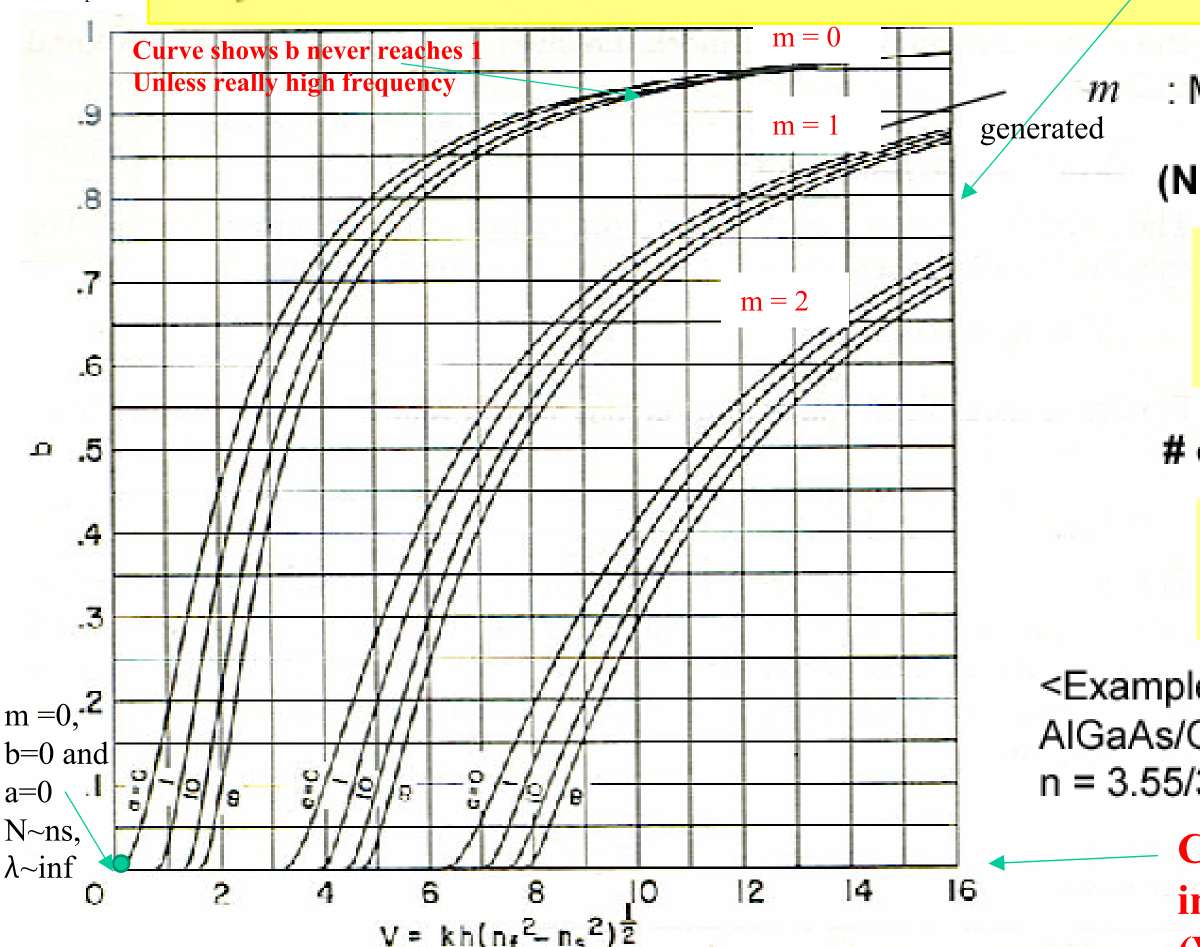


Fig. 2.8 Normalized ω-β diagram of a planar slab waveguide showing the guide index b as a function of the normalized thickness V for various degrees of asymmetry [2.20]

m : Mode number

(Normalized) cut-off frequency:

$$V_0 = \tan^{-1}\sqrt{a}$$
$$V_m = V_0 + m\pi$$

# of guided modes allowed:

$$m = \frac{2h}{\lambda}\sqrt{n_f^2 - n_s^2}$$

<Example>
AlGaAs/GaAs/AlGaAs double heterostructure
n = 3.55/3.6/3.55

Another way to find V:

Eigenvalue equations in terms of normalized frequency

TE: $\tan(hd/2 - m\pi/2) = (V^2 - h^2d^2)^{1/2}/hd$ ⇐ **Remember $h = kn_f \cos\theta$ and $d = h$**

For TE modes, dispersion relation

recall $kn_f h\cos\theta - \phi_c - \phi_s = m\pi \Rightarrow V\sqrt{1-b} = m\pi + \tan^{-1}\sqrt{\frac{b}{1-b}} + \tan^{-1}\sqrt{\frac{b+a}{1-b}}$

TM: $\tan(hd/2 - m\pi/2) = (n_1^2/n_2^2)(V^2 - h^2d^2)^{1/2}/hd$

$m = 0, 1, 2, \ldots,$

- The eigenvalue equations are in the form of *transcendental equations*, which are usually solved graphically by plotting their left- and right-hand sides as a function of hd.
- The solutions yield the *allowed values of hd* for a given value of the waveguide parameter V for TE/TM modes.

Examples

Symmetric weakly guided waveguide

- Consider a ***weakly guiding*** waveguide $n_2 - n_1 << n_2$
- Here we choose $n_2 = 3.6$ and $n_1 = 3.55$. These values are characteristic of an AlGaAs double heterojunction light-emitting diode or laser diode.

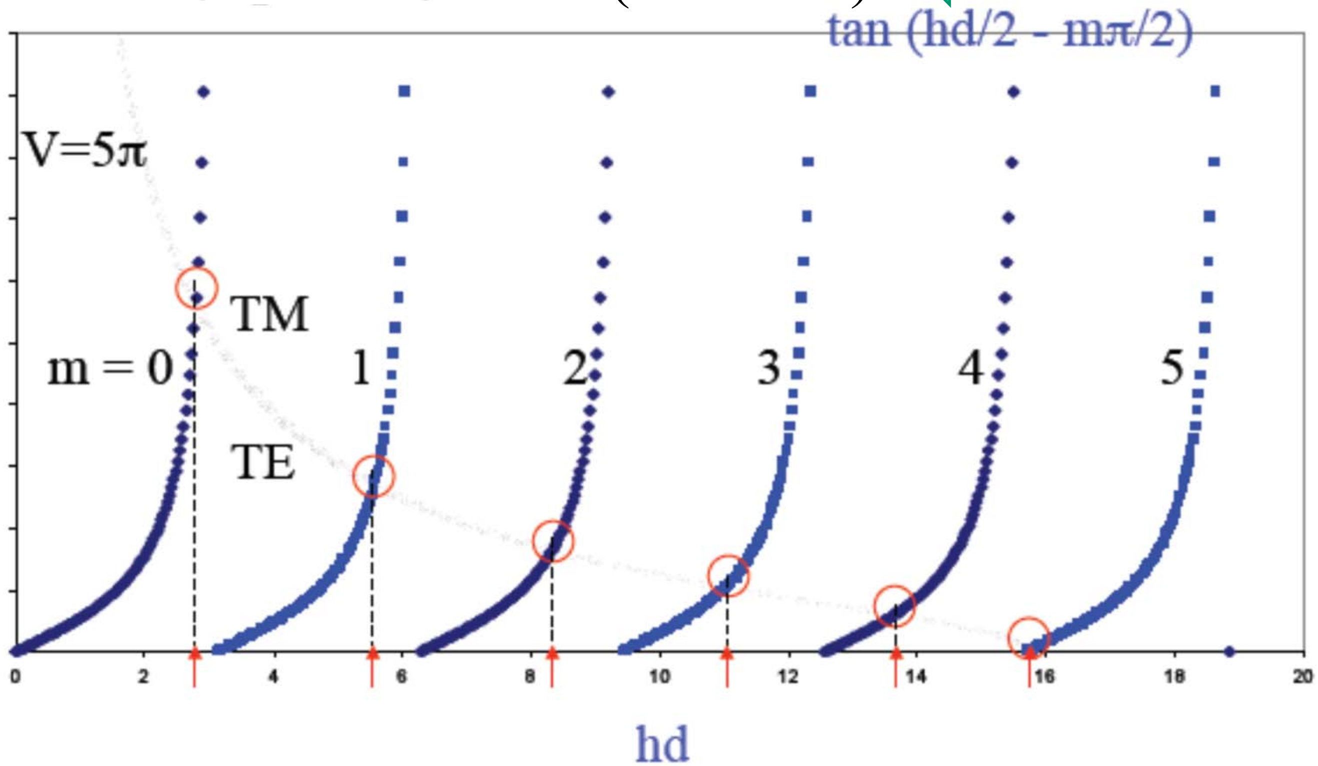
- The critical angle for this structure is $\theta_c = \sin^{-1}(n_1/n_2) \sim 80^\circ$

- The range of angles for trapped rays is then $80^\circ \leq \theta \leq 90^\circ$.

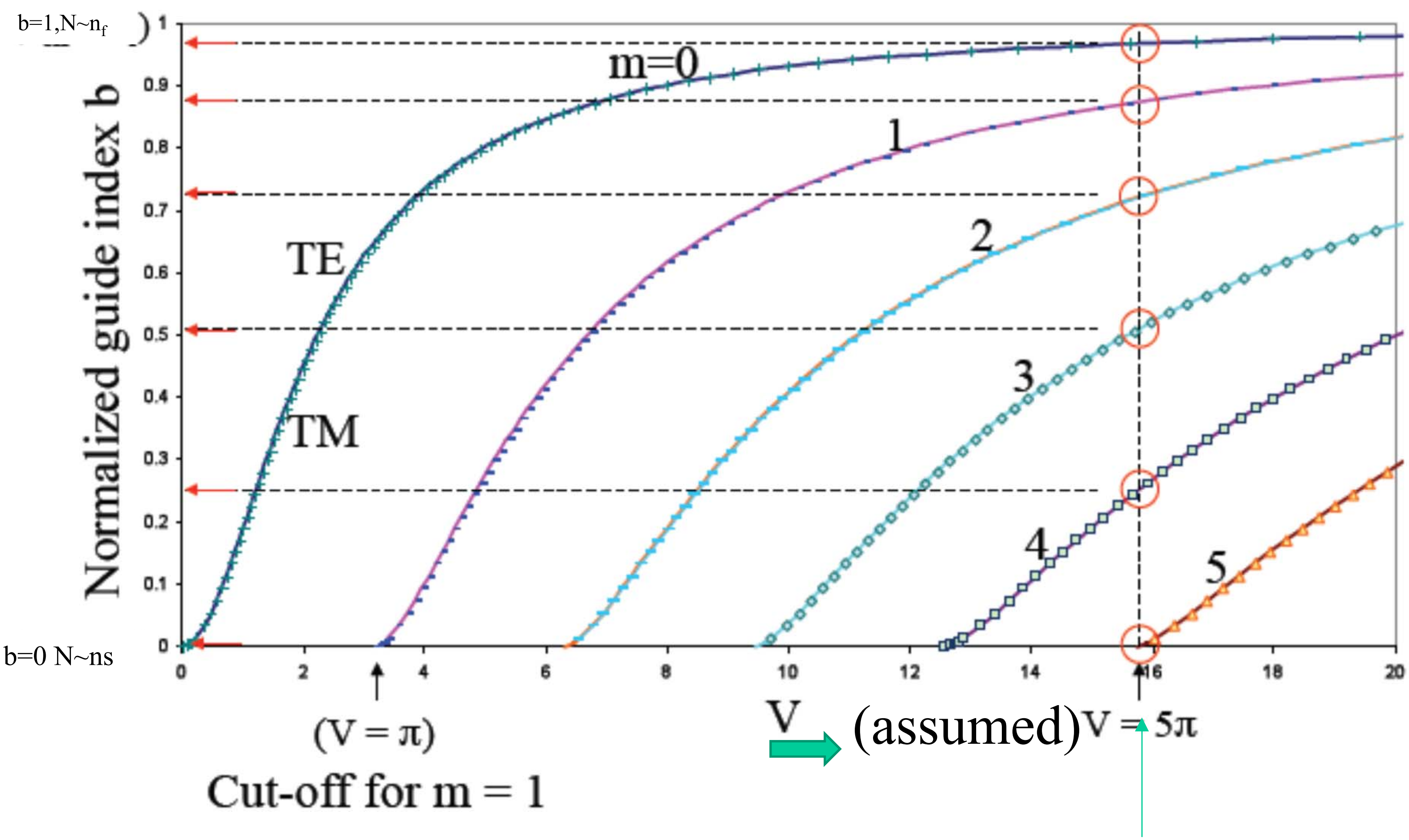
- The range of *waveguide effective refractive index* is $3.55 \leq n_{eff} \leq 3.6$

Graphic solutions for the eigenvalues of guided TE and TM modes of a weakly guiding symmetric slab waveguide

$n_2 = 3.6$, $n_1 = 3.55$, $V = 5\pi$ (assumed)



Mode chart for the first six TE and TM modes (m = 0 – 5) of symmetric slab waveguides in AlGaAs (n_1 = 3.6, n_2 = 3.55)



Know V find b and From b you can find N and from N your incident angle and modes

Examples

- For example, consider V = 15 on the mode chart, the TE_5/TM_5 modes could not propagate because V was not large enough to intersect with the b vs. V curves.

  => The TE_5/TM_5 modes, and *all higher-ordered modes*, are *cut off*.

Example: Symmetric strongly guiding slab waveguides

- Consider a strongly guiding waveguide $n_2 - n_1 >> 0$

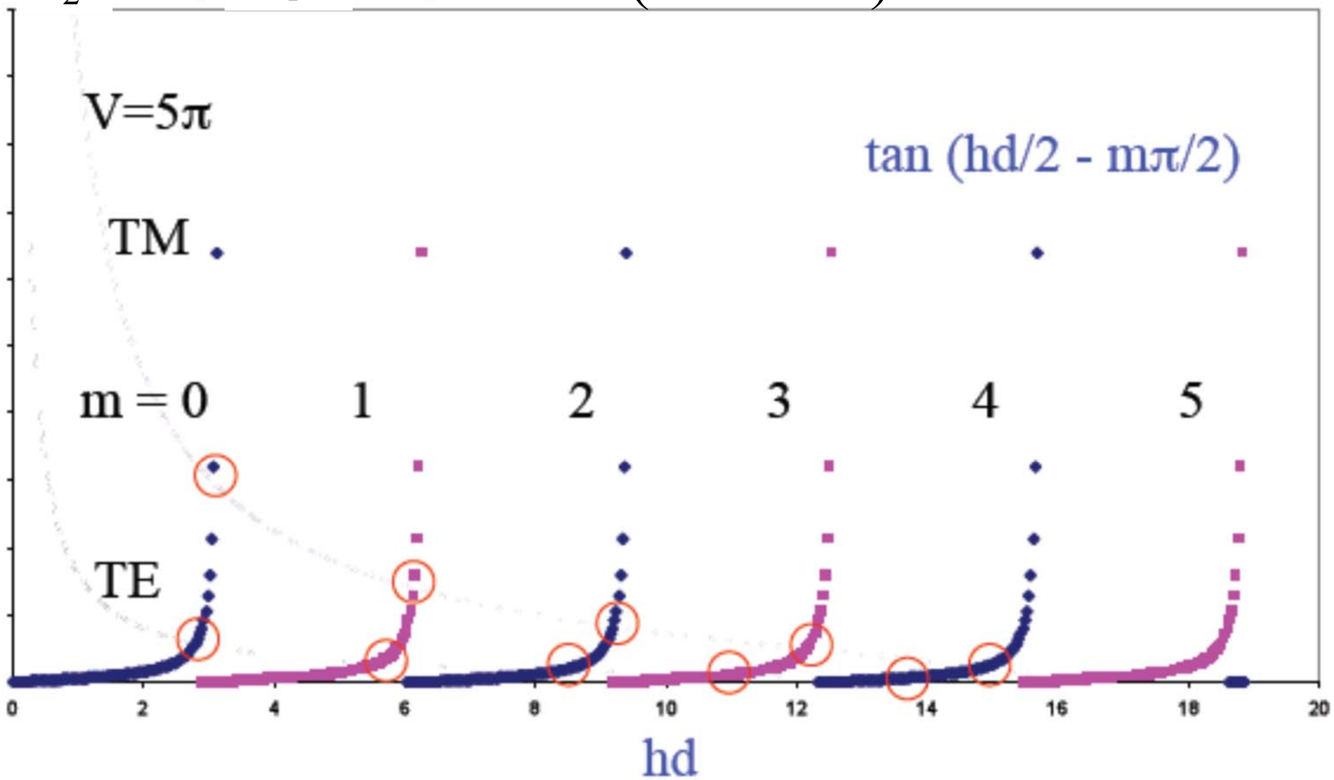
- Here we choose $n_2 = 3.5$ and $n_1 = 1.45$. These values are characteristic of an silicon-on-insulator (SOI) waveguide.

- The critical angle for this structure is $\theta_c = \sin^{-1}(n_1/n_2) \sim 24.5^o$
- The range of angles for trapped rays is then $24.5^o \leq \theta \leq 90^o$.

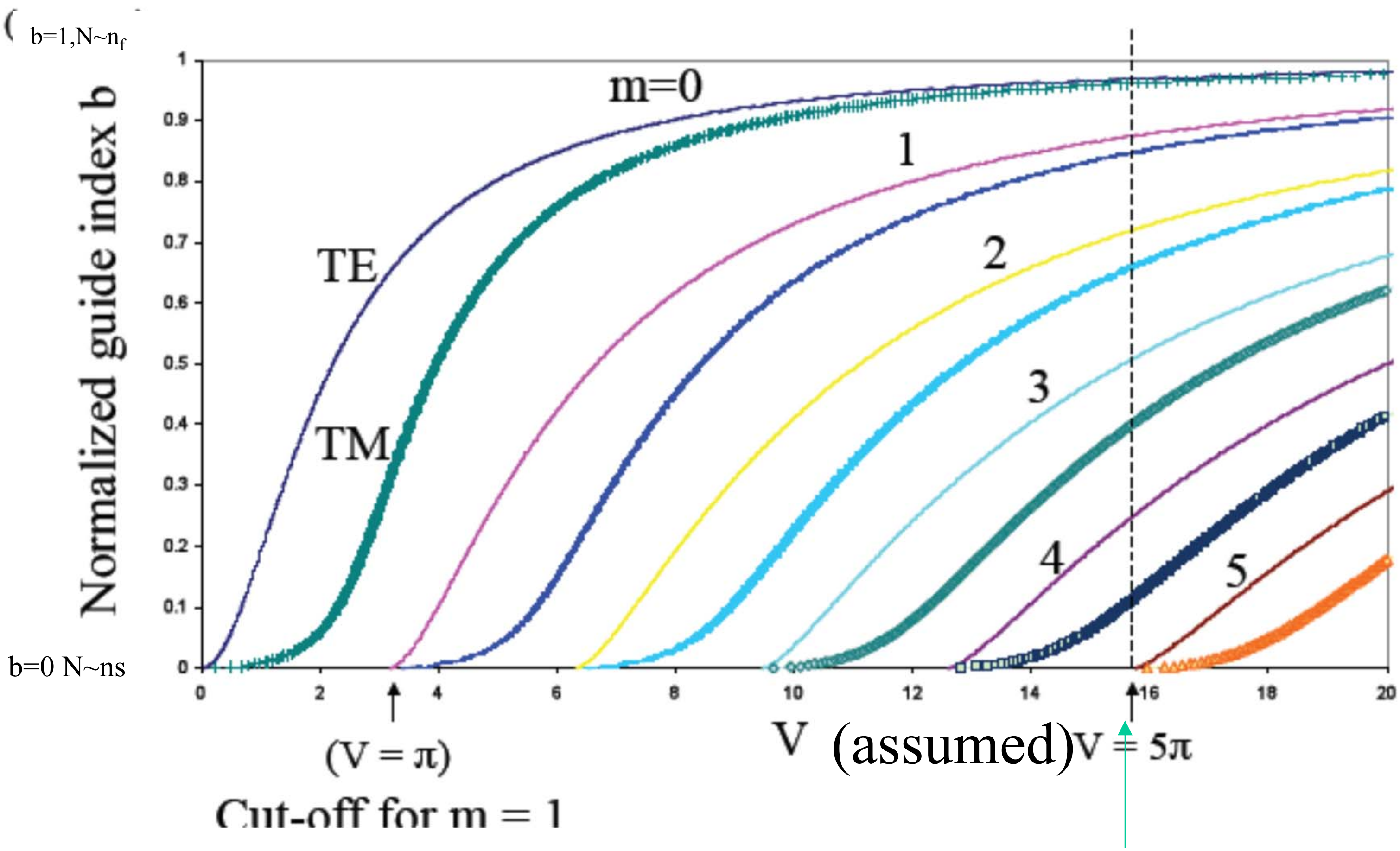
- The range of waveguide effective refractive index is $1.45 \leq n_{eff} \leq 3.5$

Graphic solutions for the eigenvalues of guided TE and TM modes of a strongly guiding symmetric slab waveguide

n_2 3.5, ,$n_{1=}$ .45, V = 5π (assumed)

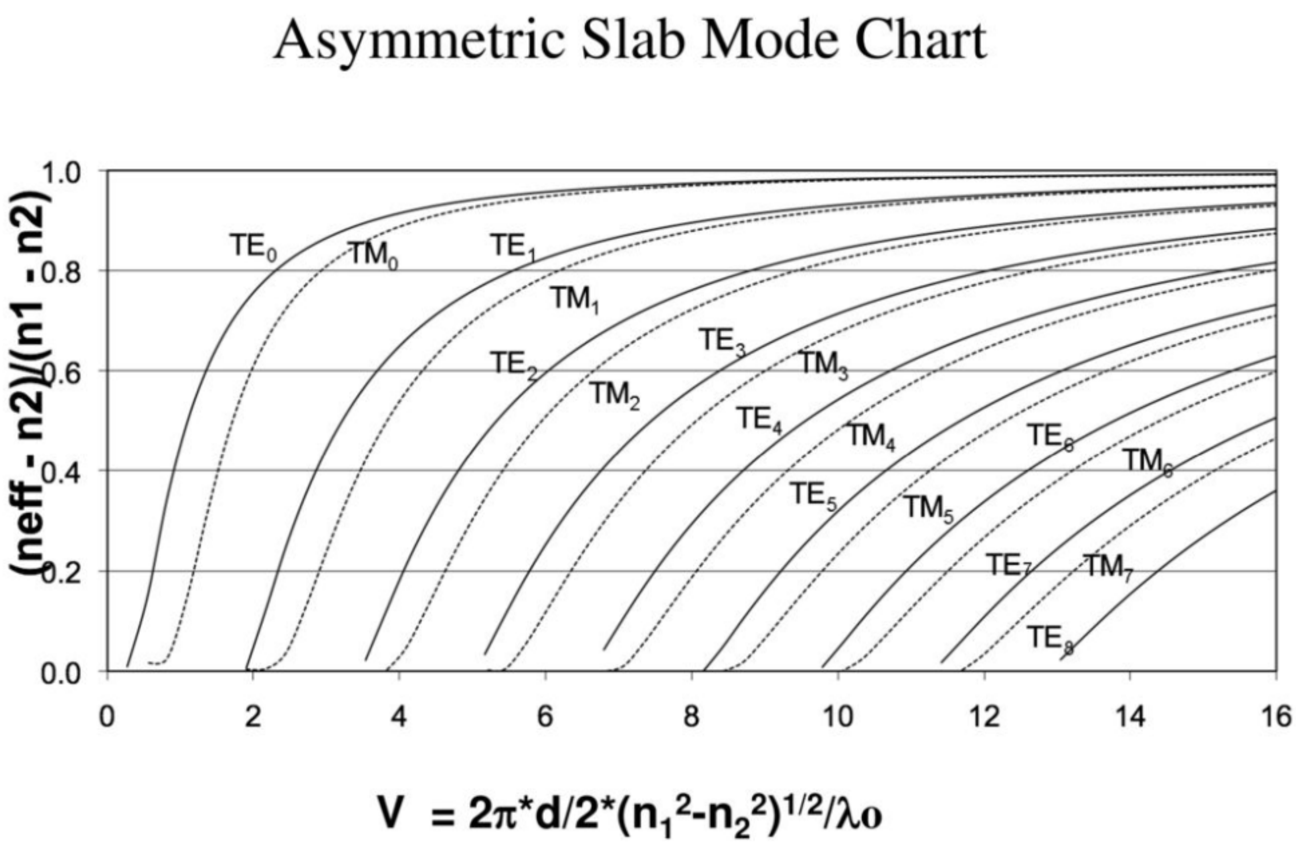


Mode chart for the first six TE and TM modes (m = 0 – 5) of symmetric slab waveguides in SOI (n_1 = 3.5, n_2 = 1.45)



Know V find b and From b you can find N and from N your incident angle and modes

Asymmetric Slab waveguide mode chart



example: use AlGaAs / GaAs / AlGaAs

3.55 3.6 3.55

$V_0 = \tan^{-1}\sqrt{a}$

$\frac{2\pi}{\lambda_0} h_0 \sqrt{n_f^2 - n_s^2} = \tan^{-1}\sqrt{\frac{n_s^2 - n_c^2}{n_f^2 - n_s^2}}$

Find h:

$\sqrt{n_f^2 - n_s^2} = 0.598$

$\lambda = 1.55\,\mu m$

$\underline{a = 0}$

$\leftarrow \pi$

$h = \frac{\lambda}{2} \cdot \frac{1}{\sqrt{n_f^2 - n_s^2}} = 1.296\,\mu m$

$m = 1$

$h \geq 2 \cdot \frac{\lambda}{2} \cdot \frac{1}{\sqrt{n_f^2 - n_s^2}} = 2.592\,\mu m$

$\rightarrow \pi$

Thicker core more modes

$m > 1$

Find N:

use $1.33\,\mu m$

$n_f = 3.6$

$n_s = 3.55$

$n_c = 1$

$n_s < N < n_f$

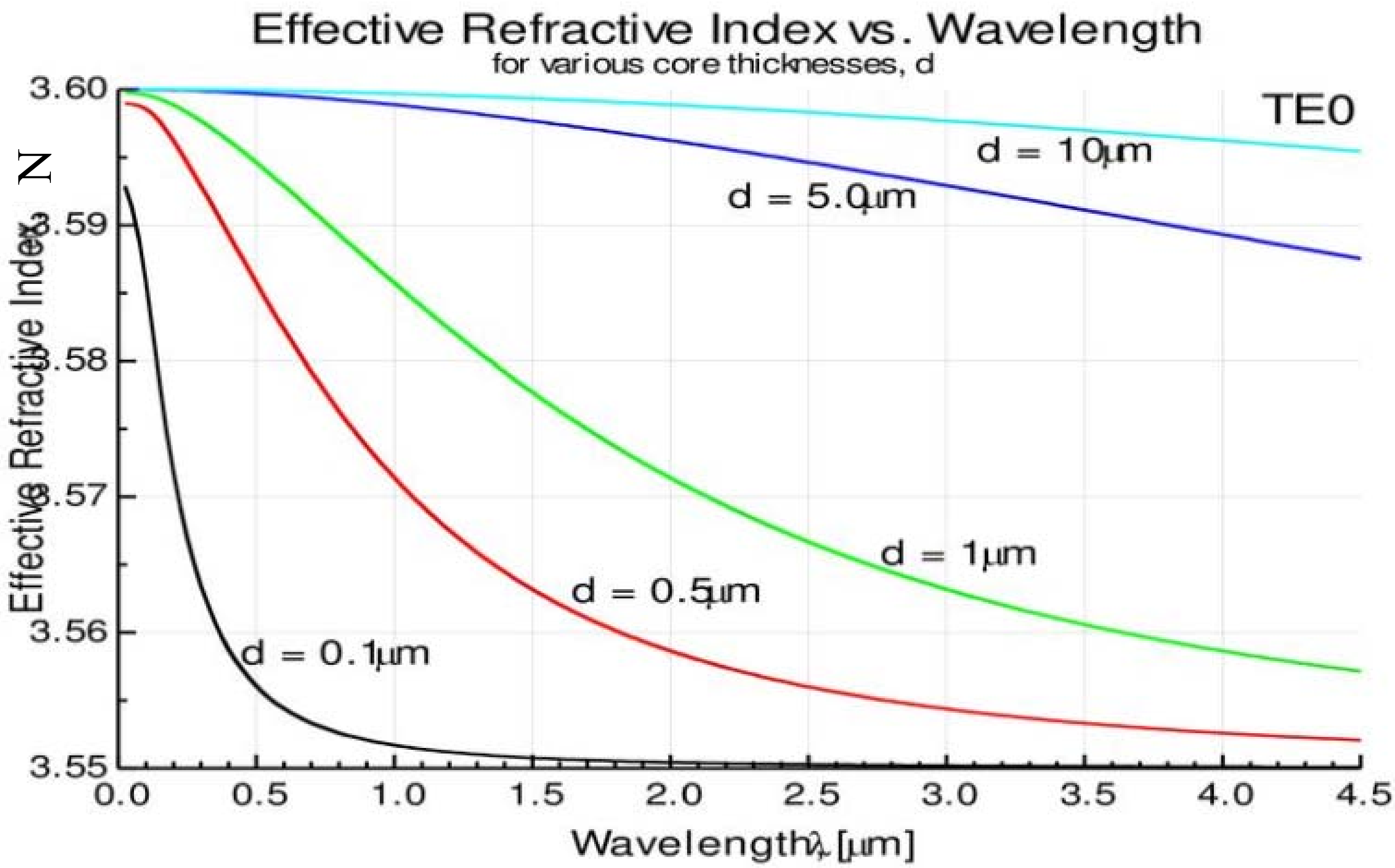
$b = \frac{N^2 - n_s^2}{n_f^2 - n_s^2} \rightarrow N = 3.58$

effective index

~ 0.7

$\leftarrow \pi$

Effective Index for fundamental mode



Thicker the waveguide less affected by wavelengths

Dispersion Equation

recall

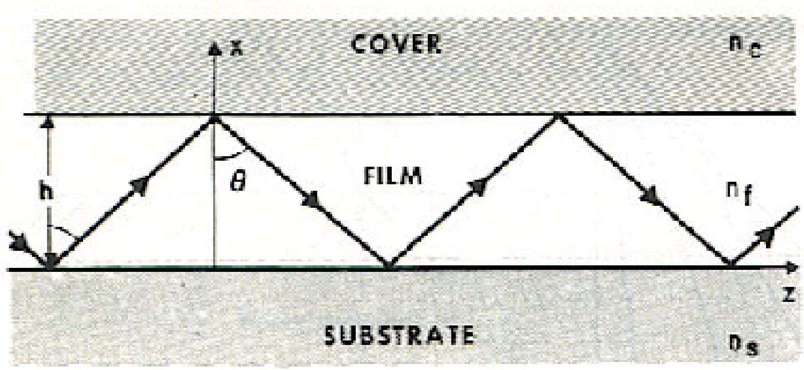
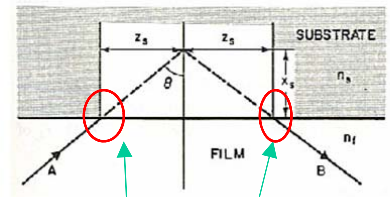


Fig. 2.5. Side-view of a slab waveguide showing wave normals of the zig-zag waves corresponding to a guided mode



Transverse resonance condition:

$$2kn_f h\cos\theta - 2\phi_c - 2\phi_s = 2m\pi$$

m : mode number

$kn_f h\cos$: phase shift for the transverse passage through the film

$2\phi_c = \phi_{TE,TM}$: phase shift due to total internal reflection from film/cover interface

$2\phi_s = \phi_{TE,TM}$: phase shift due to total internal reflection from film/substrate interface

Dispersion equation (β vs. ω):

$$kn_f h\cos\theta - \phi_c - \phi_s = m\pi$$

The phase shift can be representing the zig-zag ray at a certain depth into the confining layers 1 ad 3 before it is reflected (goos-Hanchen shifts- lateral shift)

Effective guide index $N = \beta/k = n_f \sin\theta$ $\quad n_s < N < n_f$

The Goos-Hänchen Shift

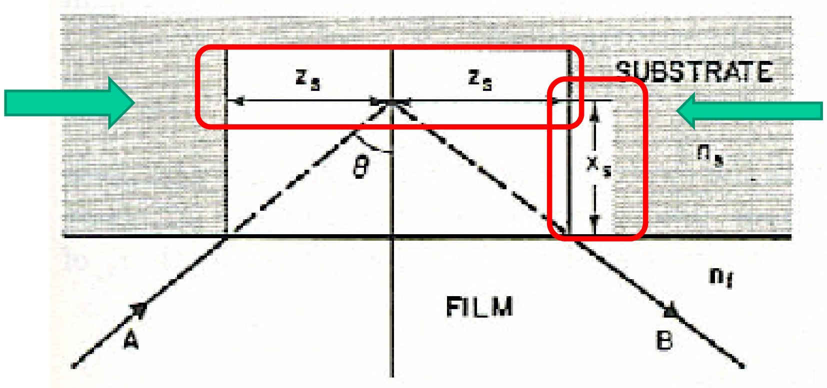


Fig. 2.9. Ray picture of total reflection at the interface between two dielectric media showing a lateral shift of the reflected ray (Goos-Hänchen shift)

For TE modes

$$kz_s = (N^2 - n_s^2)^{-1/2} \tan\theta$$

For TM modes

$$kz_s = \frac{(N^2 - n_s^2)^{-1/2} \tan\theta}{\frac{N^2}{n_s^2} + \frac{N^2}{n_f^2} - 1}$$

The phase shift can be representing the zig-zag ray at a certain depth into the confining layers 1 ad 3 before it is reflected

The lateral ray shift indicates a penetration depth:

$$x_s = \frac{z_s}{\tan\theta}$$

$$z_s = \frac{d\phi_s}{d\beta}$$

Effective Guide Thickness

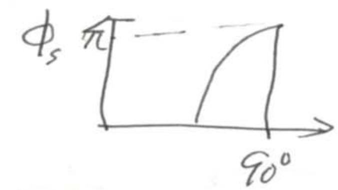
The Goos-Hanchen shift

This is basically the evanescent wave or leaky wave part which extended beyond the confined core into cladding using ray optic way to explain it.

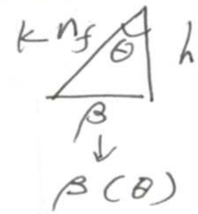
phase change = propagation constant × propagation distance

(shift) $\phi_s = \phi_s(\theta)$

$= \phi_s(\beta)$



(distance) $Z_s = \dfrac{d\phi_s}{d\beta}$



$\beta = kn_f\sin\theta = kN$

For TE

$$\phi_s = \tan^{-1}\frac{\sqrt{n_f^2\sin^2\theta - n_s^2}}{n_f\cos\theta}$$

$$= \tan^{-1}\frac{\sqrt{N^2 - n_s^2}}{\sqrt{n_f^2 - N^2}}$$

$Z_s = \dfrac{d\phi_s}{dN}\cdot\dfrac{dN}{d\beta}$

Sub identity

$$\frac{d}{dx}\tan^{-1}u = \frac{1}{1+u^2}\frac{du}{dx}$$

Total Internal Reflection for TE Wave

recall

$$\tan\frac{\phi_{TE}}{2} = \frac{\sqrt{\sin^2\theta_1 - \sin^2\theta_c}}{\cos\theta_1} = \frac{\sqrt{n_1^2\sin^2\theta_1 - n_2^2}}{n_1\cos\theta_1}$$

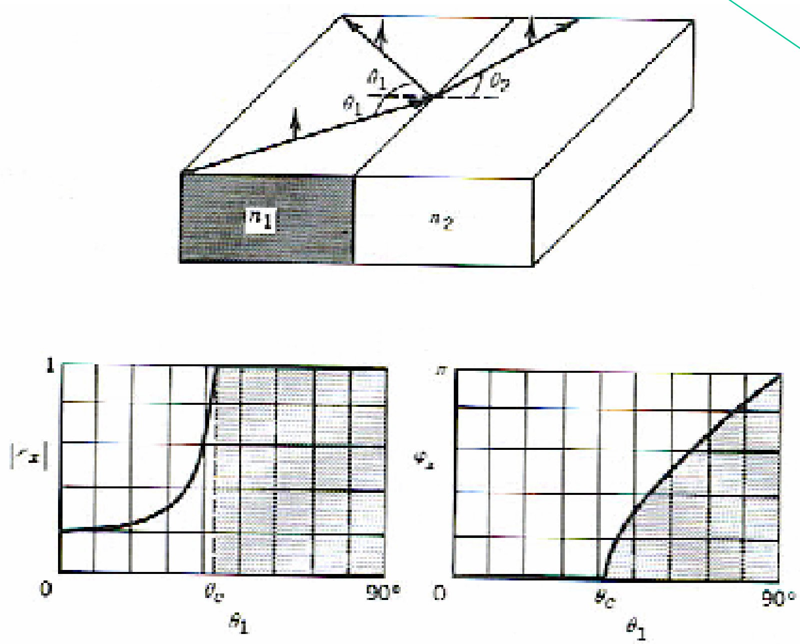


Figure 6.2-3 Magnitude and phase of the reflection coefficient for internal reflection of the TE wave ($n_1/n_2 = 1.5$).

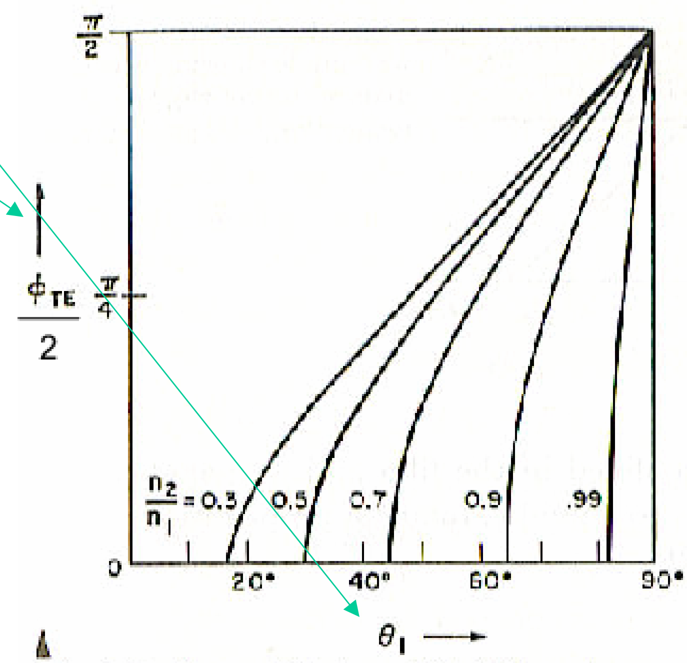


Fig. 2.3. Phase shift ϕ_{TE} of the TE mode as a function of the angle of incidence θ_1

Total Internal Reflection for TM Wave

recall

$$\tan\frac{\phi_{TM}}{2} = \frac{\sqrt{\sin^2\theta_1 - \sin^2\theta_c}}{\cos\theta_1 \sin^2\theta_c} = \frac{n_1^2}{n_2^2}\frac{\sqrt{n_1^2\sin^2\theta_1 - n_2^2}}{n_1\cos\theta_1}$$

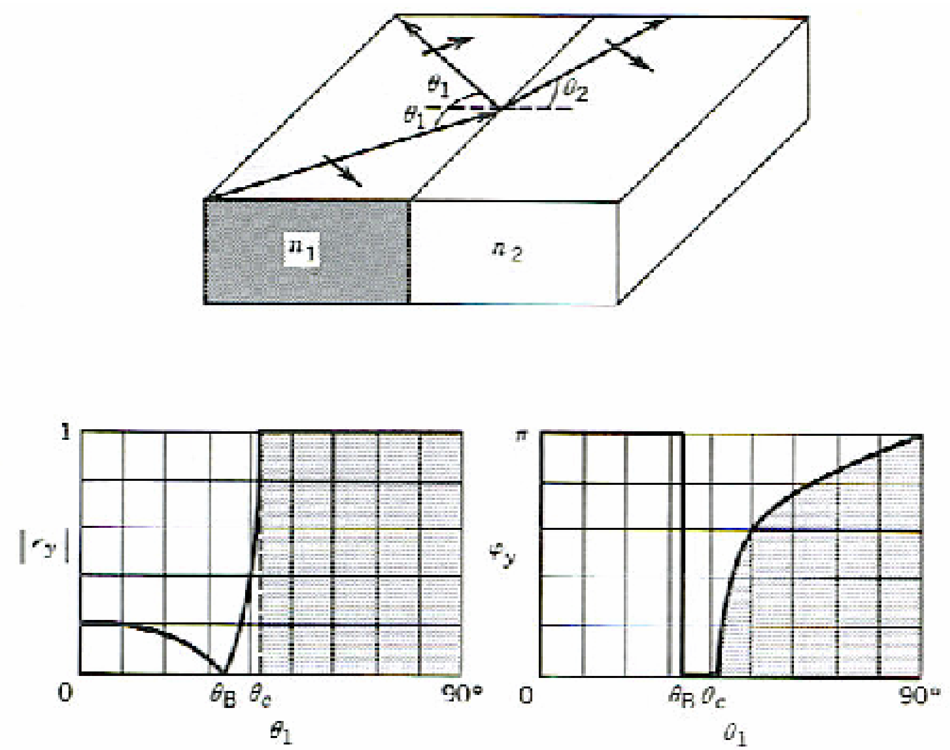


Figure 6.2-5 Magnitude and phase of the reflection coefficient for internal reflection of the TM wave ($n_1/n_2 = 1.5$).

$$kz_s = (N^2 - n_s^2)^{-1/2}\tan\theta$$

For TM

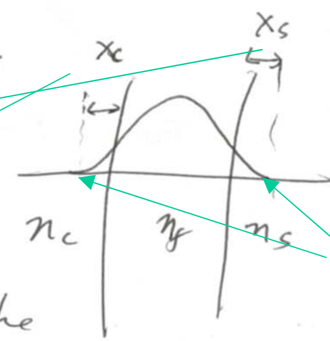
$$kz_s = \frac{(N^2 - n_s^2)^{1/2}\tan\theta}{\frac{N^2}{n_s^2} + \frac{N^2}{n_f^2} - 1}$$

The phase shift indicates a penetration depth

$$x = \frac{z_s}{\tan\theta}$$

* effective Guide Thickness

$$h_{eff} = h + x_s + x_c$$



Evanescent wave

plot this as a function of the normalized freq

$$V = kh\sqrt{n_f^2 - n_s^2}$$

Effective Waveguide Thickness

Effective thickness

$$h_{eff} = h + x_s + x_c$$

Normalized effective thickness

$$H = kh_{eff}\sqrt{n_f^2 - n_s^2}$$

For TE modes

$$H = V + \frac{1}{\sqrt{b}} + \frac{1}{\sqrt{b+a}}$$

Minimum H - > Maximum confinement

<Example> Sputtered glass, n_s = 1.515,
n_f = 1.62, n_c = 1, a = 3.9

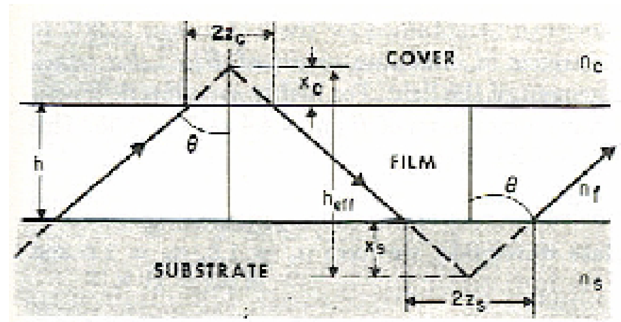


Fig. 2.10. Ray picture of zig-zag light propagation in a slab waveguide. Goos Hänchen shifts are incorporated in the model, and the effective guide thickness h_{eff} is indicated

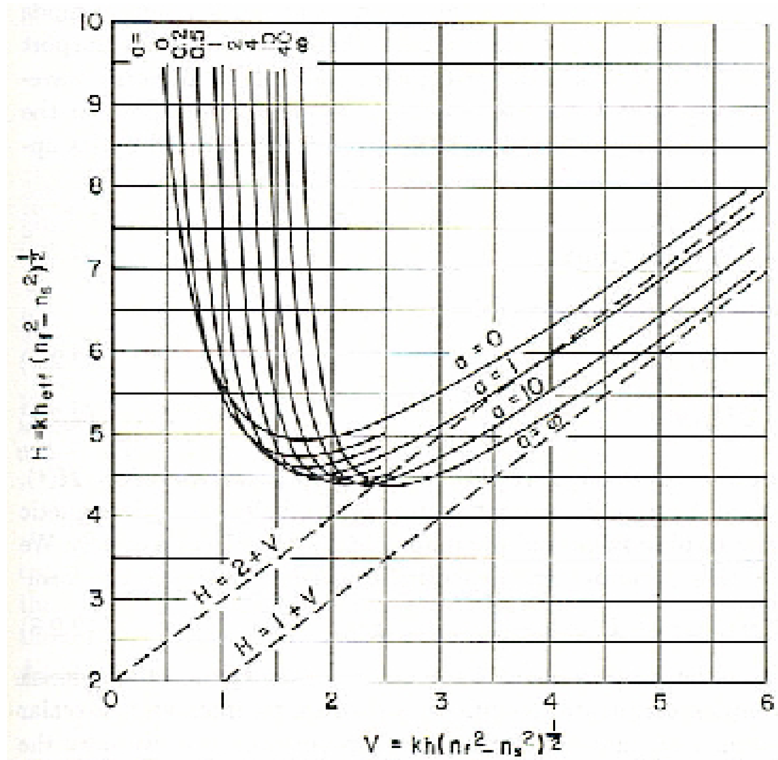


Fig. 2.11. Normalized effective thickness of a slab waveguide as a function of the normalized film thickness V for various degrees of asymmetry (after [2.20])

For TE mode

$$H = k\sqrt{n_f^2 - n_s^2}\,(h + X_s + X_c) \quad = \quad k h_{eff}\sqrt{n_f^2 - n_s^2}$$

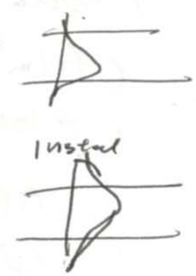
$$= V + \frac{\sqrt{n_f^2 - n_c^2}}{\sqrt{N^2 - n_s^2}} + \frac{\sqrt{n_f^2 - n_s^2}}{\sqrt{N^2 - n_c^2}}$$

since $b = \dfrac{N^2 - n_s^2}{n_f^2 - n_s^2}$

$$b + a = \frac{N^2 - n_c^2}{n_f^2 - n_s^2}$$

$$H = V + \frac{1}{\sqrt{b}} + \frac{1}{\sqrt{b+a}}$$

H allows us to pick the thickness that gives least penetration most confine field



Multilayer Structure (wave equation)

Things covered in this section

- Multilayer structure
- Analytical approach to various rectangular waveguide
- Examples

① Field distribution — Ray optics can't solve it

② multilayer waveguide

3. E-M wave approach to optical waveguide theory

Guided E-M Wave in a Planar Waveguide

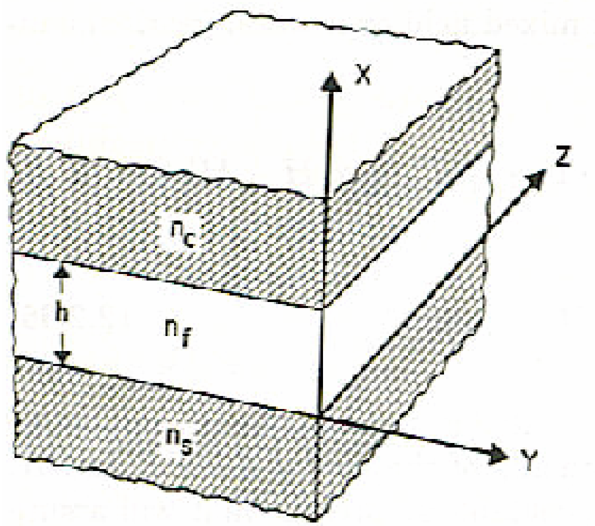


Fig. 2.14. Sketch of an "asymmetric" slab waveguide and the choice of the coordinate system. Note that the z-axis lies in the film-substrate interface

Define:

$$\kappa_c^2 = n_c^2 k^2 - \beta^2 = -\gamma_c^2$$

$$\kappa_f^2 = n_f^2 k^2 - \beta^2$$

$$\kappa_s^2 = n_s^2 k^2 - \beta^2 = -\gamma_s^2$$

Cover: $$\frac{\partial^2}{\partial x^2} E(x,y) + (n_c^2 k^2 - \beta^2) E(x,y) = 0 \Rightarrow \frac{\partial^2}{\partial x^2} E - \gamma_c^2 E = 0$$

Film: $$\frac{\partial^2}{\partial x^2} E(x,y) + (n_f^2 k^2 - \beta^2) E(x,y) = 0 \Rightarrow \frac{\partial^2}{\partial x^2} E - \gamma_f^2 E = 0$$

Substrate: $$\frac{\partial^2}{\partial x^2} E(x,y) + (n_s^2 k^2 - \beta^2) E(x,y) = 0 \Rightarrow \frac{\partial^2}{\partial x^2} E - \gamma_s^2 E = 0$$

TE Modes (I)

Modal solutions are sinusoidal or exponential, depending on the sign of $(k^2 n_i^2 - \beta^2)$

Boundary conditions: The tangential components of **E** and **H** are continuous at the interface between layers. # E_y and $\partial E_y / \partial x$ continuous at the interface.

From Maxell equation

For guided modes:

Cover: $$\frac{\partial^2}{\partial x^2} E_y - \gamma_c^2 E_y = 0 \Rightarrow E_y = E_c \exp[-\gamma_c (x - h)]$$

Film: $$\frac{\partial^2}{\partial x^2} E_y - \kappa_f^2 E_y = 0 \Rightarrow E_y = E_f \cos(\kappa_f x - \phi_s)$$

Substrate: $$\frac{\partial^2}{\partial x^2} E_y - \gamma_s^2 E_y = 0 \Rightarrow E_y = E_c \exp(\gamma_s x)$$

Applying boundary conditions, we obtain:

$$\tan\phi_s = \frac{\gamma_s}{\kappa_f}, \quad \tan\phi_c = \frac{\gamma_c}{\kappa_f}$$

We derived in last section

$$\kappa_f h - \phi_s - \phi_c = m\pi$$ -> Dispersion relation

TE Modes (II)

Relation between the peak fields:

$$E_f^2(n_f^2 - N^2) = E_s^2(n_f^2 - n_s^2) = E_c^2(n_f^2 - n_c^2)$$

E_c, E_f, and E_s can be determined by,

Optical power

$$P = \frac{1}{2}\int \mathrm{Re}\{\mathbf{E} \quad \mathbf{H}^*\} \ \hat{\mathbf{z}}dx$$

Optical confinement factor

$$\Gamma = \frac{\frac{1}{2}\int_0^h \mathrm{Re}\{\mathbf{E} \quad \mathbf{H}^*\}\cdot\hat{\mathbf{z}}dx}{\frac{1}{2}\int_{-\infty}^{\infty} \mathrm{Re}\{\mathbf{E} \quad \mathbf{H}^*\}\cdot\hat{\mathbf{z}}dx}$$

Normalized power distribution

TM Modes

Cover: $\frac{\partial^2}{\partial x^2}H_y - \gamma_c^2 H_y = 0 \Rightarrow H_y = H_c \exp[-\gamma_c(x-h)]$

Film: $\frac{\partial^2}{\partial x^2}H_y + \kappa_f^2 H_y = 0 \Rightarrow H_y = H_f \cos(\kappa_f x - \phi_s)$

Substrate: $\frac{\partial^2}{\partial x^2}H_y - \gamma_s^2 H_y = 0 \Rightarrow H_y = H_c \exp(\gamma_s x)$

Boundary conditions: H_y and E_z continuous at the interface between the layers

$\Rightarrow H_y$ and $\frac{1}{n^2}\frac{dH_y}{dx}$ continuous at the interface between the layers

Applying boundary conditions, we obtain:

$$\tan\phi_s = \left[\frac{n_f}{n_s}\right]^2 \frac{\gamma_s}{\kappa_f}, \quad \tan\phi_c = \left[\frac{n_f}{n_c}\right]^2 \frac{\gamma_c}{\kappa_f}$$

$\kappa_f h - \phi_s - \phi_c = m\pi$ -> Dispersion relation

Relation between the peak fields: $H_f^2 \frac{(n_f^2 - N^2)}{n_f^2} = H_s^2(n_f^2 - n_s^2)\frac{q_s}{n_s^2} = H_c^2(n_f^2 - n_c^2)\frac{q_c}{n_c^2}$

$$q_s = \left[\frac{N}{n_f}\right]^2 + \left[\frac{N}{n_s}\right]^2 - 1, \quad q_c = \left[\frac{N}{n_f}\right]^2 + \left[\frac{N}{n_c}\right]^2 - 1$$

$E_y(x,y,z) = E_i(x,y)\exp(-j\beta z)$ (20)

↳ c, f, s

TE →

$E_x = E_z = H_y = 0$

$\vec{H} = \frac{j}{\omega\mu}\nabla\times\vec{E} \Rightarrow H_x = \frac{j}{\omega\mu}\left(-\frac{\partial}{\partial z}E_z\right) = -\frac{\beta}{\omega\mu}E_y$

$H_z = \frac{j}{\omega\mu}\frac{\partial E_x}{\partial x}$

E_y and H_z are continuous at the boundary

→ E_y an $\frac{\partial E_x}{\partial x}$ are = =

Guided modes Solutions for each layer

(cover) $E_y = E_0 e^{-\gamma_c(x-h)}$, where $x > h$

(film) $E_y = E_f\cos(k_f x - \phi_s)$, $0 < x < h$

(substrate) $E_y = E_s e^{-\gamma_s x}$ $x < 0$

Applying B.C.: Tangential and normal fields are cont.

① $E_s = E_f\cos\phi_s$

② $E_c = E_f\cos(k_f h - \phi_s)$

$= E_f\cos\phi_c = E_f\cos(\phi_c + m\pi)$

③ $\gamma_c E_s = k_f E_f\sin\phi_s$

④ $-\gamma_c E_c = -k_f E_f\sin(k_f h - \phi_s) = -k_f E_f\sin\phi_c$ Phase matching condition

③ ÷ ① → $\tan\phi_s = \frac{\gamma_c}{k_f} = \left(\frac{\sqrt{N^2-n_s^2}}{\sqrt{k_f^2-N^2}}\right)$

④ ÷ ② → $\tan\phi_c = \frac{\gamma_c}{k_f} = \left(\frac{\sqrt{N^2-n_c^2}}{\sqrt{n_f^2-N^2}}\right)$

From ② →

(21)

Dispersion relation $k_f h - \phi_s - \phi_c = m\pi$

$k n_f h \cos\theta - \phi_s - \phi_c = m\pi$

The reason for wave eq.

relation between the peak fields (E_c, E_f, E_s)

$E_s = E_f \cos\phi_s = E_f \frac{\sqrt{n_f^2 - N^2}}{\sqrt{n_f^2 - n_s^2}}$

$\Rightarrow E_f^2 (n_f^2 - N^2) = E_s^2 (n_f^2 - n_s^2) = E_c^2 (n_f^2 - n_s^2)$

★ (Ray can only tell you quality, wave can tell you quantity)

optical power

$P = \frac{1}{2} \int (\vec{E} \times \vec{H}^*)\, dx \cdot \hat{z}$

★ optical confinement factor

$\Gamma = \frac{\frac{1}{2}\int_0^h Re\{\vec{E} \times \vec{H}^*\} \hat{z}\, dx}{\frac{1}{2}\int_{-\infty}^{\infty} Re\{\vec{E} \times \vec{E}^*\} \cdot \hat{z}\, dx}$

3.2 TM

$H_x = H_z = E_y = 0 \Rightarrow H = H_y$

$\vec{E} = -\frac{j}{\omega\epsilon} \nabla \times \vec{H} \rightarrow E_x = \frac{\beta}{\omega\epsilon} H_y$

$E_z = \frac{j}{\omega\epsilon} \frac{\partial H_y}{\partial x} \rightarrow$

$H_y = H_c e^{-\gamma_c (x-h)}$ $x > h$

$H_y = H_f \cos(k_f x - \phi_s)$ $0 < x < h$

$H_y = H_s e^{\gamma_s x}$ $x < 0$

B.C. :

① H_y & E_z continuous at Boundary → H_y & $\frac{1}{n^2} \frac{\partial H_y}{\partial x}$

$\tan\phi_s = \left(\frac{n_f}{n_s}\right)^2 \frac{\gamma_s}{k_f}$ ← s?

$\tan\phi_c = \left(\frac{n_f}{n_c}\right)^2 \frac{\gamma_c}{k_f}$

same dispersion relation (3-layer) $k_f h - \phi_s - \phi_c = m\pi$

purpose for multilayer slab guide (1)

(1) buffer layer to separate electrode from a guide

(2) use metal layer to serve as guide-wave polarization filter that separate TE from TM modes

(3) tailor the waveguide dispersion to obtain phase matching for guide wave second-harmonic generation

(4) selective filter of higher-order harmonic generation

i.e. 5 layer guide for heterostructure (SCH) laser to achieve separate confinement for charge carriers & photons

① Field distribution ← Ray optics can't solve it

② multilayer waveguide

3. E-M wave approach to optical waveguide theory

Wave propating in multilayer structure

Multilayer Stack Theory

Focusing on TE modes first,

$$U = E_y, \quad V = \omega \mu H_z$$

$$U = A\exp(-j\kappa x) + B\exp(j\kappa x)$$

$$V = \kappa\,[A\exp(-j\kappa x) - B\exp(j\kappa x)]$$

At x = 0,

$$U_0 = U(0), \quad V_0 = V(0)$$

$$\begin{bmatrix} U \\ V \end{bmatrix} = \begin{bmatrix} \cos(\kappa x) & \frac{j}{\kappa}\sin(\kappa x) \\ j\kappa\sin(\kappa x) & \cos(\kappa x) \end{bmatrix} \begin{bmatrix} U \\ V \end{bmatrix}$$

$$= \mathbf{M} \begin{bmatrix} U \\ V \end{bmatrix}$$

M: Characteristic matrix of the layer

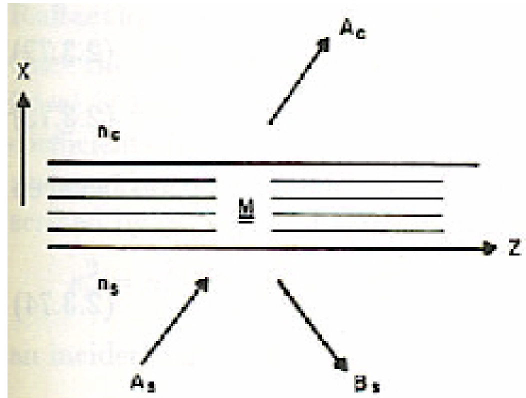


Fig. 2.15. Sketch of a multilayer stack waveguide with substrate index n_s and cover index n_c. The z-axis indicates the direction of mode propagation

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$$\mathbf{M}_i = \begin{bmatrix} \cos(\kappa_i h_i) & \frac{j}{{}_i}\sin(\kappa_i h_i) \\ j\kappa_i \sin(\kappa_i h_i) & \cos(\kappa_i h_i) \end{bmatrix}$$

$$\begin{bmatrix} U_0 \\ V_0 \end{bmatrix} = \mathbf{M} \begin{bmatrix} U_n \\ V_n \end{bmatrix}$$

W. Wang

$$\mathbf{M} = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} = \mathbf{M}_1 \; \mathbf{M}_2 \; . \; \mathbf{M}_n$$

Dispersion Relation for Multilayer Slab Waveguide

Consider guided mode. For substrate and cover,

$$U = A\exp(\gamma x) + B\exp(-\gamma x)$$

$$V = j\gamma[A\exp(\gamma x) - B\exp(-\gamma x)]$$

Imaginary

In the substrate,

$$U_0 = A_s, \quad V_0 = j\gamma_s A_s$$

In the cover,

$$U_n = A_c, \quad V_n = -j\gamma_c A_c$$

Using the multilayer stack matrix theory, we obtain:

$$j(\gamma_s m_{11} + \gamma_c m_{22}) = m_{21} - \gamma_s \gamma_c m_{12}$$

-> ***Dispersion relation for multilayer slab waveguide***

<Example> Four-layer waveguides

Multilayer Stack Theory for TM Modes

$$U = H_y, \quad V = \omega\mu_0 E_z$$

$$U = A\exp(-j\kappa x) + B\exp(j\kappa x)$$

$$V = -\frac{\kappa}{n^2}[A\exp(-j\kappa x) - B\exp(j\kappa x)]$$

Therefore,

$$TE \Rightarrow TM \qquad \kappa \text{ -> } -[\frac{\kappa}{n^2}]$$

Dispersion relation:

$$-j(m_{11}\frac{\gamma_s}{n_s^2} + m_{22}\frac{\gamma_c}{n_c^2}) = m_{21} - \frac{\gamma_s\gamma_c}{n_s^2 n_c^2} m_{12}$$

Characteristic matrix of the i-th layer:

$$\mathbf{M}_i = \begin{bmatrix} \cos(\kappa_i h_i) & -j\frac{n_i^2}{\;_i}\sin(\kappa_i h_i) \\ -j\frac{\kappa_i}{n_i^2}\sin(\kappa_i h_i) & \cos(\kappa_i h_i) \end{bmatrix}$$

Multilayer Stack Theory

(usually have cleaner eq)

Focusing on TE modes first,

Define two field variables: $U \equiv E_y, \quad V \equiv \omega\mu H_z$

General solutions:
$$\begin{cases} U = A\exp(-j\kappa x) + B\exp(j\kappa x) \\ V = \kappa[A\exp(-j\kappa x) - B\exp(j\kappa x)] \end{cases}$$

At x = 0, $\Downarrow$ $U' = -jV \Rightarrow V = jU'$

$$U_0 \equiv U(0), \quad V_0 \equiv V(0)$$

Can be derived from Maxwell eq

$$\begin{bmatrix} U_0 \\ V_0 \end{bmatrix} = \begin{bmatrix} \cos(\kappa x) & \frac{j}{\kappa}\sin(\kappa x) \\ j\kappa\sin(\kappa x) & \cos(\kappa x) \end{bmatrix} \begin{bmatrix} U \\ V \end{bmatrix}$$

$$\equiv \mathbf{M}\begin{bmatrix} U \\ V \end{bmatrix}$$

M: Characteristic matrix of the layer

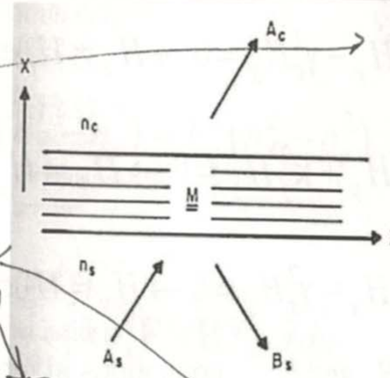


Fig. 2.15. Sketch of a multilayer stack waveguide with substrate index n_s and cover index n_c. The z-axis indicates the direction of mode propagation

* $U(x)$ & $V(x)$ are continuous at the layer boundaries

$$\frac{\partial^2 E}{\partial x^2} + (n_i^2k^2 - \beta^2)E = 0$$

$V = k[Ae^{-jkx} - Be^{jkx}]$

$k_i^2 = n_i^2 k^2 - \beta^2$ ($(kn_i)^2$)

($\beta > nk \rightarrow$ exponential decay)

$\beta < nk \rightarrow$ sinusoidal)

$U'' = (\beta^2 - n^2k^2)U$

$V'' = (\beta^2 - n^2k^2)V$

since U & V must satisfy wave eq.

Rewrite this as

(characteristic) matrix of the layer

$$\mathbf{M}_i = \begin{bmatrix} \cos(\kappa_i h_i) & \frac{j}{\kappa_i}\sin(\kappa_i h_i) \\ j\kappa_i\sin(\kappa_i h_i) & \cos(\kappa_i h_i) \end{bmatrix}$$

$k_i^2 = n_i^2k^2 - \beta^2$, i = layer #

Define
$$\begin{cases} U_0 = U(0) \quad V_0 = V(0) \\ U_0 = A + B \Rightarrow A = \frac{1}{2}(U_0 + \frac{V_0}{k}) \\ V_0 = k(A - B) \Rightarrow B = \frac{1}{2}(U_0 - \frac{V_0}{k}) \end{cases}$$

$$\begin{bmatrix} U_0 \\ V_0 \end{bmatrix} = \mathbf{M}\begin{bmatrix} U_n \\ V_n \end{bmatrix} \qquad \mathbf{M} \equiv \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} = \mathbf{M}_1 \cdot \mathbf{M}_2 \cdot \ldots \mathbf{M}_n$$

$$\begin{bmatrix} U \\ V \end{bmatrix} = \begin{bmatrix} \cos kx & -\frac{j}{k}\sin kx \\ -jk\sin kx & \cos kx \end{bmatrix}\begin{bmatrix} V_0 \\ V_0 \end{bmatrix} \Leftarrow$$

$V = k(Ae^{-jkx} - Be^{jkx}) = k(-U_0\sin(kx) + \frac{V_0}{k}\cos kx)$

$U = Ae^{-jkx} + Be^{jkx}$ [redacted] $= \frac{1}{2}(U_0 + \frac{V_0}{k})e^{\cdot} + \frac{1}{2}(U_0 - \frac{V_0}{k})e^{jkx}$

$= U_0\cos(kx) - j\frac{V_0}{k}\sin(kx)$

(2)

Derive wave equation going in transverse direction

For TE mode we define

$$U = E_y$$
$$V = \omega\mu H_z$$

Which describes transverse variation of the optical field. The $U(x)$ & $V(x)$ are choosen because they are continuous at the layer boundaries

which we obtain the relation

From Maxwell eq.

$$U' = -jV$$
$$V' = j(\beta^2 - n^2k^2)U$$

both U & V obey the transverse wave equation

From wave equation

$$U'' = (\beta^2 - n^2k^2)U$$
$$V'' = (\beta^2 - n^2k^2)V$$

U & V describes the transverse field distribution in a particular layer of constant refractive index n

(3)

The general solution of the wave eq in the layer is

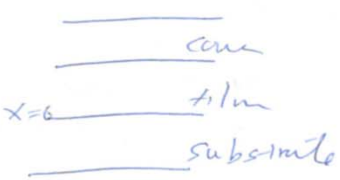
$$U = A e^{-jK_1 x} + B e^{jKx}$$

$$V = K[A e^{-jK_1 x} - B e^{jKx}]$$

where $K_1^2 = n^2 k^2 - \beta^2$

($\beta > nK$ exponential decay, $\beta < nK \rightarrow$ sinusoidal)

constants A & B can be replaced by input value $U_0 = U(0)$ & $V_0 = V(0)$ at input plane $x = 0$ of the layer



Define $\begin{cases} U_0 = A + B \\ V_0 = K(A - B) \end{cases}$

we have

$$A = \frac{1}{2}(U_0 + V_0 / K_1)$$

$$B = \frac{1}{2}(U_0 - V_0 / K_1)$$

$$U = A e^{-jK_1 x} + B e^{jK_1 x}$$
$$= \frac{1}{2}\left(U_0 + \frac{V_0}{K_1}\right) e^{-jK_1 x} + \frac{1}{2}\left(U_0 - \frac{V_0}{K_1}\right) e^{jK_1 x}$$
$$= U_0 \cos(K_1 x) - j \frac{V_0}{K_1} \sin(K_1 x)$$

$$V = K_1 (A e^{-jK_1 x} - B e^{jK_1 x})$$

(4)

$$= K_1' \left(-V_0 \sin(K_1 x) + \frac{V_0}{K_1} \cos(K_1 x)\right)$$

Rearrange

$$\begin{bmatrix} V \\ v \end{bmatrix} = \begin{bmatrix} \cos K_1 x & -\frac{j}{K} \sin K_1 x \\ -j K_1 \sin K_1 x & \cos K_1 x \end{bmatrix} \begin{bmatrix} V_0 \\ V_0 \end{bmatrix}$$

it's like an interference filter

stack along x direction

$$\begin{bmatrix} U_{i-1} \\ V_{i-1} \end{bmatrix} = M \begin{bmatrix} U_i \\ V_i \end{bmatrix} \Rightarrow \begin{bmatrix} U_0 \\ V_0 \end{bmatrix} = M \begin{bmatrix} U_n \\ V_n \end{bmatrix}$$

$$M \equiv \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} = M_1 \cdot M_2 \cdots M_n$$

Dispersion Relation for Multilayer Slab Waveguide

Consider guided mode. For substrate and cover,

$$U = A\exp(\gamma x) + B\exp(-\gamma x)$$
$$V = j\gamma[A\exp(\gamma x) - B\exp(-\gamma x)]$$

In the substrate,

$$U_0 = A_s, \quad V_0 = j\gamma_s A_s$$

$U = A_s e^{\gamma_s x}$

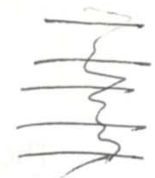
$V = j\gamma_s A_s e^{\gamma_s x}$

In the cover,

$$U_n = A_c, \quad V_n = -j\gamma_c A_c$$

$U = B_c e^{-\gamma_c x} \rightarrow U_n = B_c$

$V = -j\gamma_c B_c e^{-\gamma_c x} \rightarrow V_n = -j\gamma_c B_c$



Using the multilayer stack matrix theory, we obtain:

$$\begin{bmatrix} U_0 \\ V_0 \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} U_n \\ V_n \end{bmatrix}$$

$$j(\gamma_s m_{11} + \gamma_c m_{22}) = m_{21} - \gamma_s\gamma_c m_{12}$$

find n_{eff} — Eigenfunction

→ *Dispersion relation for multilayer slab waveguide*

$\rightarrow A_s = (m_{11} - j\gamma_c m_{12})B_c$ (1)

$j\gamma_s A_s = (m_{21} - j\gamma_c m_{22})B_c$ (2)

<Example> Four-layer waveguides

$\beta = k n_{eff}$

① ÷ ② $\rightarrow j(\gamma_s m_{11} + \gamma_c m_{22}) = m_{21} - \gamma_s\gamma_c m_{12}$

$\sqrt{\beta^2 - k^2 n_{s,c}^2} \leftarrow \gamma_s, \gamma_c$ functions of β, $k(\lambda, \omega)$

m_{ij} function of $k(\omega)$, h_i

multilayer Stack Theory

TE mode $\quad U = E_y$

$V \equiv \omega\mu H_z$

M = characteristic matrix of the layer

det M = 1

$$\rightarrow \begin{bmatrix} U_0 \\ V_0 \end{bmatrix} = M \begin{bmatrix} U \\ V \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} U \\ V \end{bmatrix} = \begin{bmatrix} \cos(kx) & -\frac{j}{k}\sin(kx) \\ -jk\sin(kx) & \cos(kx) \end{bmatrix} \begin{bmatrix} V_0 \\ V_0 \end{bmatrix}$$

one layer

$$\begin{bmatrix} U_0 \\ V_0 \end{bmatrix} = \begin{bmatrix} \cos(kx) & \frac{j}{k}\sin(kx) \\ jk\sin(kx) & \cos kx \end{bmatrix} \begin{bmatrix} U \\ V \end{bmatrix}$$

multilayer (tell you the BC)

$$\begin{bmatrix} U_{i-1} \\ V_{i-1} \end{bmatrix} = M_i \begin{bmatrix} U_i \\ V_i \end{bmatrix}$$

$$\begin{bmatrix} U_0 \\ V_0 \end{bmatrix} = M \begin{bmatrix} U_n \\ V_n \end{bmatrix}$$

$$M_i = \begin{bmatrix} \cos(k_i' x) & \frac{j}{k_i'}\sin k_i' x \\ jk_i'\sin(k_i' x) & \cos k_i' x \end{bmatrix}$$

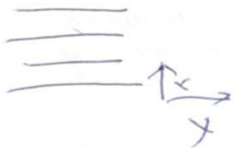
$k_i'^2 = n_i k^2 - \beta^2 \quad i = \text{layer \#}$

Dispersion relation

$$j(\gamma_s m_{11} + \gamma_c m_{22}) = m_{21} - \gamma_s \gamma_c m_{12}$$

TM mode

$U = H_y,\ V = \omega \varepsilon_0 E_z$ $\quad E_y = H_z = H_x = 0$



~~$U = H$~~

$\nabla \times \vec{H} = j\omega\epsilon\vec{E}$

$\Rightarrow \quad \frac{\partial H_x}{\partial x} = j\omega\varepsilon_r\varepsilon_0 E_z$

$\Rightarrow \quad U' = j n^2 V \quad \boxed{n^2 = \epsilon_r}$

$\nabla \times \vec{E} = -j\omega\mu H_z$

$E_x \propto e^{-j\beta z}$

$\frac{\partial E_z}{\partial x} + j\beta E_x = -j\omega\mu H_y$

$$\begin{cases} \nabla \times \vec{H} = j\omega\epsilon\vec{E} \\ -\frac{\partial H_y}{\partial z} = j\omega\epsilon E_x \\ \beta H_y = \omega\epsilon E_x \end{cases}$$

$\omega\epsilon \frac{\partial E_z}{\partial x} + j\beta^2 H_y = j\omega^2\mu\epsilon H_y$

$$\begin{cases} \epsilon = \epsilon_0 n^2 \\ \omega^2 \underset{= \frac{1}{c^2}}{\mu\epsilon} = n^2 k^2 \end{cases}$$

(26)

Comb.

$$\begin{cases} V' = j\left(k^2 - \frac{\beta^2}{n^2}\right) U \\ U' = j n^2 V \end{cases}$$

$$\boxed{U'' = \left(\frac{\beta^2}{n^2} - k^2\right) U}$$

General Solution :

$$\begin{cases} U = A e^{-jkx} + B e^{jkx} \\ U' = j n^2 V \end{cases}$$

$$V = -\left(\frac{k}{n^2}\right)\left(A e^{-jkx} - B e^{jkx}\right)$$

no chg

k in TE case

U_0 V_0 in terms of A, B

U, V as functions of U_0, V_0

TE → TM $\quad k \rightarrow -\left(\frac{k}{n^2}\right)$

Only applies to propagation constants γ & k

can't apply to M_i

Dispersion relation

$$TE = j(\gamma_s m_{11} + \gamma_c m_{22}) = M_{21} - \gamma \gamma_s m_{12}$$

$$TM = -j\left(m_{11}\frac{\gamma_s}{n_s^2} + m_{22}\frac{\gamma_c}{n_s^2}\right)$$

$$= m_{21} - \frac{\gamma_s \gamma_c}{n_s^2 n_s}$$

characteristic matrix

$$TE \quad M_i = \begin{bmatrix} \cos(k_i h_i) & \frac{j}{k_i}\sin(k_i h_i) \\ j k_i \sin(k_i h_i) & \cos(k_i h_i) \end{bmatrix}$$

$$TM \quad M_i = \begin{bmatrix} \cos(k_i h_i) & -j\left(\frac{n_i^2}{k_i}\right)\sin k_i \\ -j\frac{k_i}{n_i^2}\sin(k_i h_i) & \cos(k_i h_i \end{bmatrix}$$

Example Four layer waveguide

Dispersion relation

$$j\gamma(m_{11}+m_{22}) = m_{21} - \gamma^2 m_{12}$$

cut off cond.

$$\gamma = 0 \Rightarrow m_{21} = 0$$

Characterist matrix $M = M_1 M_2$

$$M_1 = \begin{bmatrix} \cos(k_1 h_1) & \frac{j}{k_1}\sin(k_1 h_1) \\ j k_1 \sin(k_1 h_1) & \cos(k_1 h_1) \end{bmatrix}$$

$$M_2 = \begin{bmatrix} k_2 h_2 & \frac{j}{k_2} \quad " \\ j k_2 \quad " & " \end{bmatrix}$$

$$m_{11} = \cos(k_1 h_1)\cos(k_2 h_2) - \frac{k_2}{k_1}\sin(k_1 h_1)\sin(k_2 h_2)$$

$$m_{22} = \cos(k_1 h_1)\cos(k_2 h_2) - \frac{k_1}{k_2}\sin(k_1 h_1)\sin(k_2 h_2)$$

$$m_{12} = \frac{j}{k_1}\sin(k_1 h_1)\cos(k_2 h_2) + \frac{j}{k_2}\cos(k_1 h_1)\sin(k_2 h_2)$$

$$m_{21} = j k_1 \sin(k_1 h_1)\cos(k_2 h_2) + j k_2 \cos(k_1 h_1)\sin(k_2 h_2)$$

Dispersion relation: (29)

$$\frac{\cos(k_1 h_1) - \frac{k}{\gamma^2}\sin(k_1 h_1)}{\cos(k_1 h_1) + \frac{\gamma}{k_1}\sin(k_1 h_1)} = -\frac{\cos(k_2 h_2) - \frac{k_2}{\gamma}\sin(k_2 h_2)}{\cos(k_2 h_2) + \frac{\gamma}{k_2}\sin(k_2 h_2)}$$

$$\Rightarrow \frac{\tan(k_1 h_1) - \frac{\gamma}{k_1}}{1 + \frac{\gamma}{k_1}\tan(k_1 h_1)} \cdot \frac{k_1}{\gamma} = \frac{\tan(k_2 h_2) - \frac{\gamma}{k_2}}{1 + \frac{\gamma}{k_2}\tan(k_2 h_2)} \cdot \frac{k}{\gamma}$$

Since $\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$

$A = k_1 h_1 \quad k_2 h_2$

$B = \tan^{-1}\frac{\gamma}{k}, \quad \tan \frac{\gamma}{k_2}$

$$k \tan\left[k_1 h_1 - \tan^{-1}\left(\frac{\gamma}{k_1}\right)\right] = -k_2 \tan\left[k_2 h_2 - \tan^{-1}\left(\frac{\gamma}{k_2}\right)\right]$$

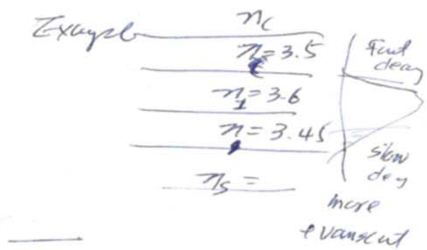
* fundametal mode $N_{eff} \approx n_{core}$

  N_{eff} is index allow wave to propagate in core

* When solve the field distribution

  – first solve N_{eff} } define k

* Look at ray theory & see why more field is in substrate



→ * optical confinement factor

→ * to find next higher order mode

* mode # $\propto \sqrt{N_{eff}^2 - n_s^2}$

Larger diff in index more modes

N_{eff} closer to "cladding"

$N_{eff} = 3.55$ (index given higher mode

$\lambda = 0.85\ \mu m$

$h = 0.5$ (given 1st mode)

$\Gamma = 0.95$

$\lambda = 10\ \mu m$
$n_1 = 3.6$
$n_2 = 3.5$
$n_s = 3.$

$\lambda = 0.85$
$n = 5.0$
n_{eff} 3.85
4th mode
wider energy spread at higher order mode

λ 0.85
$n = 5.0$
$N_{eff} = 4.8$
0 order

$N_{eff} = 3.5$
$\lambda = 0.85$
$h = 0.3$
$\Gamma = 0.89$ (given 0th)

① Field distribution ← Ray optics can't solve is

② multilayer waveguide

3. E-M wave approach to optical waveguide theory

Arbitrary Shape Waveguide

Rectangular Waveguide Geometries

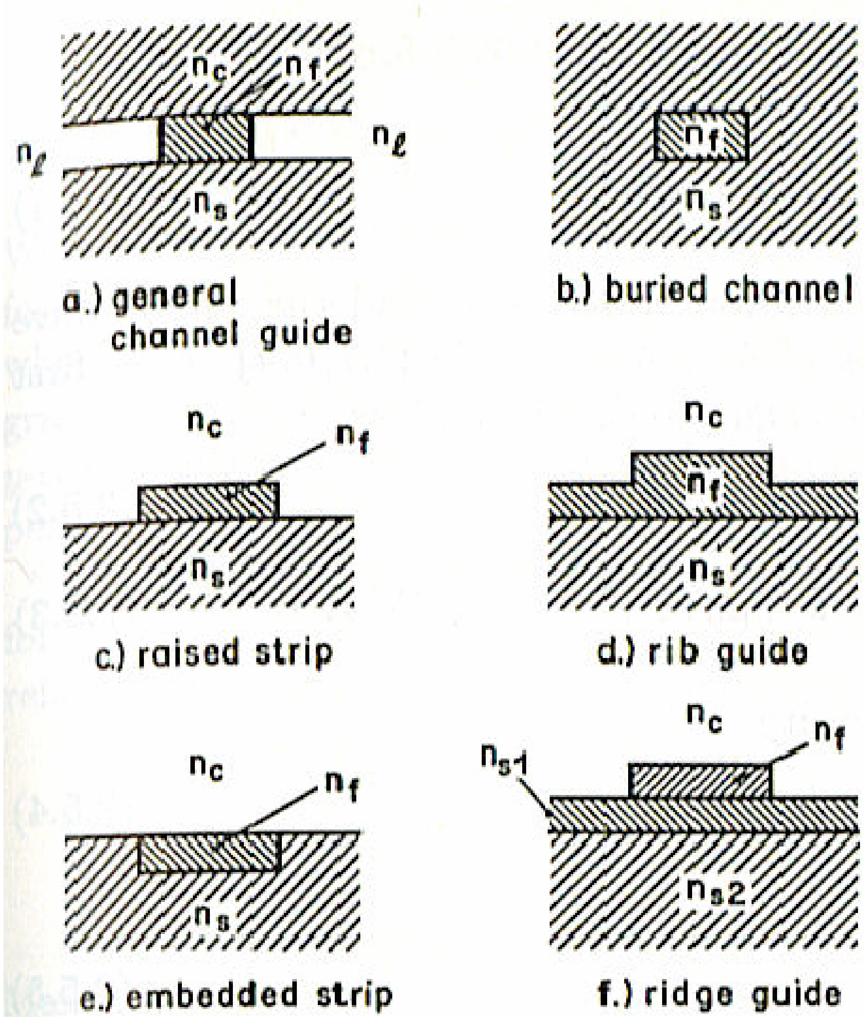


Fig. 2.24a–f. Cross-sections of six channel guide structures

The Method of Field Shadows (I)

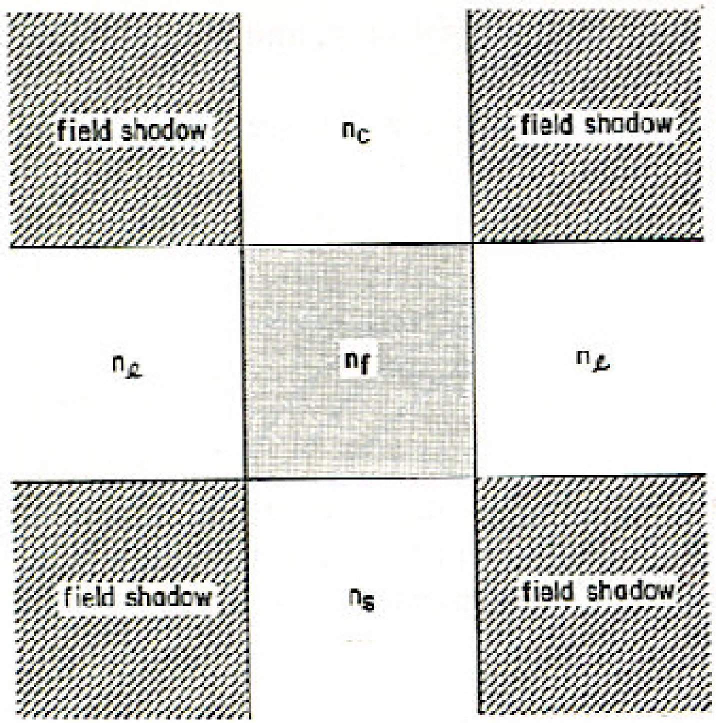


Fig. 2.26. Illustration of the method of field shadows showing the cross-section of a buried channel guide. The method ignores the fields in the shaded "shadow" areas

Ignore the fields and refractive indices in the shaded field shadow regions.

-> Results in separable index profiles.

Works well as long as the fields are well confined in the high index (n_f) region of the waveguide.

-> Not applicable at cut-off.

The Method of Field Shadows (II)

Assuming a buried channel waveguide structure.

Take the waveguide apart in x and y direction, find its effective N in each direction and then combine them to find the final N and b

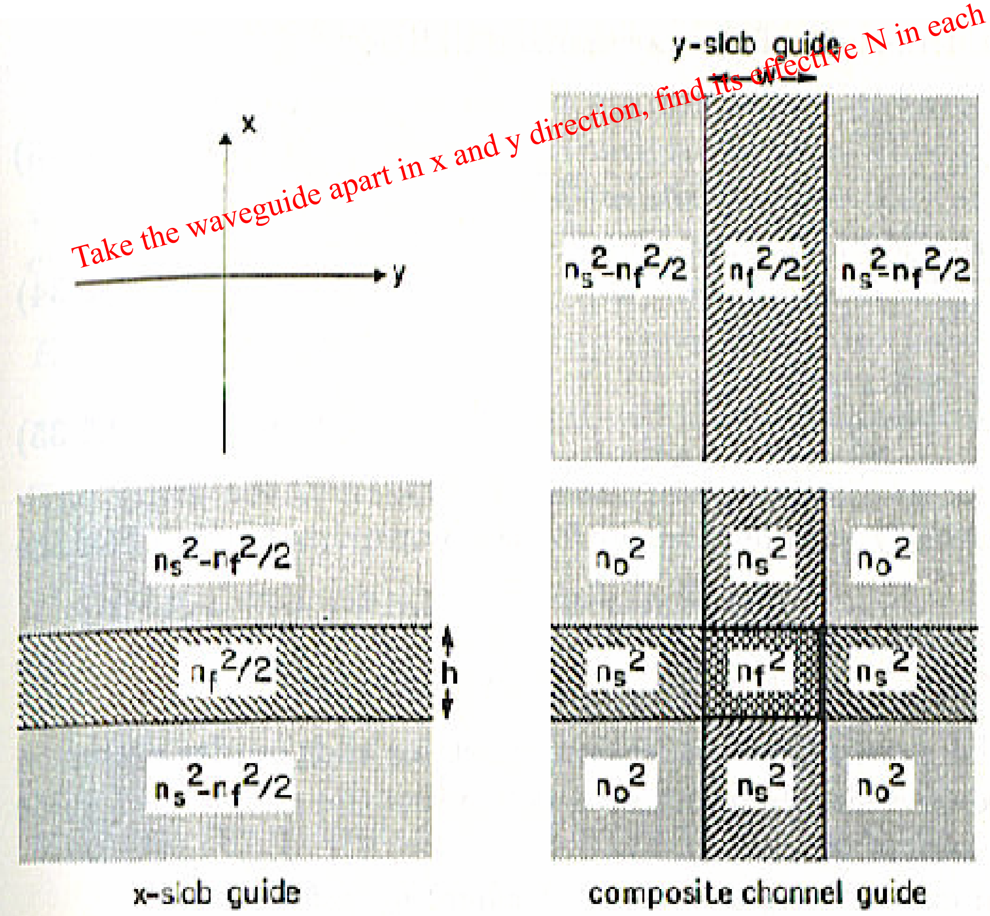


Fig. 2.27. Method of field shadows. The sketch shows the x-y cross-section of a composite guide made up by summing the permittivities (n^2) of an x-slab guide of height h and a y-slab guide of height w. The various n^2 values are indicated

$$E(x, y) = X(x)Y(y)$$

$$\beta^2 = \beta_x^2 + \beta_y^2 \qquad N^2 = N_x^2 + N_y^2$$

$$V_x = kh\sqrt{n_f^2 - n_s^2}$$

$$V_y = kw\sqrt{n_f^2 - n_s^2}$$

Obtain N_x and N_y, therefore N, by using the dispersion relation chart and

$$b_x = \frac{N_x^2 - n_s^2 + n_f^2/2}{n_f^2 - n_s^2}$$

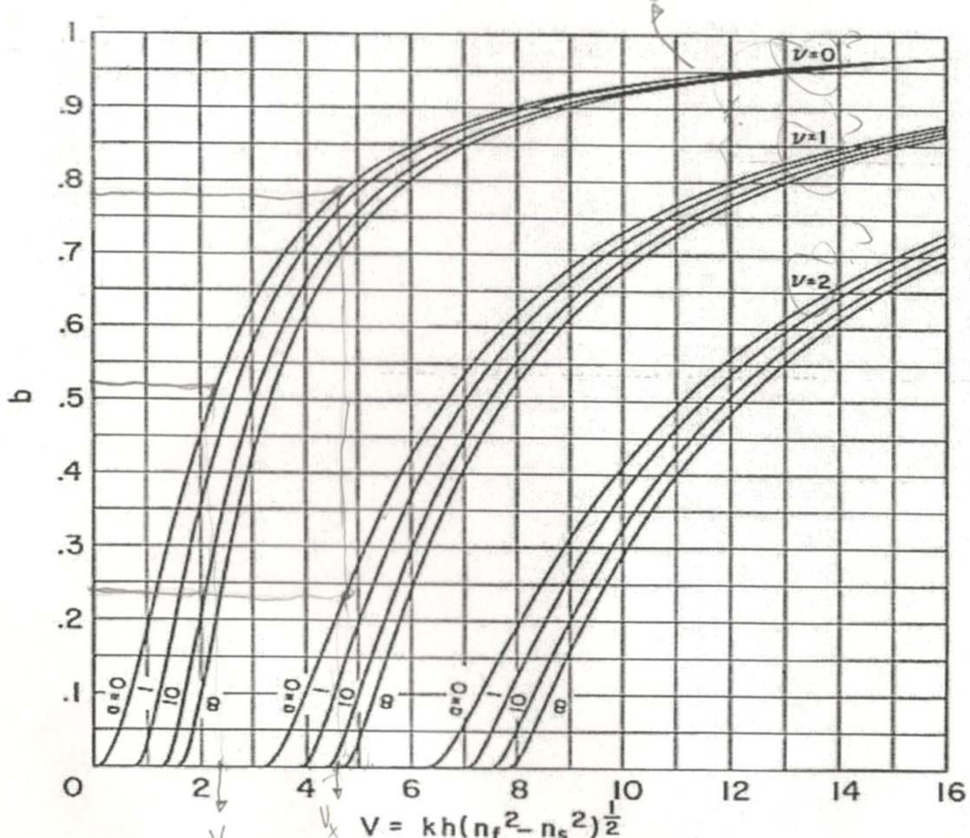
$$b_y = \frac{N_y^2 - n_s^2 + n_f^2/2}{n_f^2 - n_s^2}$$

Or instead of solving for N_x and N_y, we can use

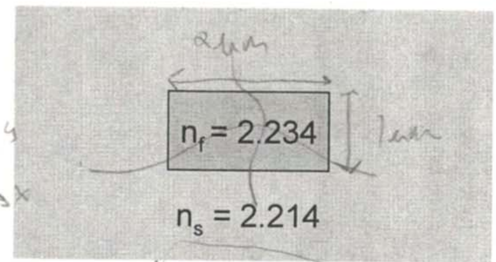
$$b = \frac{N^2 - n_s^2}{n_f^2 - n_s^2} = b_x + b_y - 1$$

Another example

Field Shadows Method Exercise



Ti:LiNbO$_3$ channel waveguide
$\lambda = 0.8\ \mu m$



Determine effective index and modal distribution

① find V_x, V_y, determine which modes
② find b
③ from b find N_{eff}

find b_x + b_y

$V_x = kh\sqrt{n_f^2 - n_s^2}$

$V_y = kh\sqrt{n_f^2 - n_s^2}$

$k = \frac{2\pi}{\lambda} = 7.853$

$V_x \Rightarrow b_x = 0.245$

$V_y \Rightarrow b_y = 0.52$

$N_x = 2.4282132$

$N_y = 2.4526772$

$N = 3.45$

$b = \frac{N - 2.406418}{.05856}$

a = 0, symmetrical

The Effective-Index Method

Treat each section as slab layer, find its effective N and then treated the effective layer as slab layer again and find its effective N again

(1) Determine the normalized thickness of the channel and lateral guides.

$$V_f = kh\sqrt{n_f^2 - n_s^2}, \quad V_l = kl\sqrt{n_f^2 - n_s^2}$$

(2) Use the dispersion relation chart to determine the normalized guide indices b_f and b_l.

Determine the corresponding effective indices by referring to the Table on

$$N_{f,l}^2 = n_s^2 + b_{f,l}(n_f^2 - n_s^2)$$

Effective index for rectangular waveguide

(3) Determine the normalized width. $V_{eq} = kw\sqrt{N_f^2 - N_l^2}$

Then determine the normalized guide index b_{eq} using the dispersion relation chart.

(4) The effective index of the waveguide can be determined from

$$b_{eq} = \frac{N^2 - N_l^2}{N_f^2 - N_l^2}$$

$$\Rightarrow \quad N^2 = N_l^2 + b_{eq}(N_f^2 - N_l^2)$$

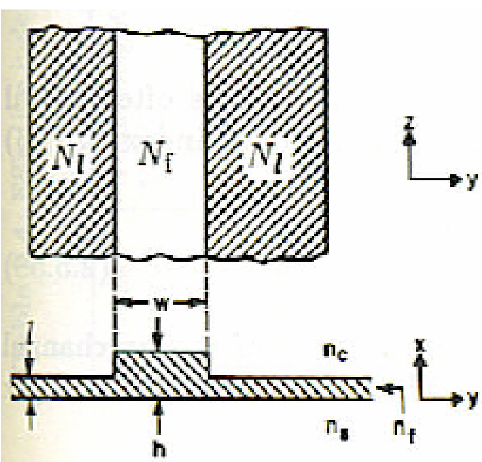
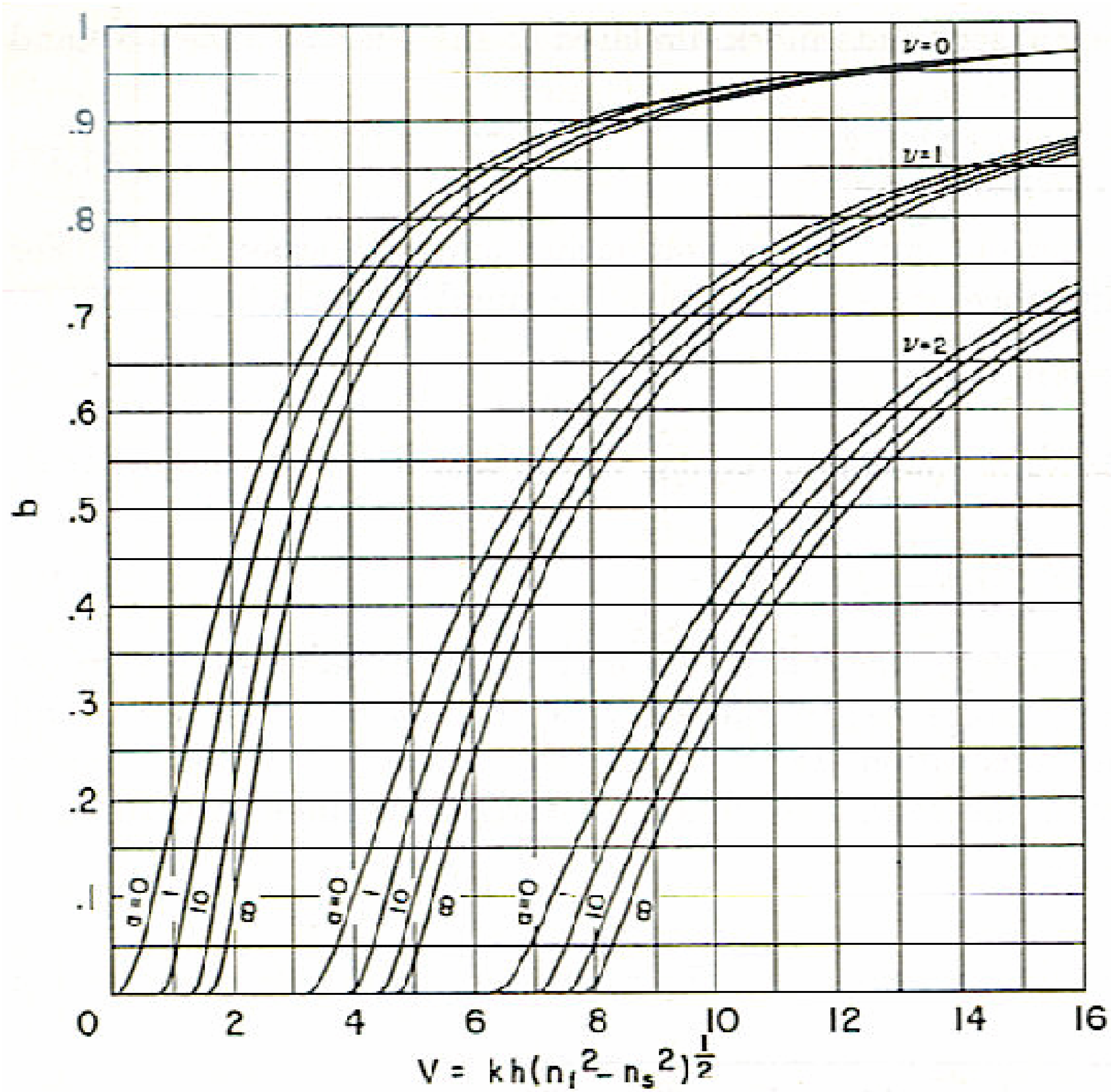


Fig. 2.28. Illustration of the effective-index method showing the top view and the cross section of a rib guide.

Note: For multi-layer waveguide structure, such as ridge waveguides, use the matrix method to determine N_f and N_l, then continue on (3).

<Example> Ti:LiNbO$_3$, λ = 0.8 μm,
n_f = 2.234, n_s = 2.2 14, n_c = 1,
h = 1 .8μm, l = " 1μm,

The Dispersion Relation Chart



Effective Index Parameters for Rectangular Waveguides

Channel structure	Guide height V_f, V_l	Eff. index N_f, N_l	$N_f^2 - N_l^2$	Channel guide index b
a) General	$V_f = kh\sqrt{n_f^2 - n_s^2}$ $V_l = kh\sqrt{n_l^2 - n_l^2}$	$N_f^2 = n_s^2 + b_f(n_f^2 - n_s^2)$ $N_l^2 = n_s^2 + b_l(n_l^2 - n_s^2)$	$b_f(n_f^2 - n_s^2) - b_l(n_l^2 - n_s^2)$	$b_f b_{eq} + b_l(1 - b_{eq})a_{ch}$
b) Buried	$V_f = kh\sqrt{n_f^2 - n_s^2}$	$N_f^2 = n_s^2 + b_f(n_f^2 - n_s^2)$ $N_l = n_s$	$b_f(n_f^2 - n_s^2)$	$b_f b_{eq}$
c) Raised	$V_f = kh\sqrt{n_f^2 - n_s^2}$	$N_f^2 = n_s^2 + b_f(n_f^2 - n_s^2)$ $N_l = n_c$	$(n_s^2 - n_c^2) + b_f(n_f^2 - n_s^2)$	$b_f b_{eq} - (1 - b_{eq})a$
d) Rib	$V_f = kh\sqrt{n_f^2 - n_s^2}$ $V_l = kl\sqrt{n_f^2 - n_s^2}$	$N_f^2 = n_s^2 + b_f(n_f^2 - n_s^2)$ $N_l^2 = n_s^2 + b_l(n_f^2 - n_s^2)$	$(b_f - b_l)(n_f^2 - n_s^2)$	$b_f b_{eq} + b_l(1 - b_{eq})$
e) Embedded	$V_f = kh\sqrt{n_f^2 - n_s^2}$	$N_f^2 = n_s^2 + b_f(n_f^2 - n_s^2)$ $N_l = n_s$	$b_f(n_f^2 - n_s^2)$	$b_f b_{eq}$
f) Ridge	$V_f = kh\sqrt{n_f^2 - n_s^2}$ $V_l = kl\sqrt{n_f^2 - n_s^2}$	$N_f^2 = n_{s1}^2 + b_f(n_f^2 - n_{s1}^2)$ $N_l^2 = n_{s2}^2 + b_l(n_{s1}^2 - n_{s2}^2)$	$(1 - b_l)(n_{s1}^2 - n_{s2}^2) + b_f(n_f^2 - n_{s1}^2)$	$b_{eq}(1 + b_f \cdot a_{ridge}) + b_l(1 - b_{eq})$

w. wang

Numerical Comparisons Between Different Methods

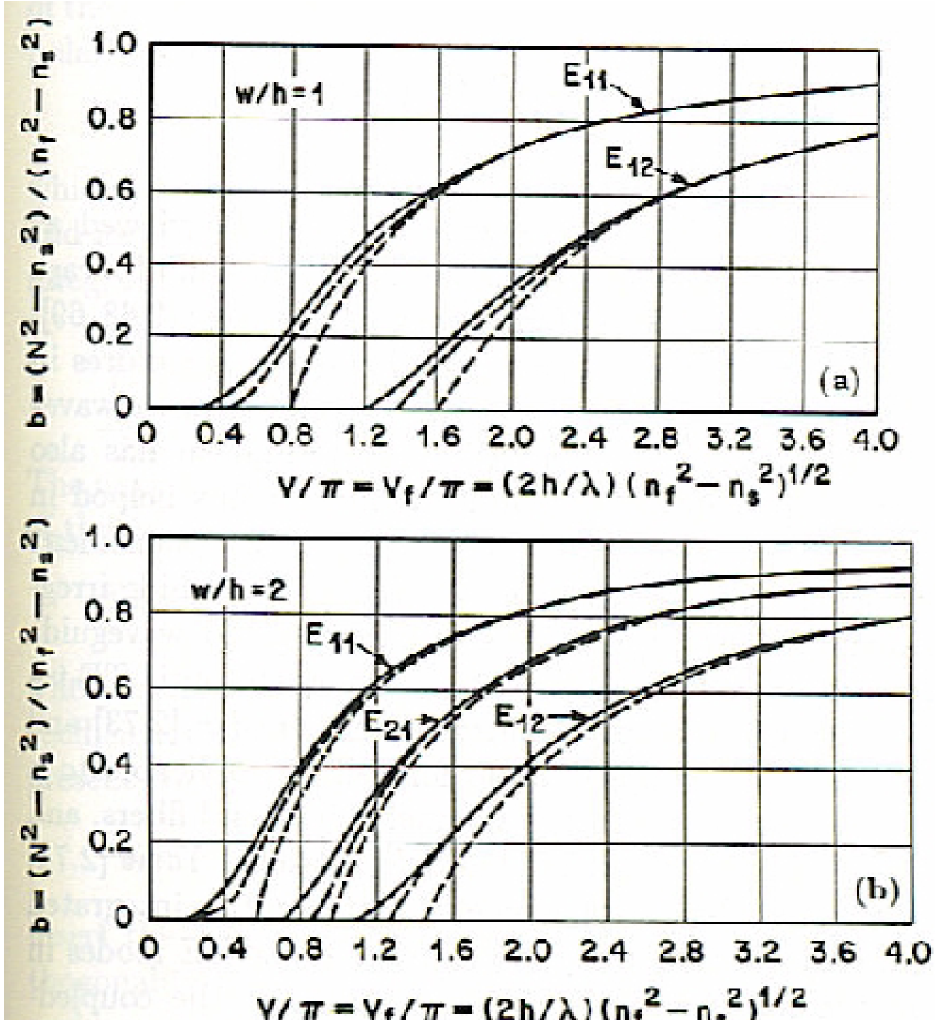
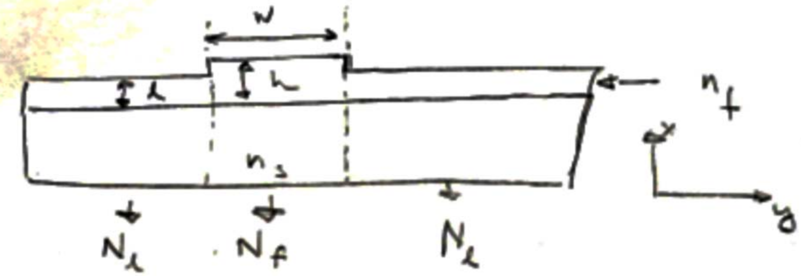


Fig. 2.29a,b. Normalized dispersion curves for a buried channel guide comparing the predictions of the numerical calculations (*dot-dashed lines*), of the effective index method (*solid lines*), and of the field-shadow method (*dashed lines*). Comparisons are shown for the aspect ratios of $w/h = 1$ and $w/h = 2$. (After [2.66])

Effective index method provides good approximation even near cut-off.

Effective-Index Method



$N_f > N_l$

Four Steps

1) Determine normalized thickness

V_f, V_l

$$V_f = kh\sqrt{n_f^2 - n_s^2}$$

$$V_l = kl\sqrt{n_f^2 - n_s^2}$$

Example: Ti : $LiNbO_3$ $\lambda = 0.8\,\mu m$

$n_f = 2.234$

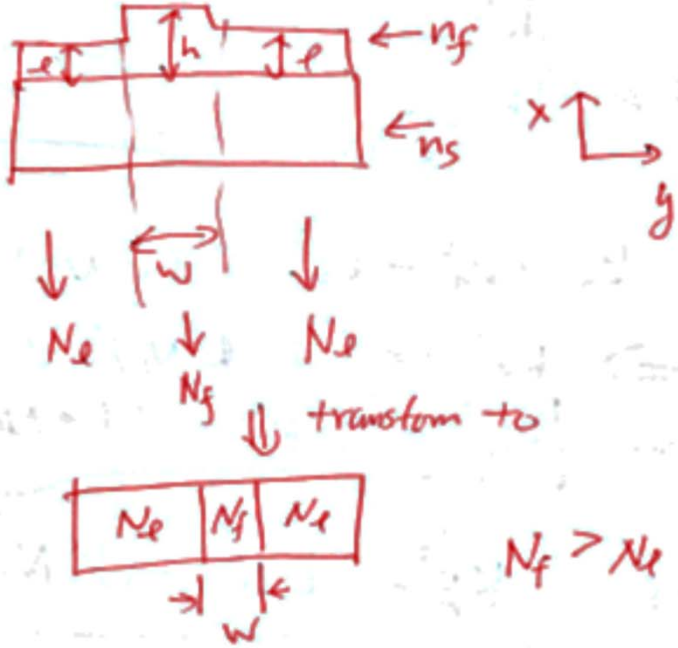
$n_s = 2.214$ $n_c = 1$

$h = 1.8\,\mu m$, $l = 1\,\mu m$

$W = 2\,\mu m$

$V_f = 4.2$

$V_l = 2.3$



$N_f > N_\ell$

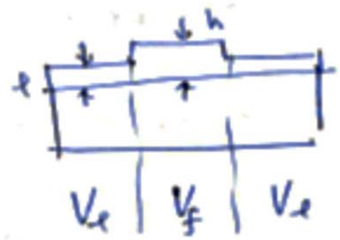
Four steps (Using dispersion relation chart to determine normalized guide indices)

1) determine normalized thickness

V_f, V_ℓ

$$V_f = k h \sqrt{n_f^2 - n_s^2}$$

$$V_\ell = k \ell \sqrt{n_f^2 - n_s^2}$$

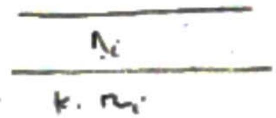


Example: $Ti = LiNbO_2$ $\quad W = 2\,\mu m$

$n_f = 2.234$ $\quad \lambda = 0.8\,\mu m$

$n_s = 2.214$ $\quad n_c = 1$

$h = 1.8\,\mu m$ $\quad \ell = 1\,\mu m$



→ $\beta = kN$

$$k_x^2 + k_y^2 + k_z^2 = (k n_i)^2$$

$$\Rightarrow k_x^2 + k_y^2 = (k n_i)^2 - (kN)^2$$

$$k_x^2 = n_i^2 k^2 - (k \cdot N_s)^2$$

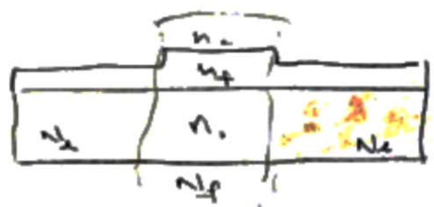
$$\Rightarrow k_y^2 = (N_{f,\ell})^2 \cdot k^2 - (kN)^2$$

Break down to these two equations

$N_f > N$ → N_f or N_e

$N_\ell < N$ k_y imaginary

→ Exponential decay in lateral region



$N_\ell < N < N_f$

$n_s < N_f < n_f$ $n_s < N < n_f$

---

homework → use matrix approach → Multi-layer stack theory

matrix approach → N_f, N_e

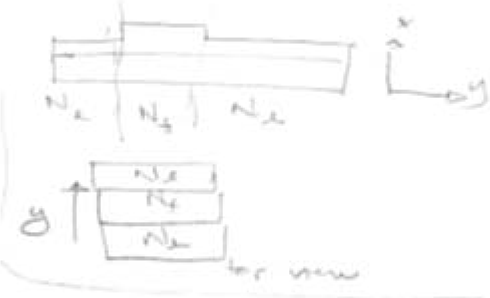
x direction $$K_x^2 = n_i^2 k^2 - (k \cdot N_f)^2$$

y direction $$K_y^2 = k^2 N_f^2 - (k N)^2$$

2) Dispersion relation chart to determine the normalized indices

$b_f = 0.65$, $b_e = 0.2$ for TE

Determine corresponding effective indices N_f, N_e



$$N_f^2 = N_s^2 + b_f (n_f^2 - n_s^2)$$

$$N_e^2 = n_s^2 + b_e (n_f^2 - n_s^2)$$

$$\rightarrow N_f = 2.227 \, , \quad N_e = 2.218$$

$\Rightarrow$ lateral $k_x^2 = n$; guided $k_x^2 = n_f^2 k^2 - k^2 N_f^2$

3) determine the normalized width

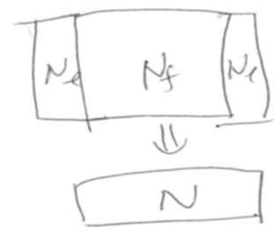
$$V_{eq} = kW \sqrt{N_f^2 - N_e^2}$$

$$V_{eq} = 3.14$$

use dispersion relation chart to determine b_{eq}

$$a = 0$$
$$b_{eq} = 0.64$$

4)
$$b = \frac{N^2 - n_s^2}{n_f^2 - n_s^2}$$

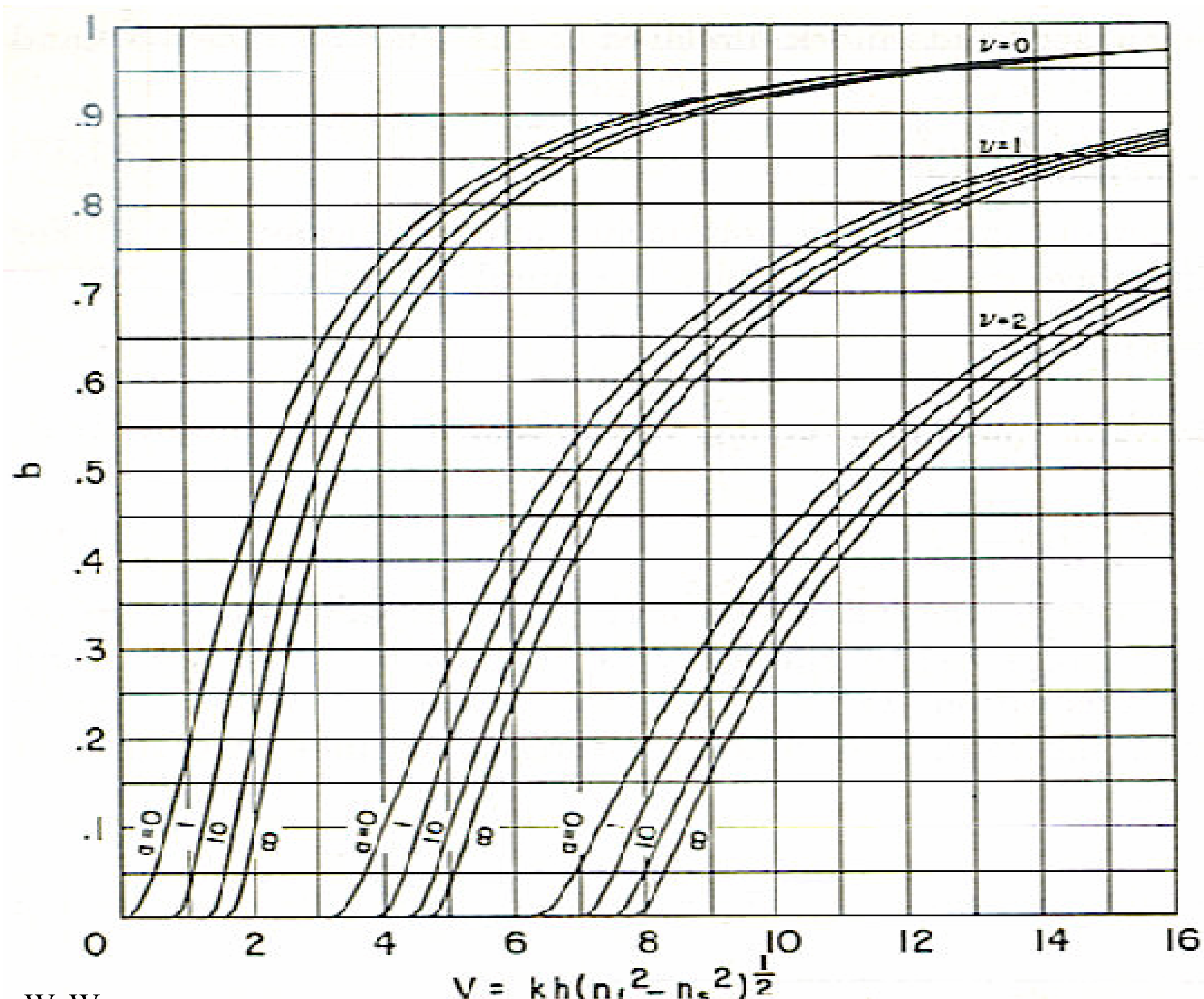


$$\Rightarrow b_{eq} = \frac{N^2 - N_e^2}{N_f^2 - N_e^2}$$

$$N \Rightarrow 2.224$$

$$\Rightarrow N^2 = N_e^2 + b_{eq} (N_f^2 - N_e^2)$$

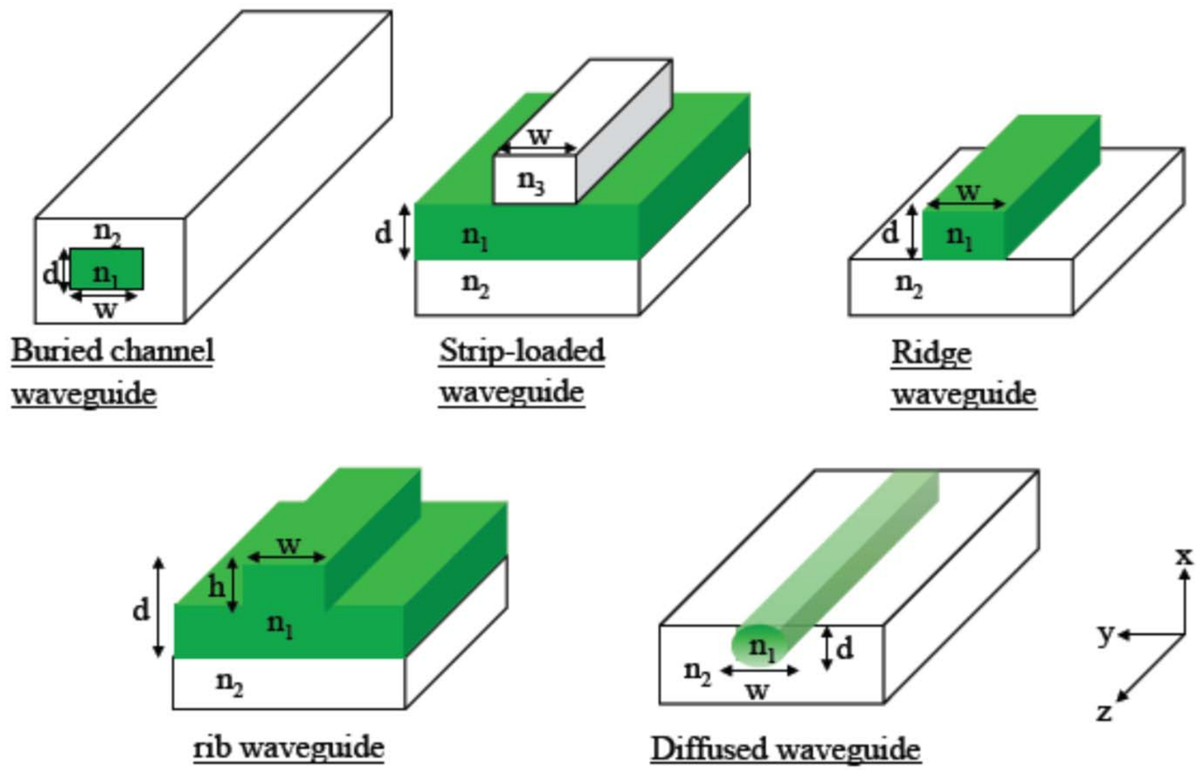
The Dispersion Relation Chart



Channel waveguides

- Most waveguides used in device applications are *nonplanar* waveguides.
- For a nonplanar waveguide, the index profile n(x, y) is a function of both transverse coordinates x and y.
- There are many different types of nonplanar waveguides that are differentiated by the distinctive features of their index profiles.
- One very unique group is the *circular optical fibers* (to be discussed in Lecture 5).
- Another important group of nonplanar waveguides is the *channel waveguides*, which include
  - The buried channel waveguides
  - The strip-loaded waveguides
  - The ridge waveguides
  - The rib waveguides
  - The diffused waveguides.

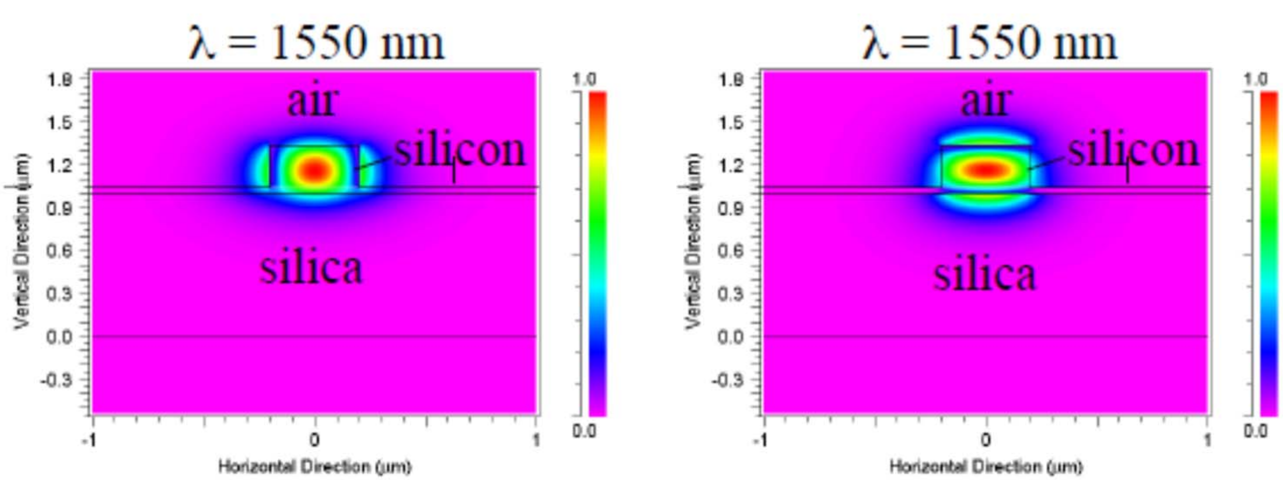
Representative channel waveguides



Numerical Analysis

• Except for those few exhibiting special geometric structures, such as circular optical fibers, non-planar dielectric waveguides generally do not have analytical solutions for their guided mode characteristics.

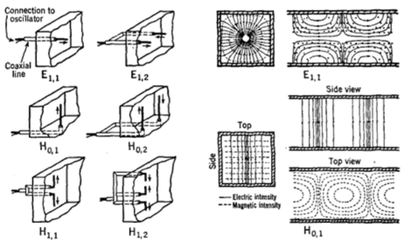
• Numerical methods, such as the beam propagation method, are typically used for analyzing such waveguides (e.g. silicon-on-insulator waveguides modes, TE and TM mode electric field distributions)

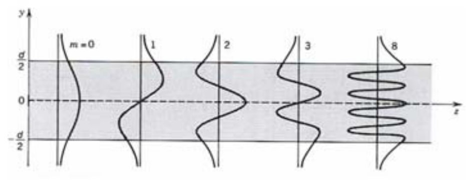


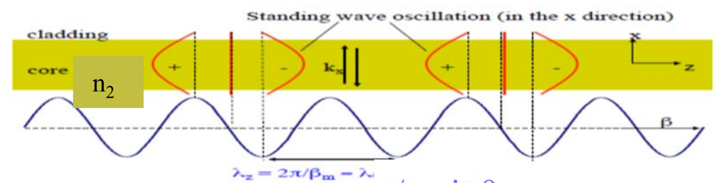
Take away

- Total Internal Reflection
- Metal (low frequency, well confine beam) versus dielectric (high frequency low loss, leaky wave)

(skin depth for dielectric) $\delta_p = 1/\alpha = \frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}}$ (skin depth for conductor) $\delta_p = 1/\alpha = \sqrt{(\frac{2}{\omega\mu\sigma})} = \delta$

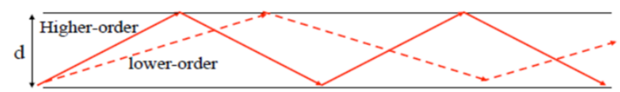






$/n_2\sin\theta_m$

- Propagating modes are discrete and function of incident angle relating particle and wave theory.. We need to know both
- Modes are determined by transverse resonating condition



$$2k_1 d\cos\theta + \varphi_2(\theta) + \varphi_3(\theta) = 2m\pi$$

where m is an integer = 0, 1, 2, …

- Wave equation determined by Maxwell's equations and B.C. (see next page)
- Operating frequency, mode numbers, effective index, waveguide dimension, can be determined by dispersion equation

Dispersion equation (β vs. ω):

$$kn_f h\cos\theta - \phi_c - \phi_s = m\pi$$

Effective guide index $N = \beta/k = n_f \sin\theta$ $\quad n_s < N < n_f$

- Later a dimensionless dispersion equation

Normalized frequency and film thickness

$$V = kh\sqrt{n_f^2 - n_s^2}$$

Normalized guide index

$$b = \frac{N^2 - n_s^2}{n_f^2 - n_s^2}$$

b = 0 at cut-ooff ($N = n_s$), and approaches 1 as $N \to n_f$.

Measure for the asymmetry

$$a = \frac{n_s^2 - n_c^2}{n_f^2 - n_s^2} \text{ for TE,} \quad a = \frac{n_f^4}{n_c^4}\frac{n_s^2 - n_c^2}{n_f^2 - n_s^2} \text{ for TM}$$

a = 0 for perfect symmetry ($n_s = n_c$), and a approaches infinity for strong asymmetry ($n_s \neq n_c$, $n_s \sim n_f$).

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Take away

- Using slab waveguide to figure out other waveguide configuration

(1) Determine the normalized thickness of the channel and lateral guides.

$$V_f = kh\sqrt{n_f^2 \quad n_s^2}, \quad V_l = kl\sqrt{n_f^2 \quad n_s^2}$$

(2) Use the dispersion relation chart to determine the normalized guide indices b_f and b_l.
Determine the corresponding effective indices by referring to the Table on page 1b-14.

$$N_{f,l}^2 = n_s^2 + b_{f,l}(n_f^2 - n_s^2)$$

(3) Determine the normalized width. $V_{eq} = kw\sqrt{N_f^2 - N_l^2}$

Then determine the normalized guide index b_{eq} using the dispersion relation chart.

(4) The effective index of the waveguide can be determined from

$$b_{eq} = \frac{N^2 - N_l^2}{N_f^2 - N_l^2}$$

$$\Rightarrow \quad N^2 = N_l^2 + b_{eq}(N_f^2 - N_l^2)$$

Note: For multi-layer waveguide structure, such as ridge waveguides, use the matrix method to determine N_f and N_l, then continue on (3).

Three two layers slabs

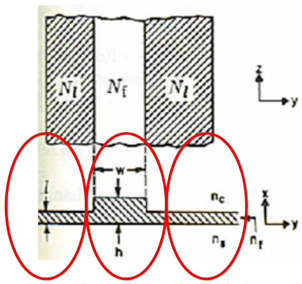


Fig. 2.28. Illustration of the effective-index method showing the top view and the cross section of a rib guide.

<Example> Ti:LiNbO$_3$, λ = 0.8 µm,
n_f = 2.234, n_s = 2.2 14n_c = 1,
h = 1 .8µm, l = " 1µm,

Assuming a buried channel waveguide structure.

Two orthogonal three layers slabs

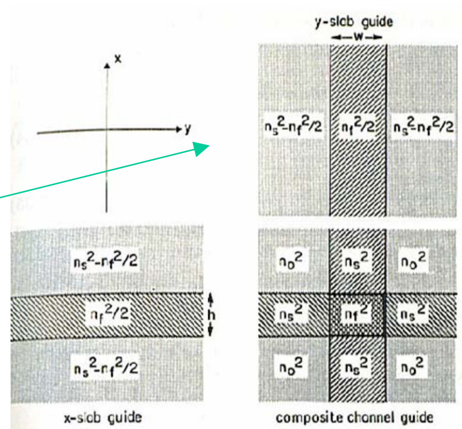


Fig. 2.27. Method of field shadows. The sketch shows the x-y cross-section of a composite guide made up by summing the permittivities (n^2) of an x-slab guide of height h and a y-slab guide of height w. The various n^2 values are indicated

$$E(x,y) = X(x)Y(y)$$

$$\beta^2 = \beta_x^2 + \beta_y^2 \qquad N^2 = N_x^2 + N_y^2$$

$$V_x = kh\sqrt{n_f^2 - n_s^2}$$

$$V_y = kw\sqrt{n_f^2 - n_s^2}$$

Obtain N_x and N_y, therefore N, by using the dispersion relation chart and

$$b_x = \frac{N_x^2 - n_s^2 + n_f^2/2}{n_f^2 \quad n_s^2}$$

$$b_y = \frac{N_y^2 - n_s^2 + n_f^2/2}{n_f^2 \quad n_s^2}$$

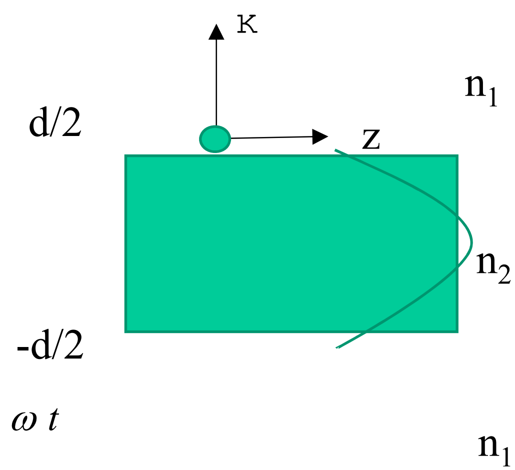
Or instead of solving for N_x and N_y, we can use

$$b = \frac{N^2 - n_s^2}{n_f^2 - n_s^2} = b_x + b_y - 1$$

Evanescent fields in the waveguide cladding

Evanescent wave outside the waveguide core decay exponentially
With an attenuation factor given by

$$\kappa = ((n_2 k \sin\theta_2)^2 - (n_1 k)^2)^{0.5}$$



In the upper layer (x $\geq$ d/2)

$$E = E_1 e^{-\kappa(x-d/2)} e^{-j\beta z+\omega t}$$

In the lower layer (x $\leq$ -d/2)

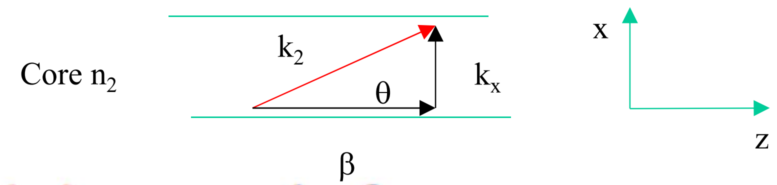
$$E = E_1 e^{\kappa(x+d/2)} e^{-j\beta z+\omega t}$$

Where E_1 peak value of te electric field at lower (x=-d/2) and upper (x=d/2) boundaries.

Waveguide effective index

- We can define the waveguide phase velocity v_p as

$$v_p = \omega / \beta$$



- We now define an effective refractive index n_{eff} as the free-space velocity divided by the waveguide phase velocity.

$$n_{eff} = c / v_p$$

Or $n_{eff} = c\beta / \omega = \beta / k$

$$\Rightarrow n_{eff} = n_2 \sin\theta$$

- The effective refractive index is a key parameter in *guided propagation*, just as the refractive index is in unguided wave travel.

Dispersion Equation

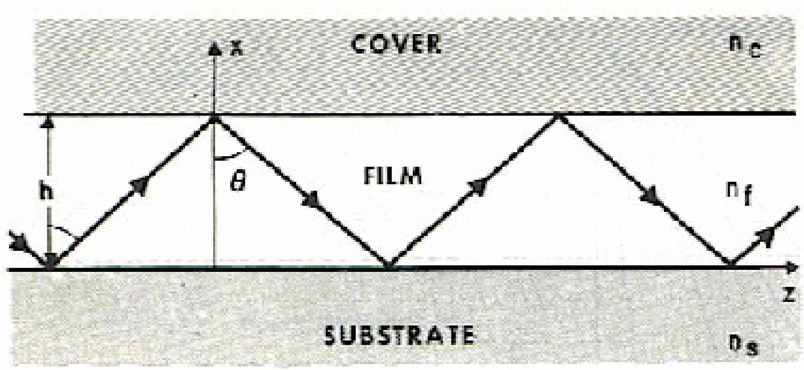
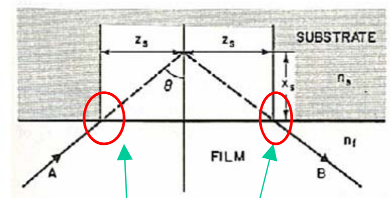


Fig. 2.5. Side-view of a slab waveguide showing wave normals of the zig-zag waves corresponding to a guided mode



Transverse resonance condition:

$$2kn_f h\cos\theta - 2\phi_c - 2\phi_s = 2m\pi$$ m : mode number

$kn_f h\cos$: phase shift for the transverse passage through the film

$2\phi_c (= \phi_{TE,TM})$: phase shift due to total internal reflection from film/cover interface

$2\phi_s (= \phi_{TE,TM})$: phase shift due to total internal reflection from film/substrate interface

The phase shift can be representing the zig-zag ray at a certain depth into the confining layers 1 ad 3 before it is reflected (goos-Hanchen shifts- lateral shift)

Dispersion equation (β vs. ω):

$$kn_f h\cos\theta - \phi_c - \phi_s = m\pi$$

Effective guide index $N = \beta/k = n_f \sin\theta$ $n_s < N < n_f$

Graphical Solution of the Dispersion Equation

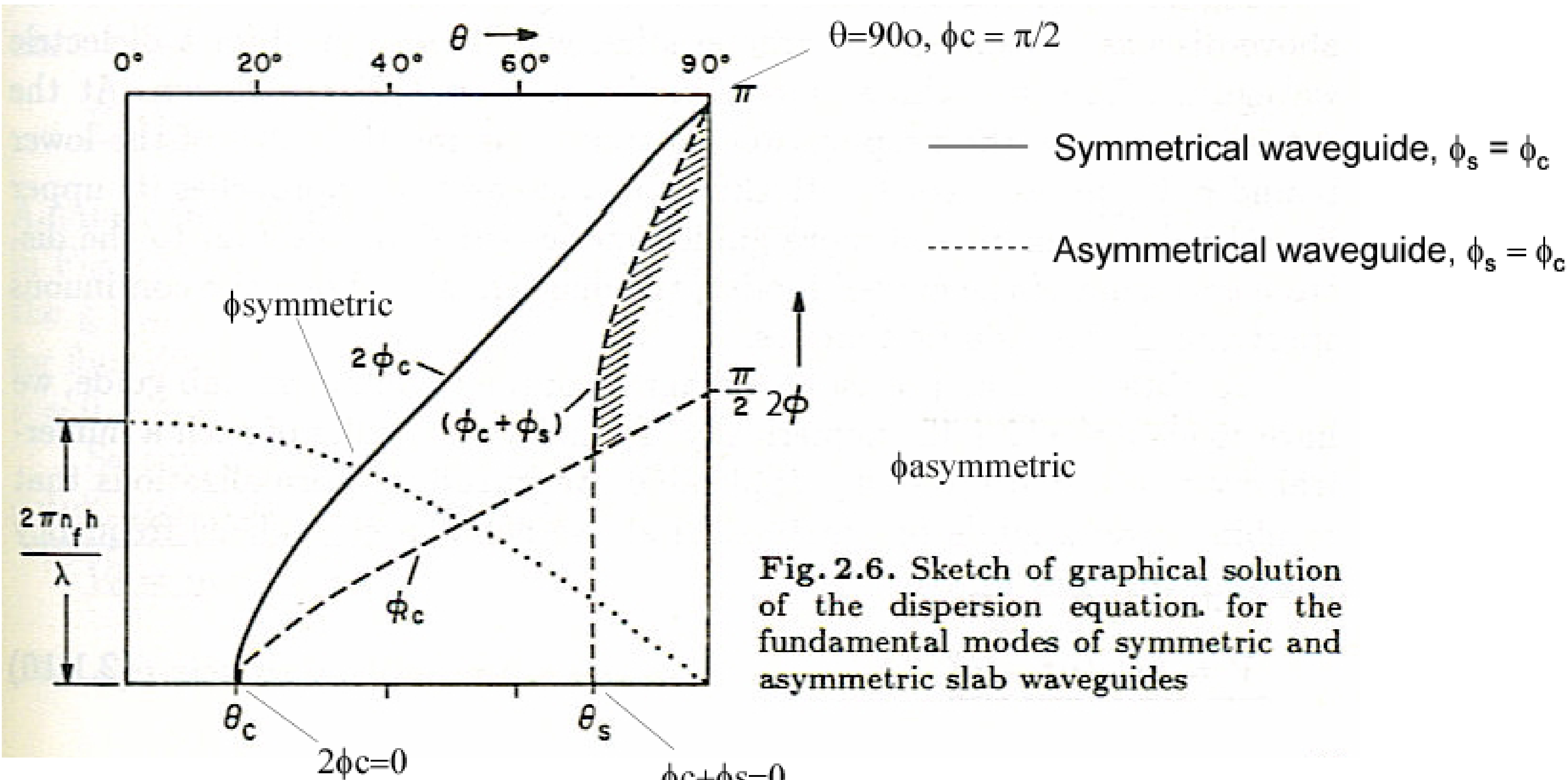
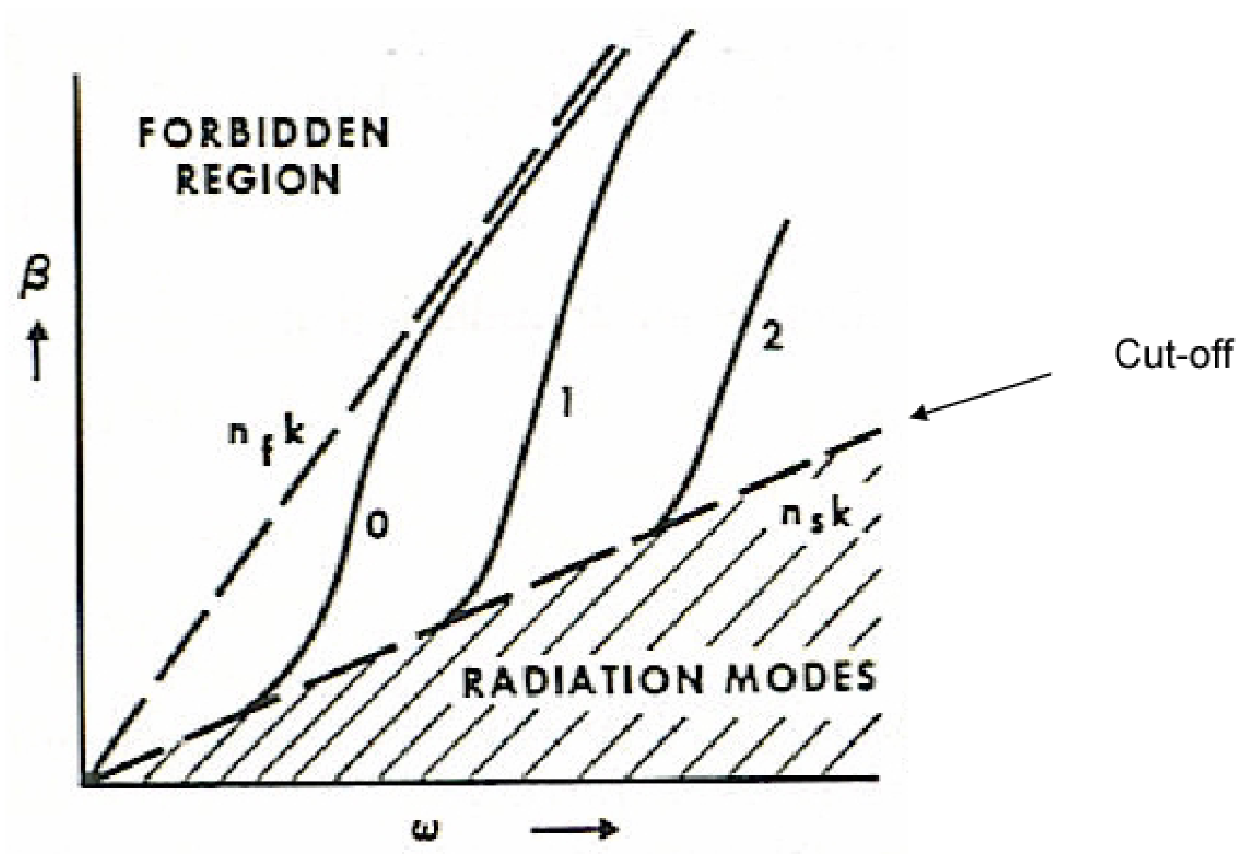


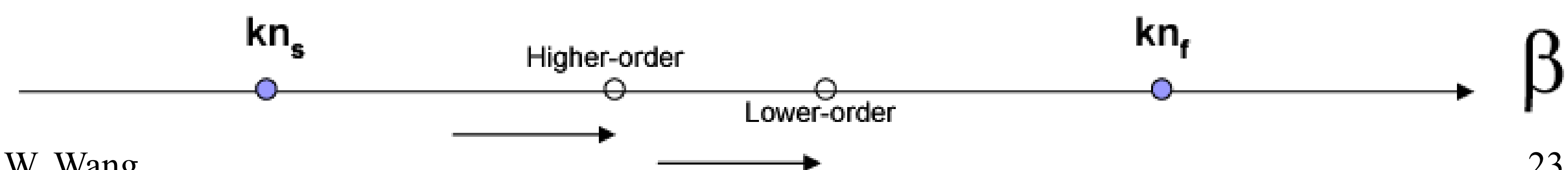
Fig. 2.6. Sketch of graphical solution of the dispersion equation. for the fundamental modes of symmetric and asymmetric slab waveguides

For fundamental mode (m = 0), there is always a solution (no cut-off) for symmetrical waveguide.

Increasing h (and/or decreasing λ) will support more modes.

Typical β – ω diagram





Numerical Solution for Dispersion Relation (I)

Define:

Normalized frequency and film thickness

$$V = kh\sqrt{n_f^2 - n_s^2}$$

Equations for slab waveguide

Normalized guide index

$$b = \frac{N^2 - n_s^2}{n_f^2 - n_s^2}$$

b = 0 at cut-ooff (N = n_s), and approaches 1 as N -> n_f.

Measure for the asymmetry

$$a = \frac{n_s^2 - n_c^2}{n_f^2 - n_s^2} \text{ for TE}, \quad a = \frac{n_f^4}{n_c^4}\frac{n_s^2 - n_c^2}{n_f^2 - n_s^2} \text{ for TM}$$

a = 0 for perfect symmetry (n_s = n_c), and a approaches infinity for strong asymmetry ($n_s \neq n_c$, $n_s \sim n_f$).

Table 2.2. Asymmetry measures for the TE modes (a_E) and the TM modes (a_M) of slab waveguides

Waveguide	n_s	n_f	n_c	a_E	a_M
GaAlAs, double heterostructure	3.55	3.6	3.55	0	0
Sputtered glass	1.515	1.62	1	3.9	27.1
Ti-diffused $LiNbO_3$	2.214	2.234	1	43.9	1093
Outdiffused $LiNbO_3$	2.214	2.215	1	881	21206

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Numerical Solution for Dispersion Relation (II)

For TE modes, dispersion relation

$$kn_f h\cos\theta - \phi_c - \phi_s = m\pi \;\Rightarrow\; V\sqrt{1-b} = m\pi + \tan^{-1}\sqrt{\frac{b}{1-b}} + \tan^{-1}\sqrt{\frac{b+a}{1-b}}$$

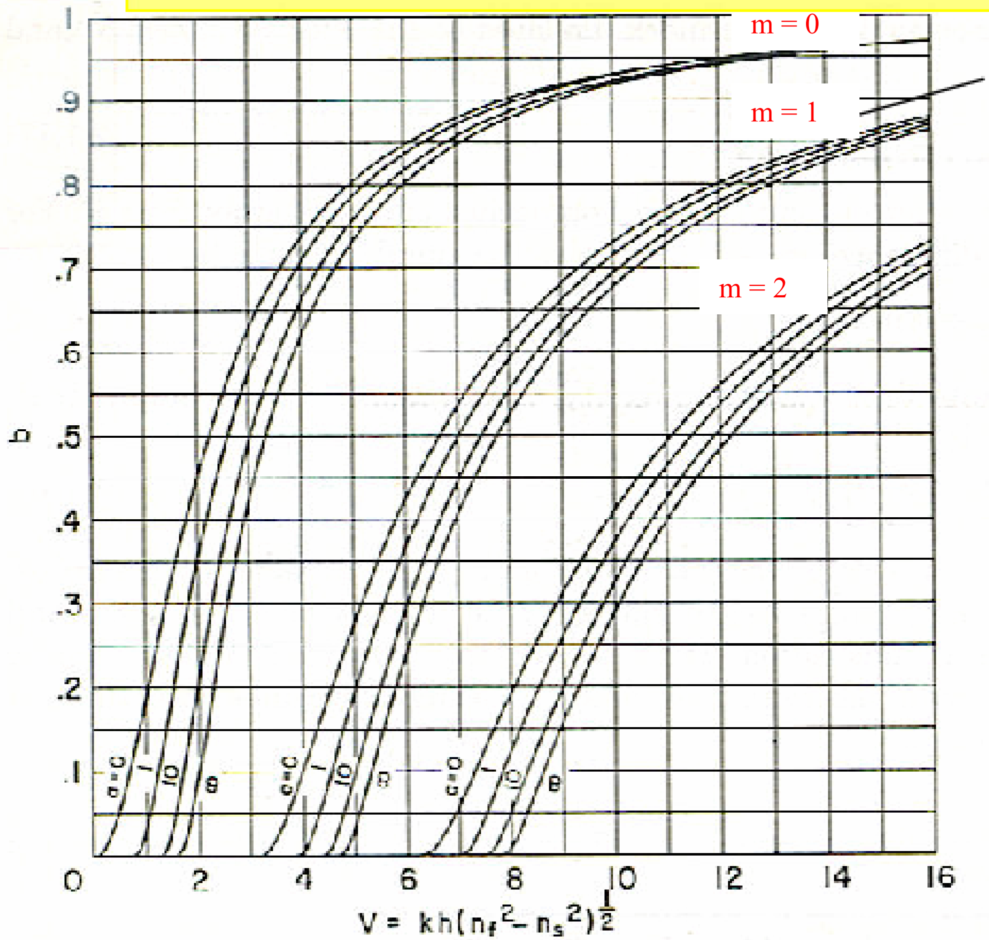

Fig. 2.8 Normalized ω-β diagram of a planar slab waveguide showing the guide index b as a function of the normalized thickness V for various degrees of asymmetry [2.20]

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m : Mode number

(Normalized) cut-off frequency:

$$V_0 = \tan^{-1}\sqrt{a}$$
$$V_m = V_0 + m\pi$$

# of guided modes allowed:

$$m = \frac{2h}{\lambda}\sqrt{n_f^2 - n_s^2}$$

<Example>
AlGaAs/GaAs/AlGaAs double heterostructure
n = 3.55/3.6/3.55

Multilayer Stack Theory

Focusing on TE modes first,

$$U = E_y, \quad V = \omega \mu H_z$$

$$U = A\exp(-j\kappa x) + B\exp(j\kappa x)$$

$$V = \kappa [A\exp(-j\kappa x) - B\exp(j\kappa x)]$$

At x = 0,

$$U_0 = U(0), \quad V_0 = V(0)$$

$$\begin{bmatrix} U \\ V \end{bmatrix} = \begin{bmatrix} \cos(\kappa x) & \frac{j}{\kappa}\sin(\kappa x) \\ j\kappa\sin(\kappa x) & \cos(\kappa x) \end{bmatrix} \begin{bmatrix} U \\ V \end{bmatrix}$$

$$= \mathbf{M} \begin{bmatrix} U \\ V \end{bmatrix}$$

M: Characteristic matrix of the layer

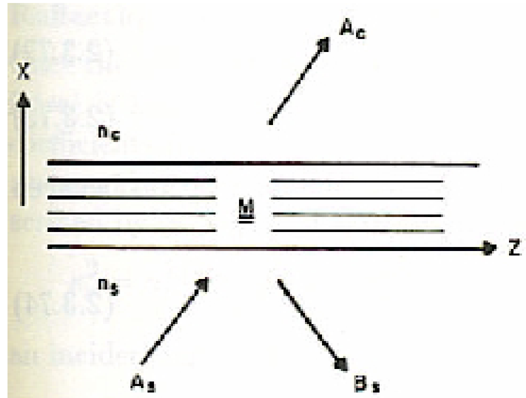

Fig. 2.15. Sketch of a multilayer stack waveguide with substrate index n_s and cover index n_c. The z-axis indicates the direction of mode propagation

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$$\mathbf{M}_i = \begin{bmatrix} \cos(\kappa_i h_i) & \frac{j}{{}_i}\sin(\kappa_i h_i) \\ j\kappa_i \sin(\kappa_i h_i) & \cos(\kappa_i h_i) \end{bmatrix}$$

$$\begin{bmatrix} U_0 \\ V_0 \end{bmatrix} = \mathbf{M} \begin{bmatrix} U_n \\ V_n \end{bmatrix}$$

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$$\mathbf{M} = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} = \mathbf{M}_1 \ \mathbf{M}_2 \ . \ \mathbf{M}_n$$

Dispersion Relation for Multilayer Slab Waveguide

Consider guided mode. For substrate and cover,

$$U = A\exp(\gamma x) + B\exp(-\gamma x)$$

$$V = j\gamma[A\exp(\gamma x) - B\exp(-\gamma x)]$$

In the substrate,

$$U_0 = A_s, \quad V_0 = j\gamma_s A_s$$

In the cover,

$$U_n = A_c, \quad V_n = -j\gamma_c A_c$$

Using the multilayer stack matrix theory, we obtain:

$$j(\gamma_s m_{11} + \gamma_c m_{22}) = m_{21} - \gamma_s \gamma_c m_{12}$$

-> ***Dispersion relation for multilayer slab waveguide***

<Example> Four-layer waveguides

Multilayer Stack Theory for TM Modes

$$U = H_y, \quad V = \omega\mu_0 E_z$$

$$U = A\exp(-j\kappa x) + B\exp(j\kappa x)$$

$$V = -\frac{\kappa}{n^2}[A\exp(-j\kappa x) - B\exp(j\kappa x)]$$

Therefore,

$$TE \geqslant TM \qquad \kappa \text{ -> -[} \frac{\kappa}{n^2} \text{]}$$

Dispersion relation:

$$-j\left(m_{11}\frac{\gamma_s}{n_s^2} + m_{22}\frac{\gamma_c}{n_c^2}\right) = m_{21} - \frac{\gamma_s\gamma_c}{n_s^2 n_c^2}m_{12}$$

Characteristic matrix of the i-th layer:

$$\mathbf{M}_i = \begin{bmatrix} \cos(\kappa_i h_i) & -j\frac{n_i^2}{{}_i}\sin(\kappa_i h_i) \\ -j\frac{\kappa_i}{n_i^2}\sin(\kappa_i h_i) & \cos(\kappa_i h_i) \end{bmatrix}$$

The Effective-Index Method

Treat each section as slab layer, find its effective N and then treated the effective layer as slab layer again and find its effective N again

(1) Determine the normalized thickness of the channel and lateral guides.

$$V_f = kh\sqrt{n_f^2 - n_s^2}, \quad V_l = kl\sqrt{n_f^2 - n_s^2}$$

(2) Use the dispersion relation chart to determine the normalized guide indices b_f and b_l.

Determine the corresponding effective indices by referring to the Table on **Effective index for rectangular waveguide**

$$N_{f,l}^2 = n_s^2 + b_{f,l}(n_f^2 - n_s^2)$$

(3) Determine the normalized width. $V_{eq} = kw\sqrt{N_f^2 - N_l^2}$

Then determine the normalized guide index b_{eq} using the dispersion relation chart.

(4) The effective index of the waveguide can be determined from

$$b_{eq} = \frac{N^2 - N_l^2}{N_f^2 - N_l^2}$$

$$\Rightarrow \quad N^2 = N_l^2 + b_{eq}(N_f^2 - N_l^2)$$

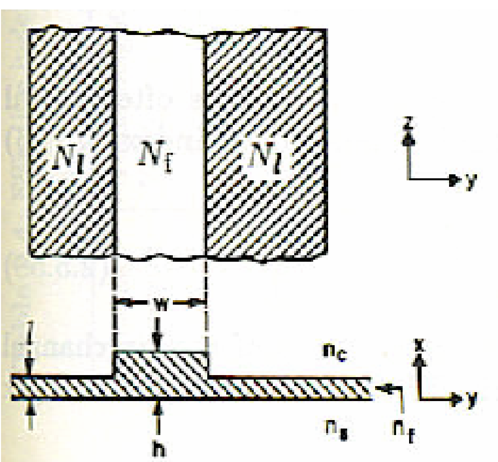
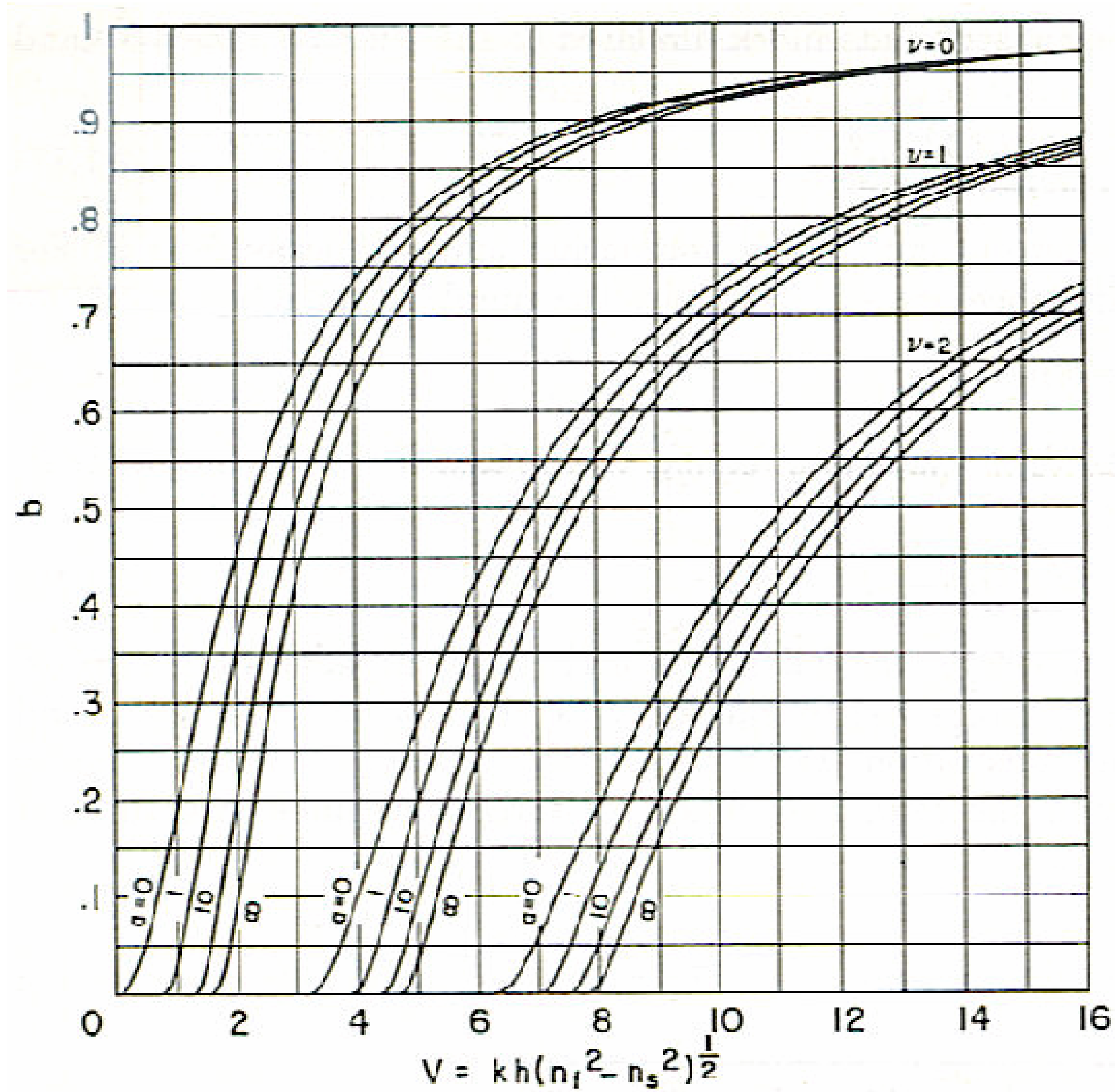


Fig. 2.28. Illustration of the effective-index method showing the top view and the cross section of a rib guide.

Note: For multi-layer waveguide structure, such as ridge waveguides, use the matrix method to determine N_f and N_l, then continue on (3).

<Example> Ti:LiNbO$_3$, λ = 0.8 µm,
n_f = 2.234, n_s = 2.2 14n_c = 1,
h = 1 .8µm, l = " 1µm,

The Dispersion Relation Chart



The Method of Field Shadows (II)

Assuming a buried channel waveguide structure.

Treat each section as slab layer, find its effective N and then treated the effective layer as slab layer again and find its effective N again

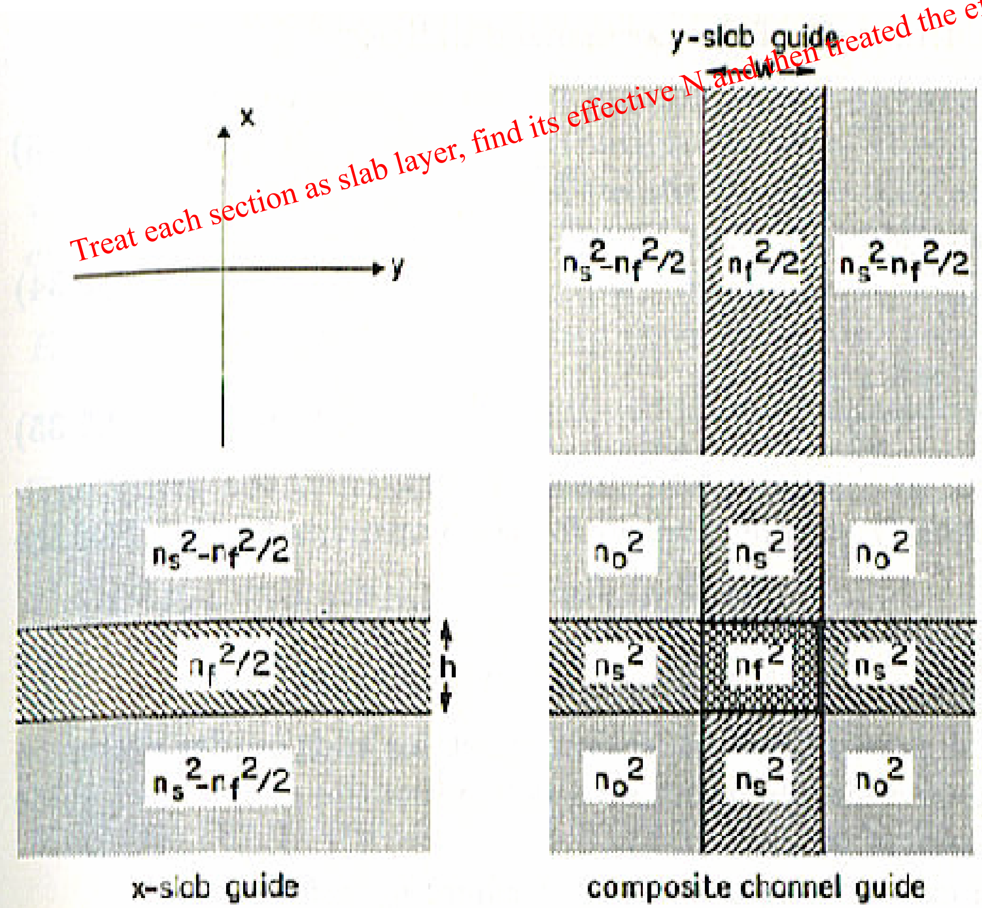


Fig. 2.27. Method of field shadows. The sketch shows the x-y cross-section of a composite guide made up by summing the permittivities (n^2) of an x-slab guide of height h and a y-slab guide of height w. The various n^2 values are indicated

$$E(x,y) = X(x)Y(y)$$

$$\beta^2 = \beta_x^2 + \beta_y^2 \qquad N^2 = N_x^2 + N_y^2$$

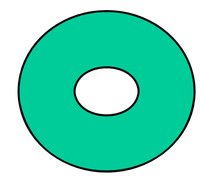
$$V_x = kh\sqrt{n_f^2 - n_s^2}$$

$$V_y = kw\sqrt{n_f^2 - n_s^2}$$

Obtain N_x and N_y, therefore N, by using the dispersion relation chart and

$$b_x = \frac{N_x^2 - n_s^2 + n_f^2/2}{n_f^2 - n_s^2}$$

$$b_y = \frac{N_y^2 - n_s^2 + n_f^2/2}{n_f^2 - n_s^2}$$

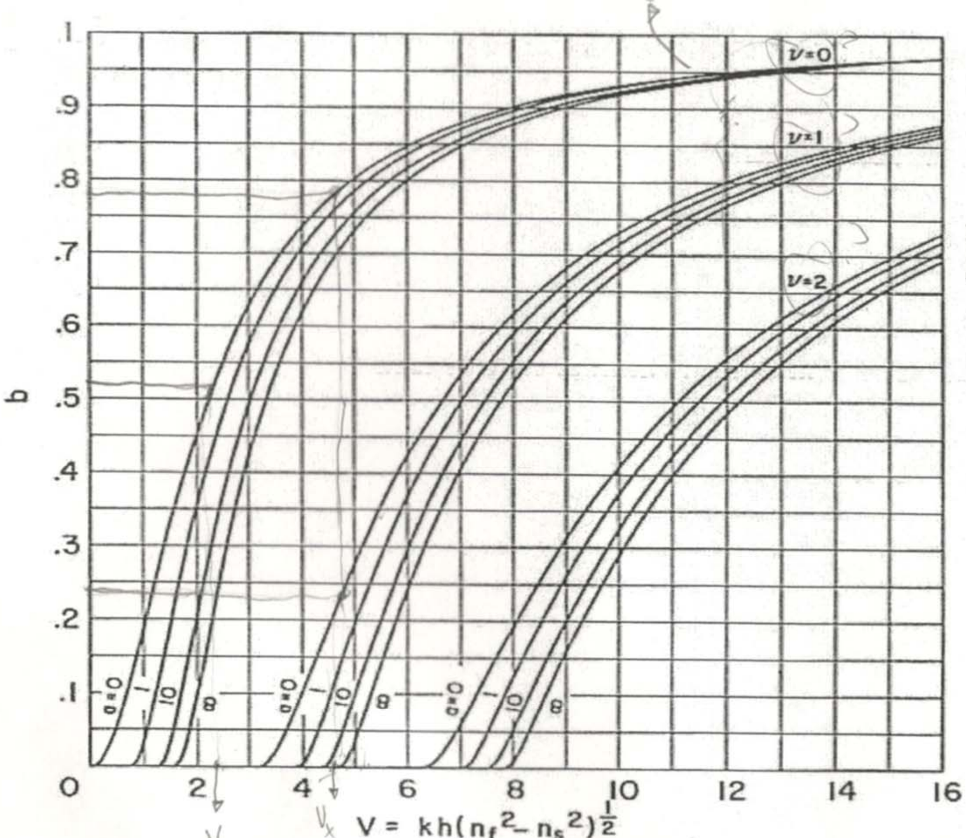


Or instead of solving for N_x and N_y, we can use

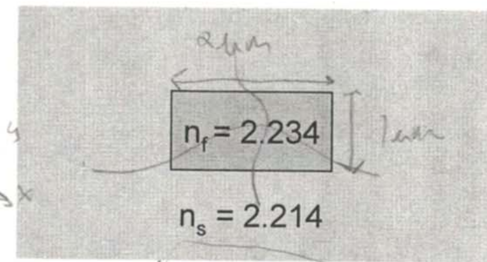
$$b = \frac{N^2 - n_s^2}{n_f^2 - n_s^2} = b_x + b_y - 1$$

Another example

Field Shadows Method Exercise



Ti:LiNbO$_3$ channel waveguide
$\lambda = 0.8\ \mu m$



Determine effective index and modal distribution

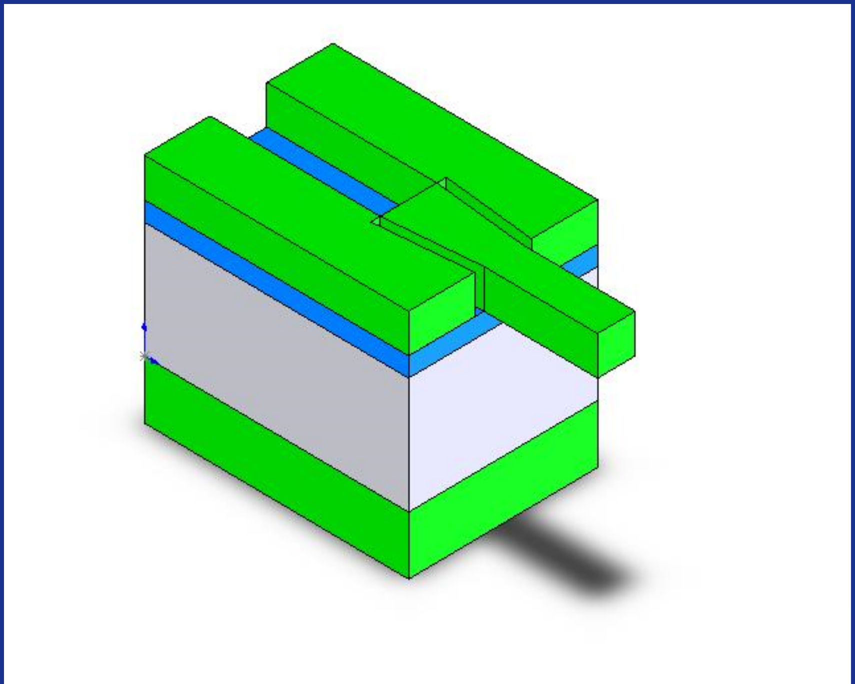
Effective Index Parameters for Rectangular Waveguides

Channel structure	Guide height V_f, V_l	Eff. index N_f, N_l	$N_f^2 - N_l^2$	Channel guide index b
a) General	$V_f = kh\sqrt{n_f^2 - n_s^2}$ $V_l = kh\sqrt{n_l^2 - n_f^2}$	$N_f^2 = n_s^2 + b_f(n_f^2 - n_s^2)$ $N_l^2 = n_s^2 + b_l(n_f^2 - n_s^2)$	$b_f(n_f^2 - n_s^2) - b_l(n_l^2 - n_s^2)$	$b_f b_{eq} + b_l(1 - b_{eq})a_{ch}$
b) Buried	$V_f = kh\sqrt{n_f^2 - n_s^2}$	$N_f^2 = n_s^2 + b_f(n_f^2 - n_s^2)$ $N_l = n_s$	$b_f(n_f^2 - n_s^2)$	$b_f b_{eq}$
c) Raised	$V_f = kh\sqrt{n_f^2 - n_s^2}$	$N_f^2 = n_s^2 + b_f(n_f^2 - n_s^2)$ $N_l = n_c$	$(n_s^2 - n_c^2) + b_f(n_f^2 - n_s^2)$	$b_f b_{eq} - (1 - b_{eq})a$
d) Rib	$V_f = kh\sqrt{n_f^2 - n_s^2}$ $V_l = kl\sqrt{n_f^2 - n_s^2}$	$N_f^2 = n_s^2 + b_f(n_f^2 - n_s^2)$ $N_l^2 = n_s^2 + b_l(n_f^2 - n_s^2)$	$(b_f - b_l)(n_f^2 - n_s^2)$	$b_f b_{eq} + b_l(1 - b_{eq})$
e) Embedded	$V_f = kh\sqrt{n_f^2 - n_s^2}$	$N_f^2 = n_s^2 + b_f(n_f^2 - n_s^2)$ $N_l = n_s$	$b_f(n_f^2 - n_s^2)$	$b_f b_{eq}$
f) Ridge	$V_f = kh\sqrt{n_f^2 - n_s^2}$ $V_l = kl\sqrt{n_f^2 - n_s^2}$	$N_f^2 = n_{s1}^2 + b_f(n_f^2 - n_{s1}^2)$ $N_l^2 = n_{s2}^2 + b_l(n_{s1}^2 - n_{s2}^2)$	$(1 - b_l)(n_{s1}^2 - n_{s2}^2) + b_f(n_f^2 - n_{s1}^2)$	$b_{eq}(1 + b_f \cdot a_{ridge}) + b_l(1 - b_{eq})$

Rectangular Waveguide

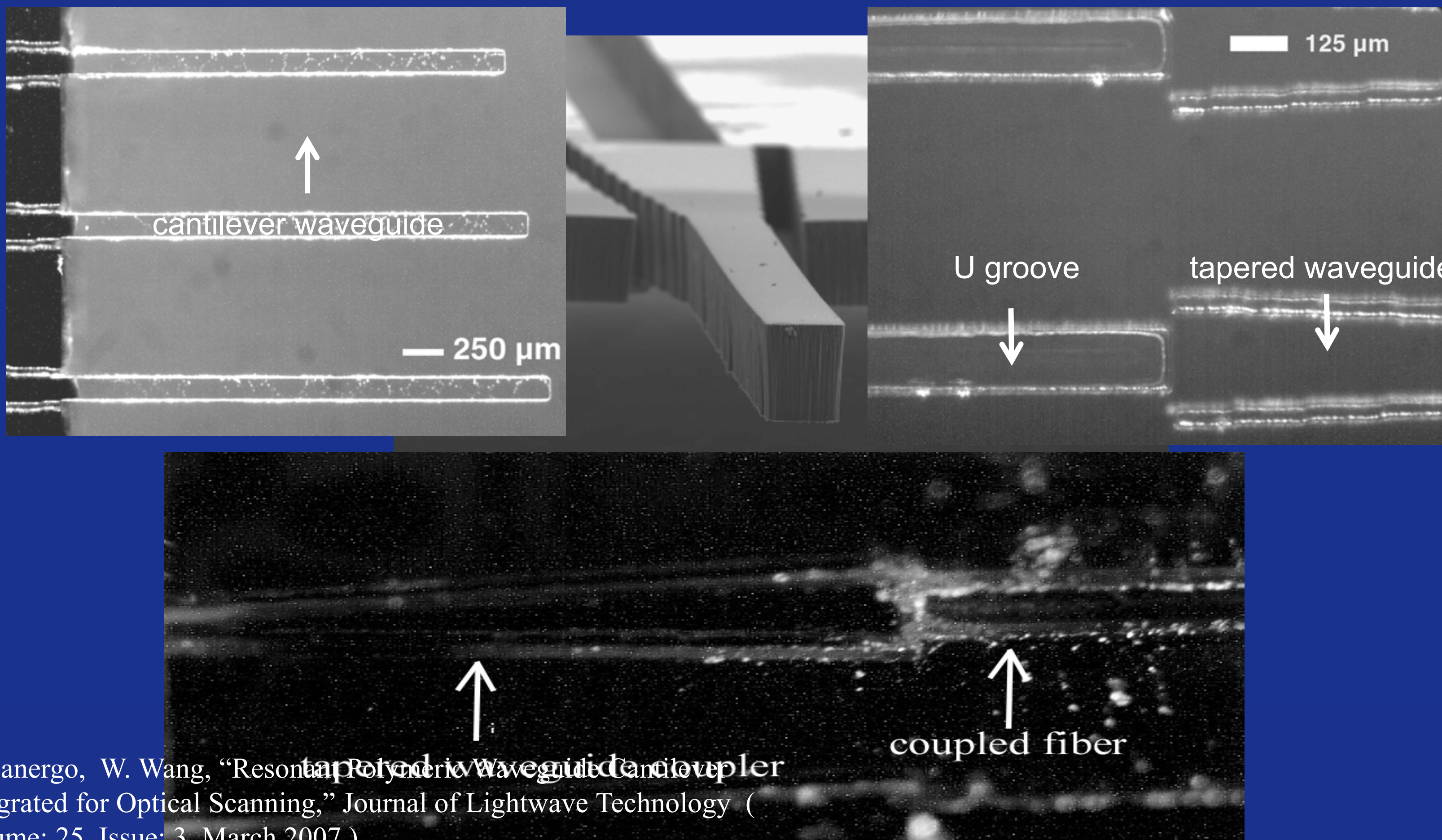
Optical Analysis

- Find Modes
- Find Maximum Coupling Efficiency
- Total Power Out



R. Panergo, W. Wang, "Resonant Polymeric Waveguide Cantilever Integrated for Optical Scanning," Journal of Lightwave Technology (Volume: 25, Issue: 3, March 2007)

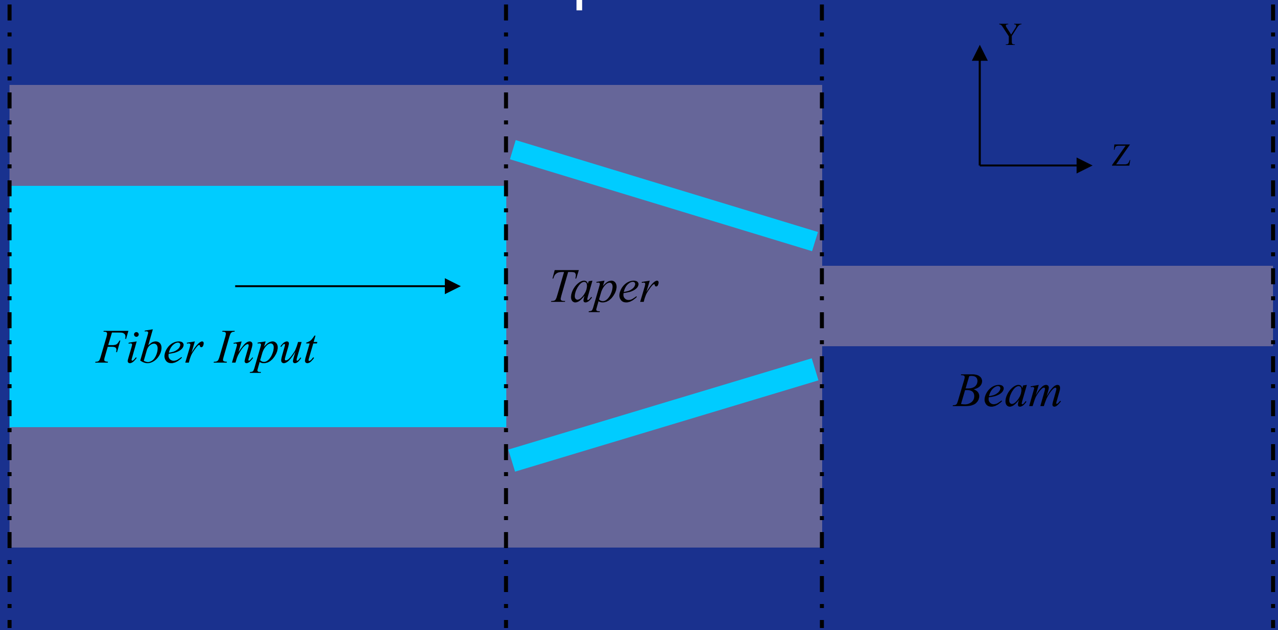
SU-8 optical scanner



R. Panergo, W. Wang, “Resonant Polymeric Waveguide Cantilever Integrated for Optical Scanning,” Journal of Lightwave Technology (Volume: 25, Issue: 3, March 2007)

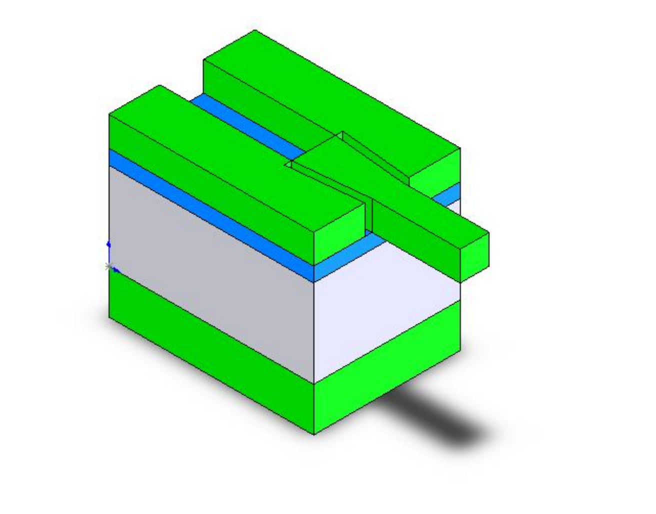
Mode Coupling (MC)

- Divided into 3 sections
  - Fiber input to facet of the waveguide
  - Taper section
  - Interface between taper to the beam



Optical Parameters

- Index of Refraction
  - n_{su8} = 1.596, n_{SiOx} = 1.46, n_{air} = 1
- Input Source/tapered Fiber (D_{core}=62.5μm)
  - Single Mode, 633nm wavelength
- Film Thickness
  - Thickness 100 micron
  - Initial Width 100 micron
  - Final Width 50 micron



MICRO TECH LAB

DEPARTMENT OF MECHANICAL ENGINEERING

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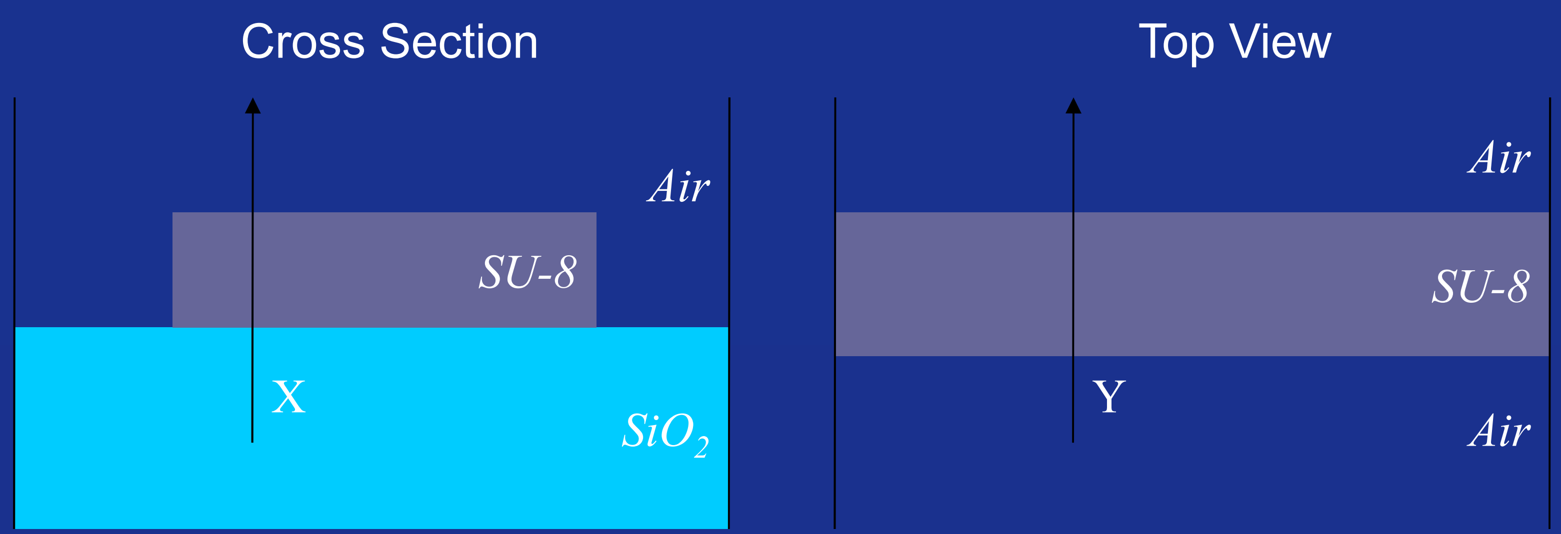
MC – Fiber to Waveguide

- Initial Assumptions
  - Input is a single mode Gaussian beam (end butt coupled)
  - Ignore loss due to scattering and absorption
- SU-8 Waveguide with 85x230μm cross section
- 633nm light source through a 62.5μm core fiber



Fiber to Waveguide continued …

- Coupling efficiency determined by overlap integral:

$$\eta_m = \frac{\left| \int\int A(x,y) B_m^*(x,y) dxdy \right|^2}{\int\int \left| A(x,y) \right|^2 dxdy \int\int \left| B_m(x,y) \right|^2 dxdy}$$

- *A(x,y)*: Amplitude distribution of input source
- $B_m(x,y)$: Amplitude distribution of the *m*th mode

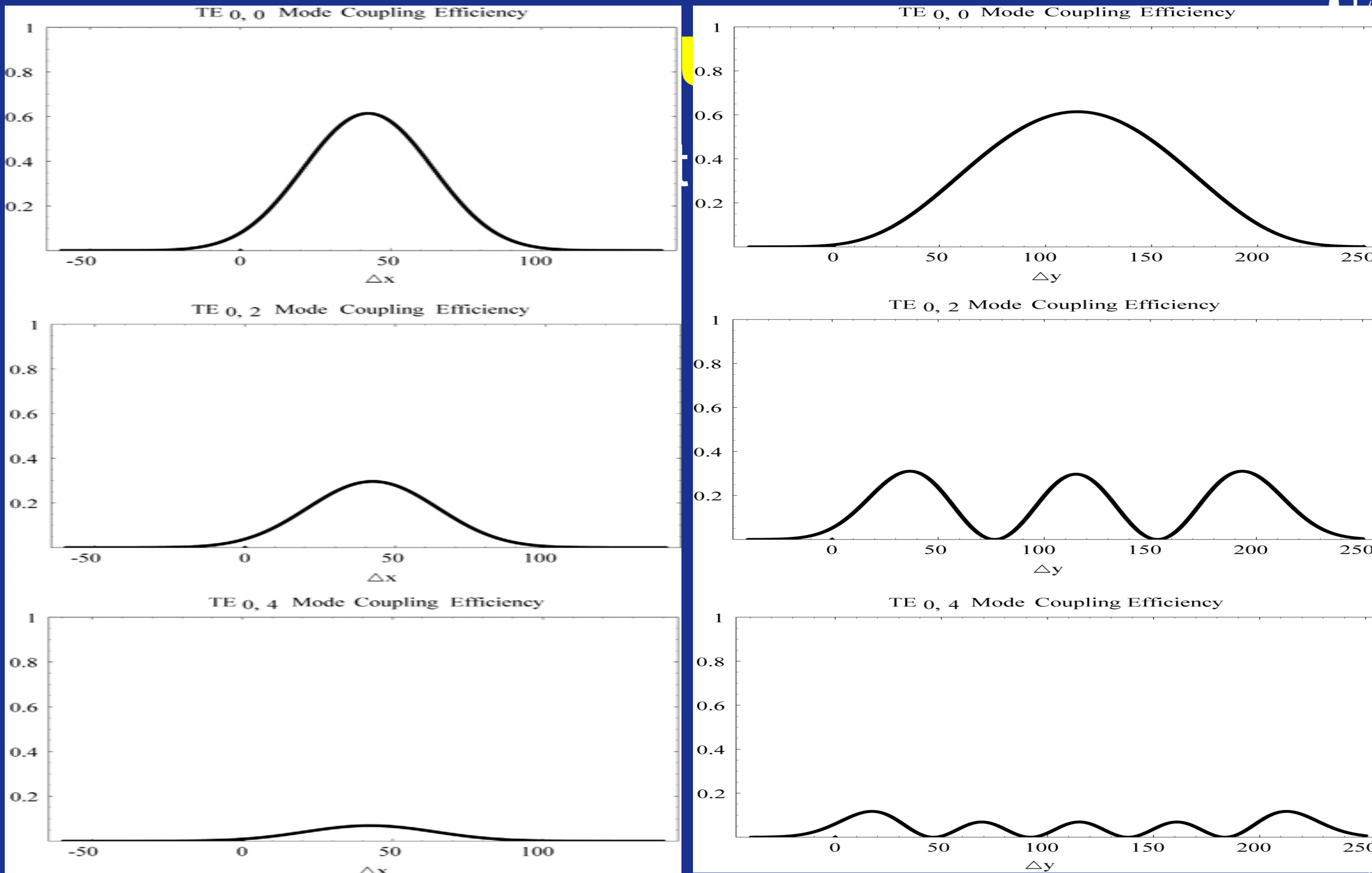
Fiber to Waveguide continued …

MICRO TECH LAB

- The first 100 combinations of modes were examined
- $TE_{0,0}$, $TE_{0,2}$, and $TE_{0,4}$ couple 98% of the light
- All 100 combinations couple 99% of the light
- Assume that 100% coupling

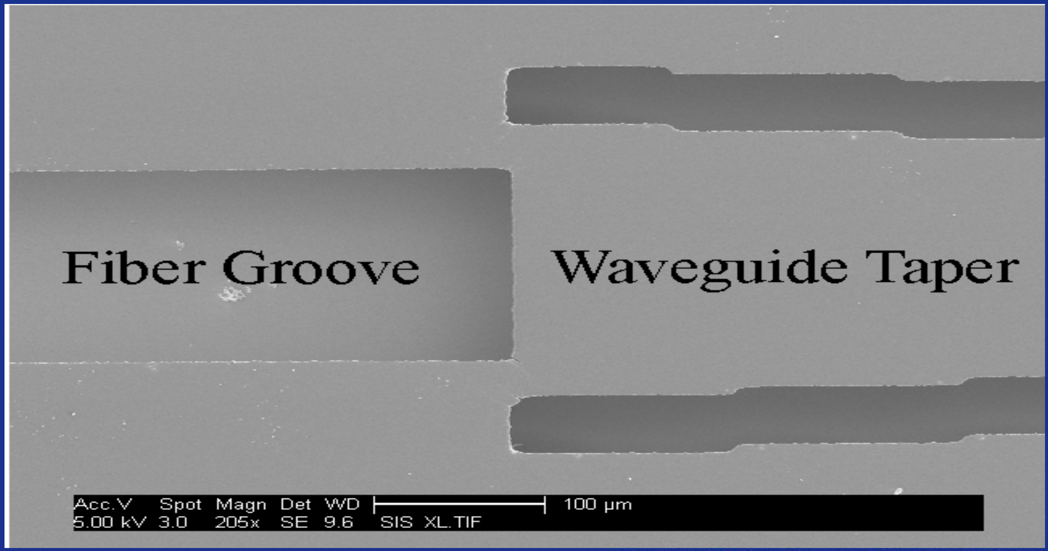
Mode	Coupling Eff. (%)
$TE_{0,0}$	**61.44**
$TE_{0,2}$	**29.66**
$TE_{0,4}$	**6.92**

Fiber to Waveguide

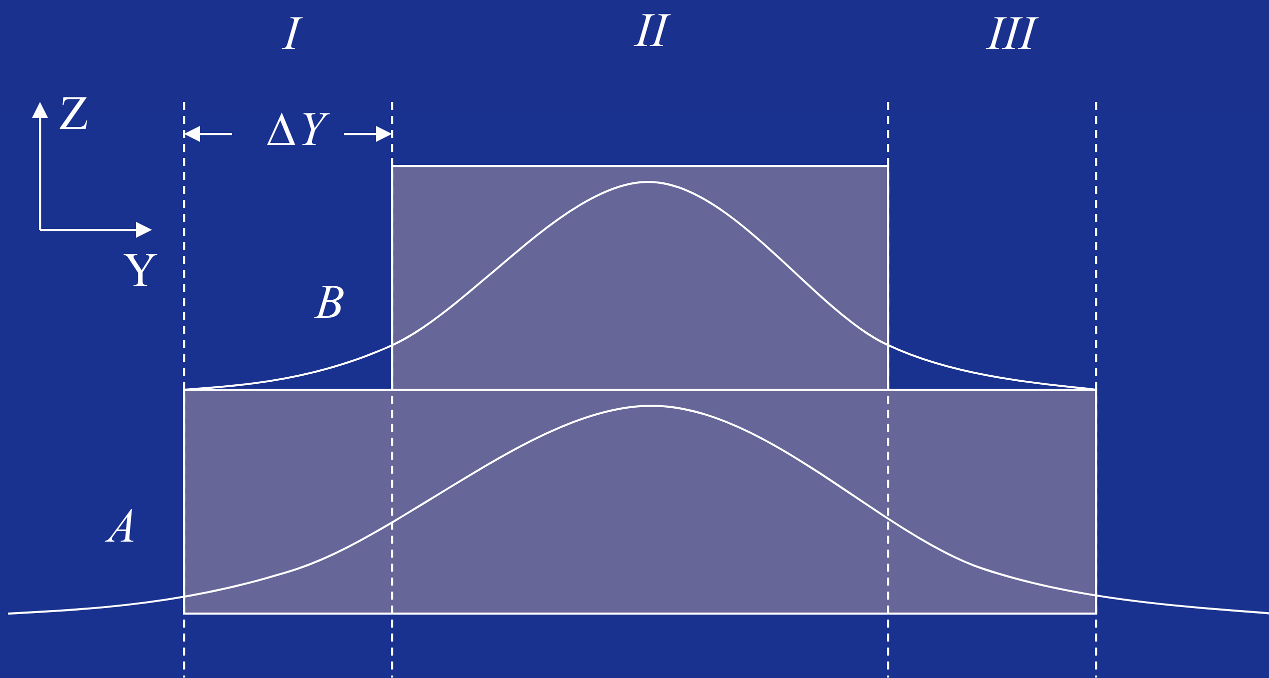


MC – Taper Section

- Photolithography process produced step-like features
- Mask for process was printed with a high resolution printer
  - Resolution: 2450 dpi horizontal, 300 dpi vertical

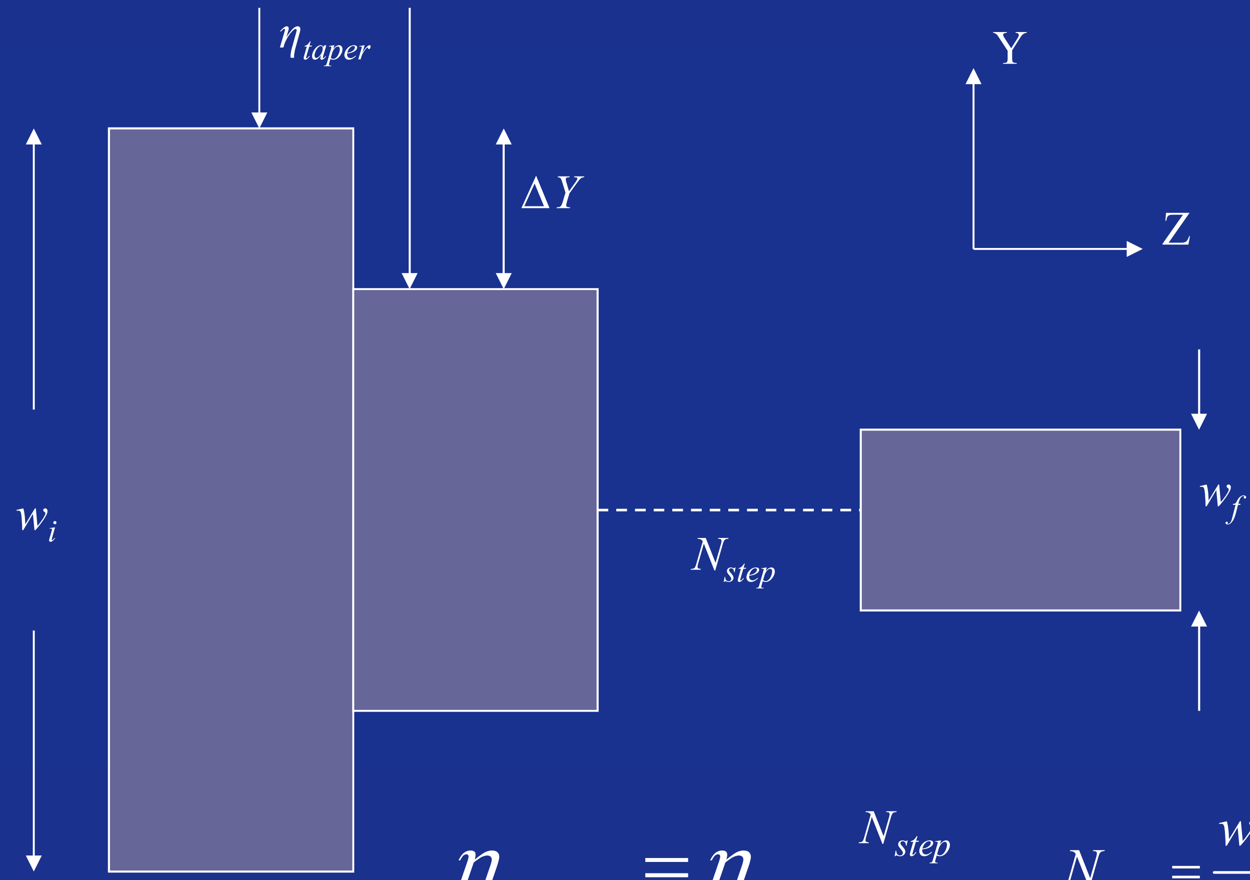


Taper Section continued …



$$\eta_{taper} = \frac{\left| \int A_n(y) B_m^*(y) dy \right|^2}{\int |A_n(y)|^2 dy \int |B_m(y)|^2 dy}$$

Taper Section continued …



$$\eta_{ttaper} = \eta_{taper}^{N_{step}} \qquad N_{step} = \frac{w_i - w_f}{2\Delta y}$$

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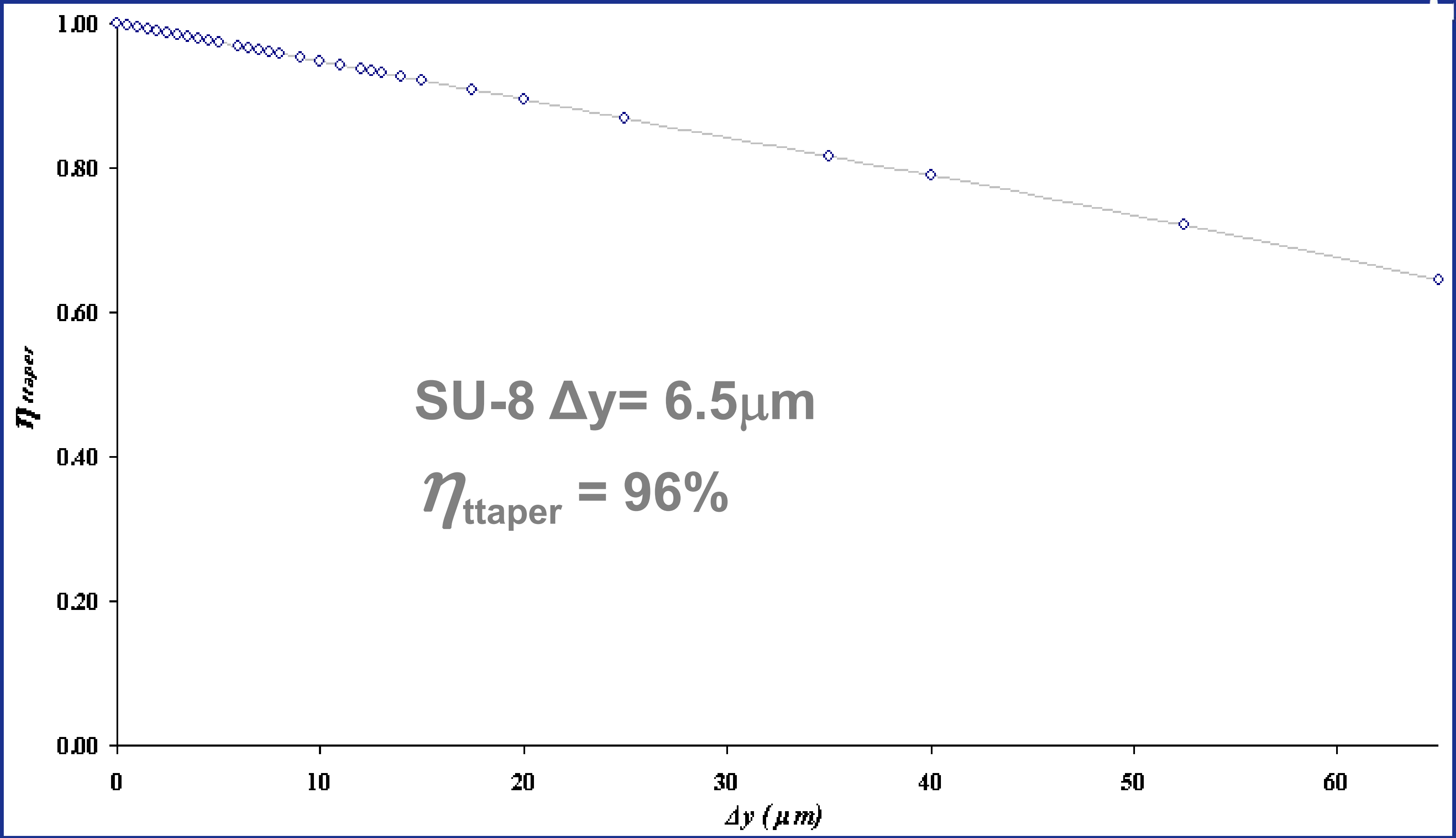
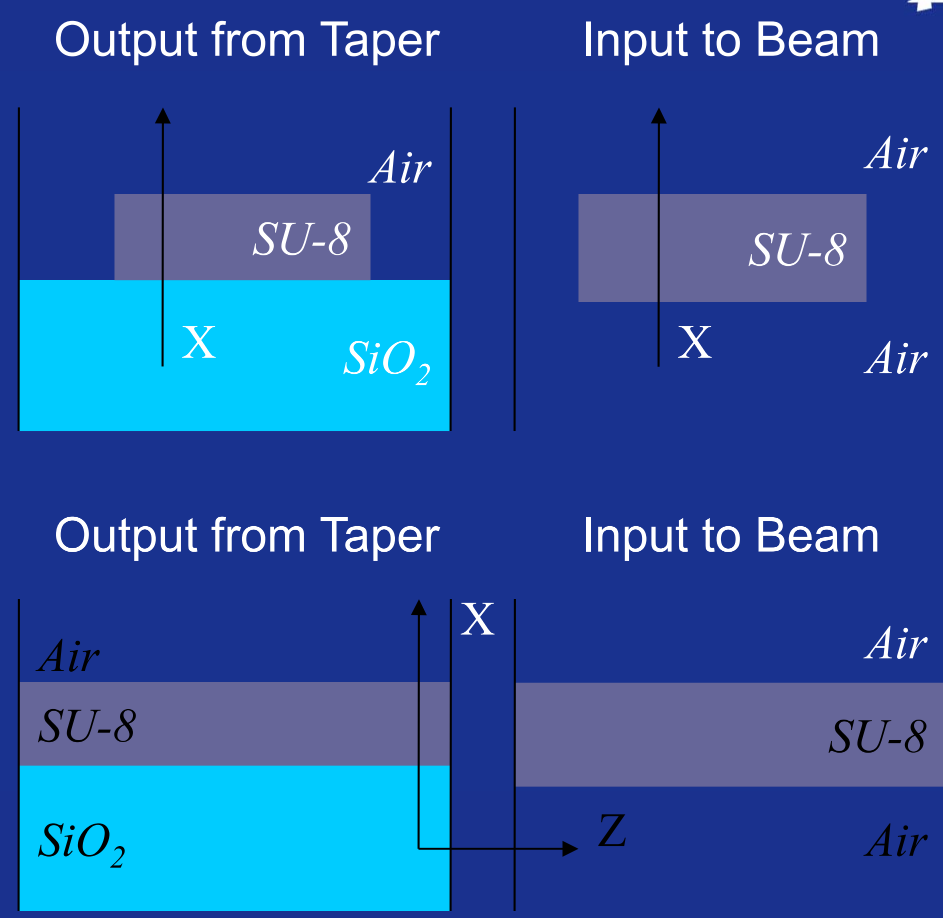
Taper Section continued …



MC - Taper to Beam

- Index change from taper section to beam

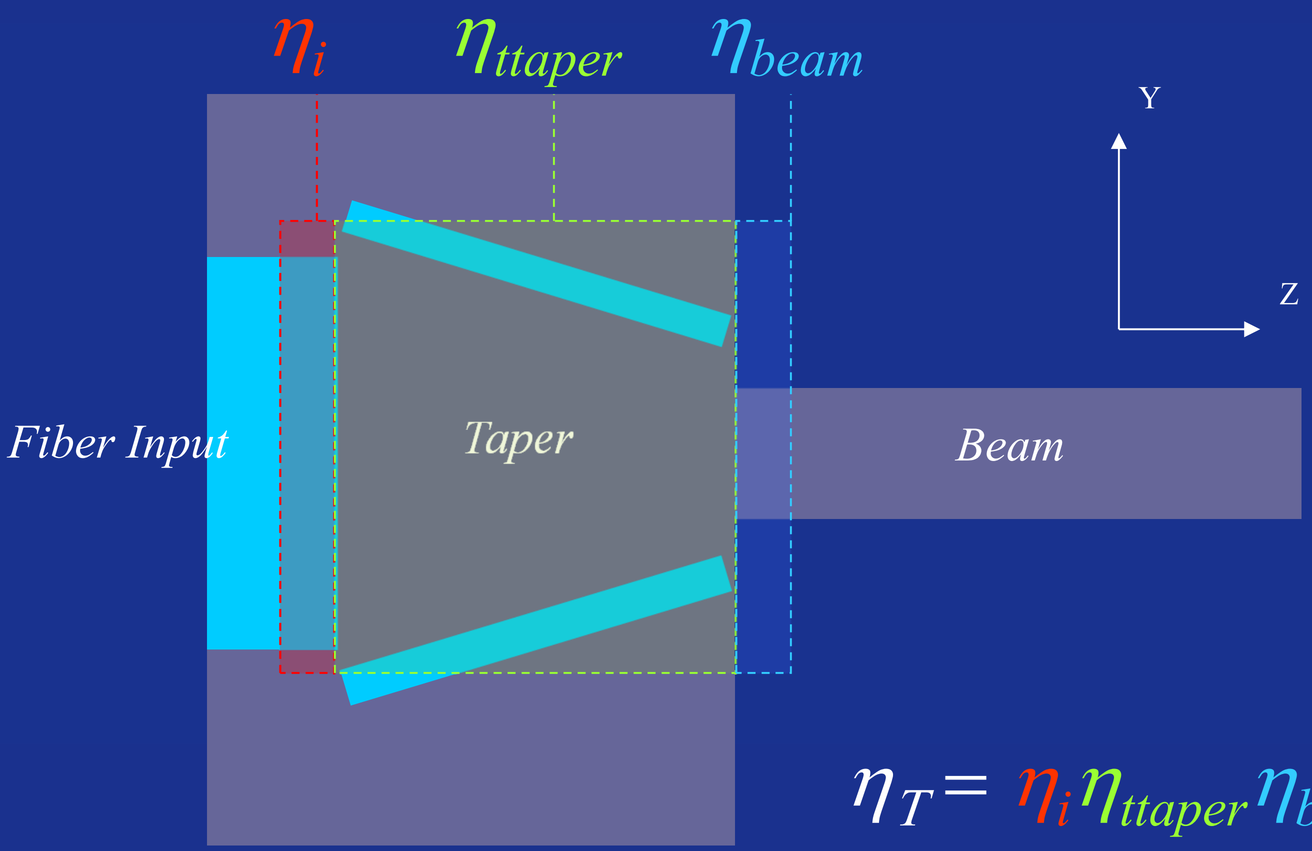
$$\eta_{beam} = \frac{\left|\int A_n(x) B_m^*(x) dx\right|^2}{\int |A_n(x)|^2 dx \int |B_m(x)|^2 dx}$$



- Y direction remains unchanged

- loss is <<1% and assumed to be negligible

MC – Total Coupling

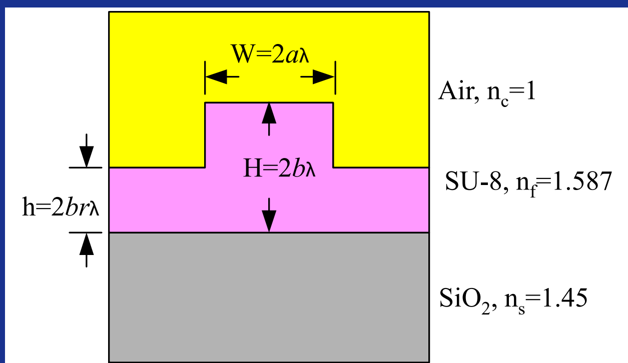


$$\eta_T = \eta_i \eta_{ttaper} \eta_{beam} = 96\%$$

Rib Waveguide

Rib Waveguide Design

The rib waveguide is based on the single mode conditions proposed by Soref as following:



$$\frac{H}{\lambda}\sqrt{n_f^2 - n_s^2} \geq 1$$

$$0.5 \leq r \equiv \frac{h}{H} < 1$$

$$\frac{a}{b} = \frac{W}{H} \leq \left(\frac{q+4\pi b}{4\pi b}\right)\frac{1+0.3\sqrt{\left(\frac{q+4\pi b}{q+4\pi rb}\right)^2 - 1}}{\sqrt{\left(\frac{q+4\pi b}{q+4\pi rb}\right)^2 - 1}}$$

Based on $H \geq \frac{\lambda}{\sqrt{n_f^2 - n_s^2}}$

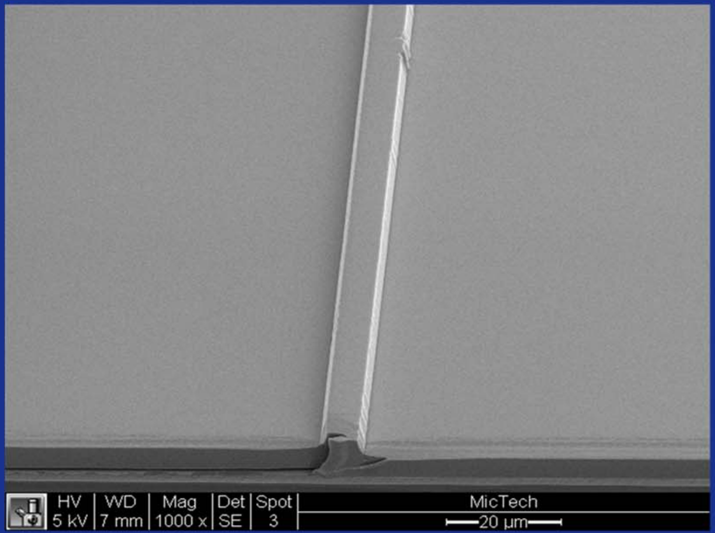
800nm: H>1.24 μm
900nm: H>1.4 μm
1300nm: H>2.02μm

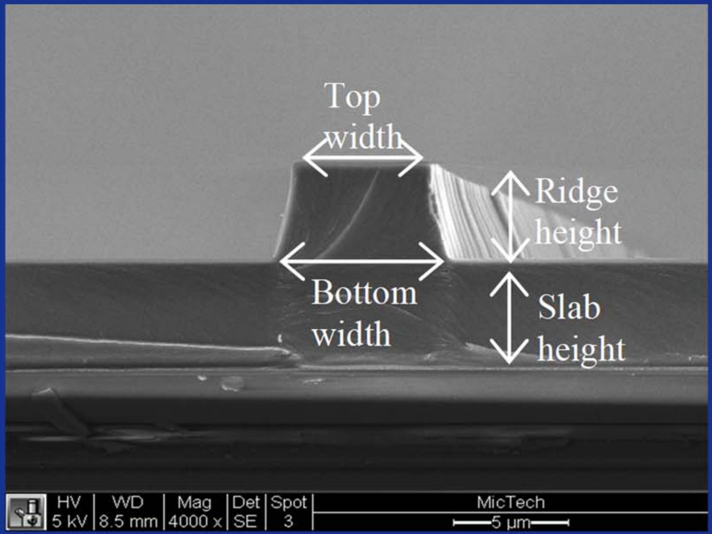
q is defined as: $q = \frac{\gamma_c}{\sqrt{n_f^2 - n_c^2}} + \frac{\gamma_s}{\sqrt{n_f^2 - n_s^2}}$

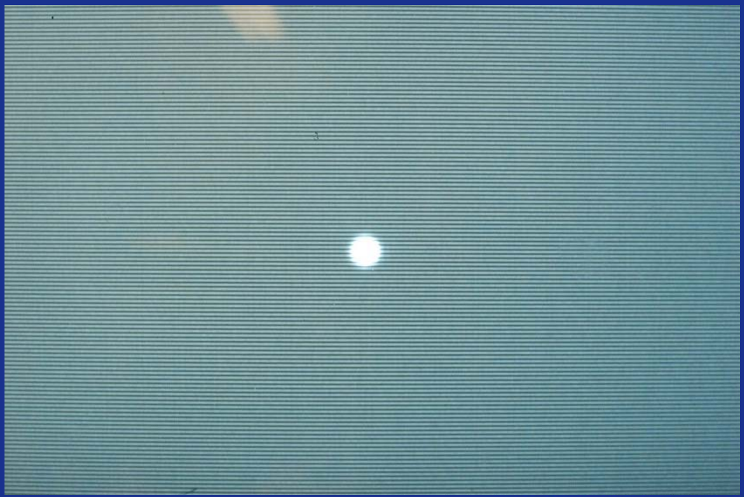
C. S. Huang, Y. B. Pun, W. C. Wang, "Fabrication of a flexible rib waveguide with Bragg grating filter," Journal of Optical Society of America B, 26(6), 1256-1262, 2009. [OSA]

Where $\gamma_c = \gamma_s = 1$ for TE mode and $\gamma_c = \frac{n_c^2}{n_f^2}$ and $\gamma_s = \frac{n_s^2}{n_f^2}$ defines for TM mode

Result





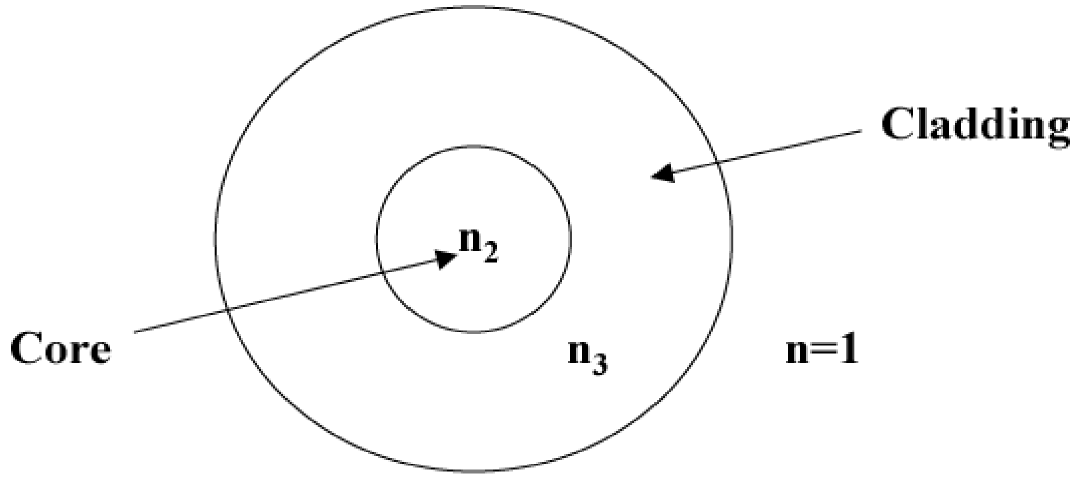


C. S. Huang, Y. B. Pun, W. C. Wang, "Fabrication of a flexible rib waveguide with Bragg grating filter," Journal of Optical Society of America B, 26(6), 1256-1262, 2009. [OSA]

Optical Fiber

Optical Fiber

Silica (S_iO_2) glass



Impurities

a) Increase n of core

OR

b) Decrease n of cladding

Increase	Decrease
Ge	B
F	F
Na-B	

Wave Analysis:

Cylindrical dielectric waveguide
(step fiber)

assume all fields proportional to $e^{j(\omega t-\beta z)}$

$$E = (E_r, E_\phi, E_z)$$
$$H = (H_r, H_\phi, H_z)$$

E_i and H_i are function of (r, ϕ)

$$\nabla^2 \begin{pmatrix} E \\ H \end{pmatrix} = \omega^* \mu\varepsilon \begin{pmatrix} E \\ H \end{pmatrix}$$

But now need to use cylindrical coordinates:

$$d^2E_z/dr^2 + 1/r\ dE_z/dr + 1/r^2\ d^2E_z/d\phi^2 + (n_1{}^2k^2 - \beta^2)E_z = 0$$

Assume E_z proportional to $E(r)\ h(\phi)$ separation of variables

Since $h(2p + \phi) = h(\phi)$ =>try $h(\phi) =$ $\sin l\ \phi$

$\cos(l\ \phi)$

$e^{jl\phi}$

where l= integers

Substitute back into

$d^2E_z/dr^2 + 1/r\ dE_z/dr + [(n_1{}^2k^2 - \beta^2)-l^2/r^2]E_z = 0$ => Bessel function

Solutions closer to match physical situation.

For guided solutions:

In core, solutions must be finite
In cladding, solutions must approach 0 as r --> ∞

For r < a: $E(r) \propto J_l\,(UR)$ " Bessel function of 1st kind"

For r > a: $E(r) \propto K_l\,(WR)$ " modified Bessel function of 2[nd] kind"

$$UR = (n_1^2k^2 - \beta^2)^{0.5}\, r = \underbrace{a(n_1^2k^2 - \beta^2)^{0.5}}_{U}\ \underbrace{r/a}_{R}$$

$$WR = (\beta^2 - n_1^2k^2)^{0.5}\, a$$

Let $V^2 = U^2 + W^2 = a^2\,[n_1^2k^2 - \beta^2 + \beta^2 - n_2^2k^2] = a^2k^2[n_1^2 - n_2^2]$

$$\therefore\ V = a \cdot (2\pi/\lambda)\,[n_1^2 - n_2^2]^{0.5} \quad \text{(Normalized frequency)}$$
$$= a \cdot (2\pi/\lambda) \cdot NA$$

Solution procedure for step-index fiber modes:

1.

$$\begin{cases} E_z \\ \quad = A\, J_l\, (UR)\, e^{jl\phi} e^{j(\omega t - \beta z)} \qquad r < a \\ H_z \end{cases}$$

$$= B\, K_l\, (WR)\, e^{jl\phi} e^{j(\omega t - \beta z)} \qquad r > a$$

2. Match E_z and H_z at $r = a$

3. Use Maxwell's curl equations to find E_θ and H_θ E_z and H_z and E_θ and H_θ must match for $r = +a$ and $-a$. Solve all four equations simultaneously to yield eigenvalues

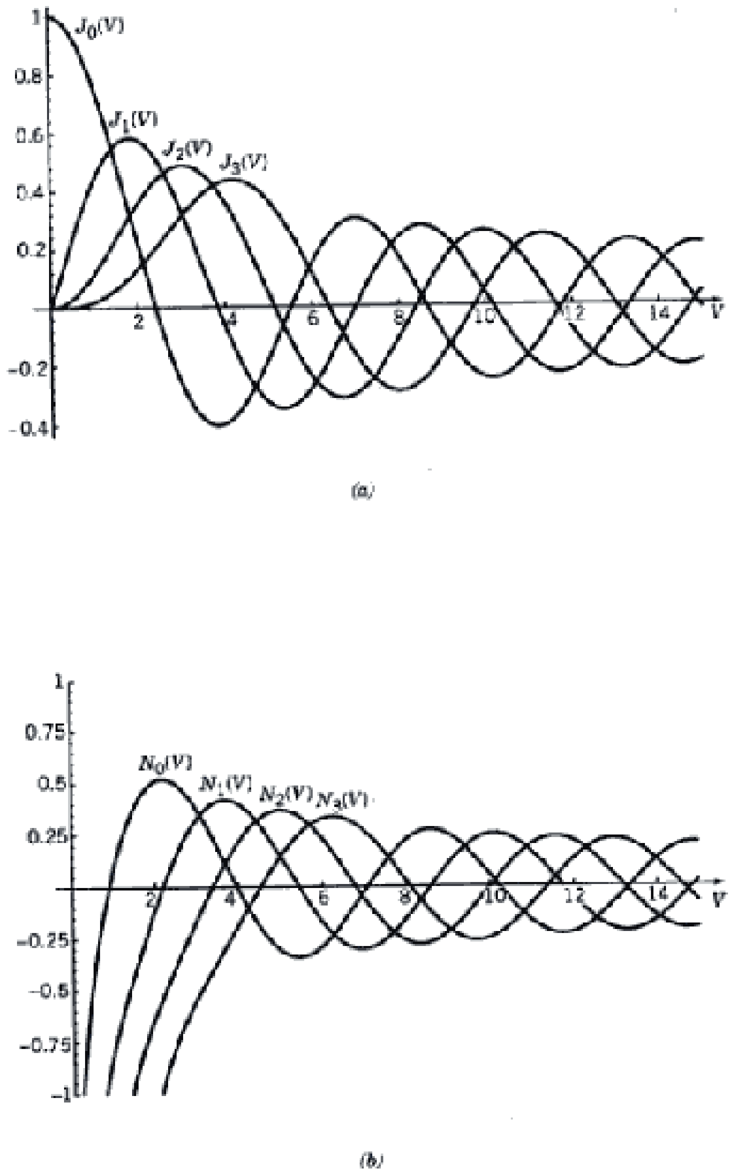


Figure 3.3. Ordinary Bessel functions.

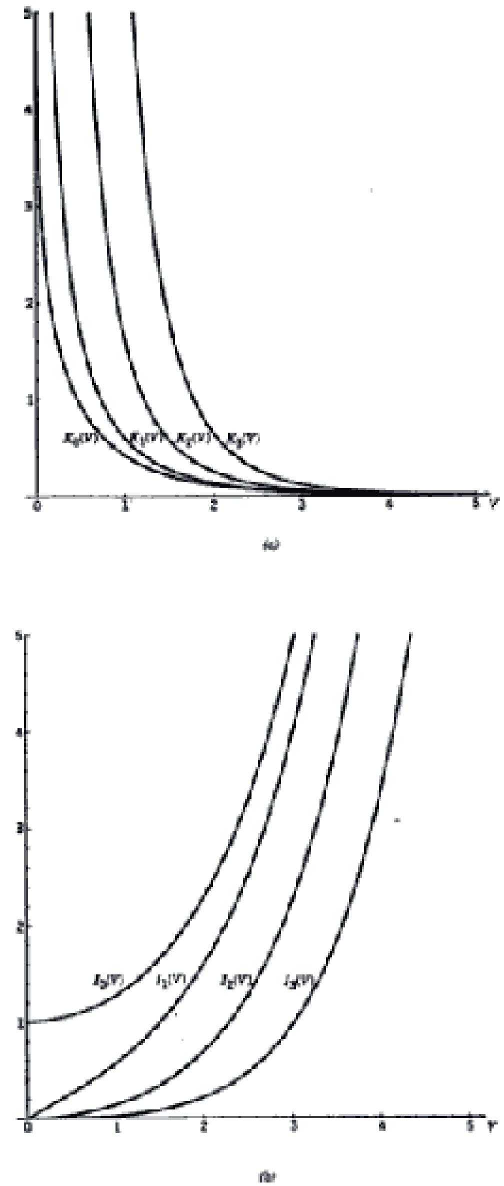


Figure 3.4. Modified Bessel functions.

A major simplification in math results if $(n_1-n_2)/n_1 << 1$
(weakly-guiding approximation $\Delta << 1$)

The eigenequations reduces to

$$J_{l\pm 1}(U) / J_l(U) = \pm (W/U)\, (K_{l\pm 1}(W) / K_l(W) \quad (+ \text{ only for } l = 0)$$

There are *m* possible solutions for each value of *l*

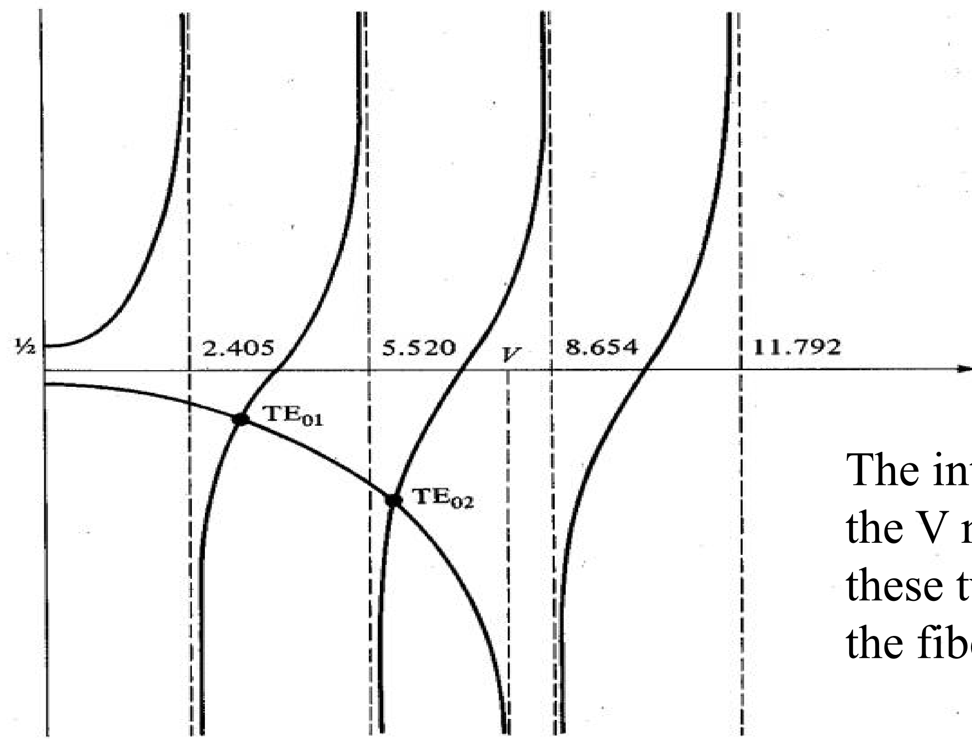
$\therefore$ U_{lm} are solutions

From definition of U, knowing U_{lm} permits calculation of β

$$\beta_{lm} = (n_1^2 k^2 - U_{lm})^{0.5}$$

The resulting system of equations can only be solved graphically. The graphical solutions represent the mode cutoffs for the different modes that can propagate in the fiber for any given V, where V is a convenient parameter determined by the properties of the fiber and wavelength of incident light.

$$V = 2*\pi/\lambda*a*NA$$



The intersections represent the V numbers at which these two modes turn on in the fiber.

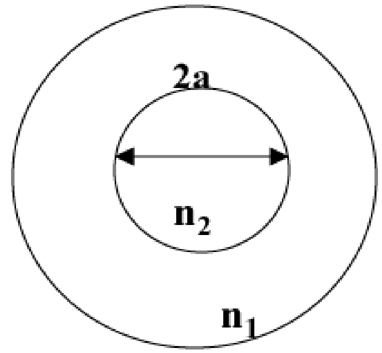
Normalization Parameters for Cylindrical Waveguides

Normalized frequency

$$V = \frac{2\pi a}{\lambda}\sqrt{n_2^2 - n_1^2}$$

$$\Delta = \frac{n_2^2 - n_1^2}{2n_2^2} \cong \frac{n_2 - n_1}{n_2}$$

$$V \cong \frac{2\pi a}{\lambda} n_2 \sqrt{2\Delta}$$



Normalized propagation constant

$$b = \frac{\left(\beta / k_0\right)^2 - n_1^2}{n_2^2 - n_1^2}$$

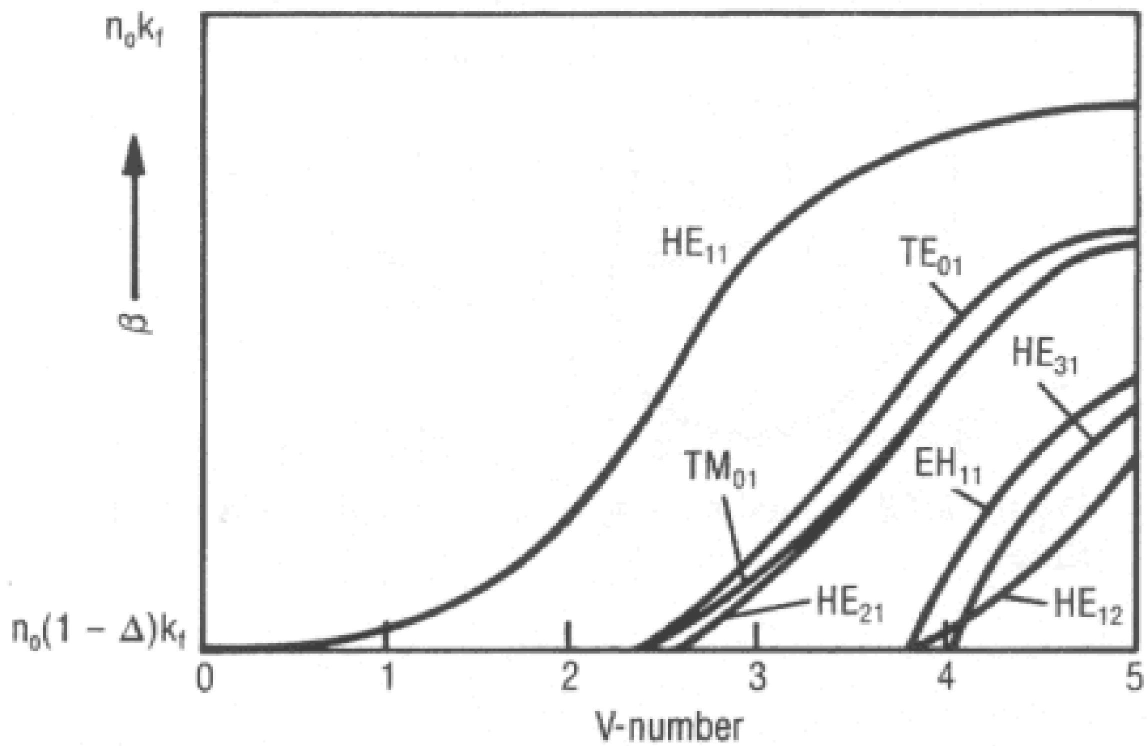
$$b = \frac{\beta / k_0 - n_1}{n_2 - n_1}$$

$$b\ \max = 1$$

$$b\ \min = 0$$

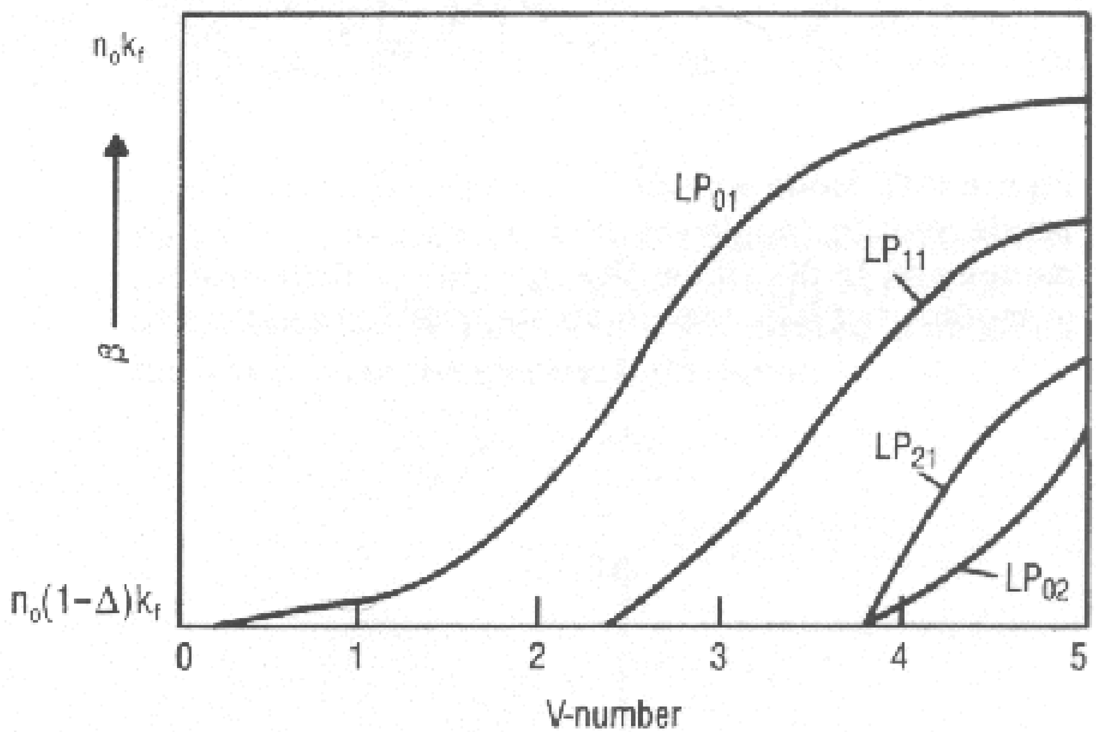
W. Wang

Same as slab waveguide except a=0 (almost always)



The normalized wave number, or V-number of a fiber is defined as V $= k_f\, a$ NA. Here k_f, is the free space wave number, $2\pi/\lambda_0$, a is the radius of the core, and NA is the numerical aperture of the fiber, NA = $(n_{core}{}^2 - n_{cladding}{}^2)^{1/2} \approx n_{core}(2\Delta)^{1/2}$, with $\Delta = (n_{core}-n_{cladding})/n_{core}$. Many fiber parameters can be expressed in terms of V. The TE and TM modes have non-vanishing cut-off frequencies. The cutoff frequency is found from $V = a\omega(2\Delta)^{½}/c = 2.405$. Only the lowest HE mode, HE_{11}, has no cutoff frequency. For $0 < V < 2.405$ it is the only mode that propagates in the fiber.

In the weakly-guiding approximation ($\Delta << 1$), the modes propagating in the fiber are linearly polarized (LP) modes characterized by two subscripts, m and n. (The longitudinal components of the fields are small when $\Delta << 1$.) The LP modes are combinations of the modes found from the exact theory of the wave guide. The HE_{11} mode becomes the LP_{01} mode in the weakly-guiding approximation.



The following table presents the first ten cutoff frequencies in a step-index fiber, as well as their fundamental modes.

V_c	Bessel function	q	Modes	LP desig.
0	–	0	HE_{11}	LP_{01}
2.405	J_0	1	TE_{01}, TM_{01}, HE_{21}	LP_{11}
3.832	J_1	2	EH_{11}, HE_{31}	LP_{21}
3.832	J_{-1}	0	HE_{12}	LP_{02}
5.136	J_2	3	EH_{21}, HE_{41}	LP_{31}
5.520	J_0	1	TE_{02}, TM_{02}, HE_{22}	LP_{12}
6.380	J_3	4	EH_{31}, HE_{51}	LP_{41}
7.016	J_1	2	EH_{12}, HE_{32}	LP_{22}
7.016	J_{-1}	0	HE_{13}	LP_{03}
7.588	J_4	5	EH_{41}, HE_{61}	LP_{51}

Single mode fiber

Single mode (SM) fiber is designed such that all the higher order waveguide modes are cut-off by a proper choice of the waveguide parameters as given below.

$$V = \frac{2\pi a}{\lambda}\sqrt{n_1^2 - n_2^2}$$

where, λ is the wavelength, *a* is the core radius, and n_1 *and* n_2 *are the core and cladding refractive indices, respectively. When* $V < 2.405$ *single mode condition is ensured. SM fiber is an essential requirement for interferometric sensors. Due to the small core size (~4 μm) alignment becomes a critical factor.*

Figure 3.10 shows plots of b, calculated using (3.76) and (3.77) in (3.78), for several LP modes as functions of V. As plotted on this scale, these results are essentially indistinguishable from those given by the exact numerical solution [5].

When considering single-mode fibers, the accuracy of the LP_{01} curve in Fig. 3.10 is of increased importance. It was in fact found to be accurate to within 5% over the range of V between 2.0 and 3.0, with the error increasing to around 10% as V decreases to 1.5 [7]. Numerous other approximate formulas exist as alternatives to (3.77) for determining b. The best of these was found by Rudolf and Neumann [8], who recognized that w is a nearly linear function of V over the range $1.3 < V < 3.5$. They were thus able to approximate w over this range by the simple function

$$w \approx 1.1428V - 0.9960 \tag{3.80}$$

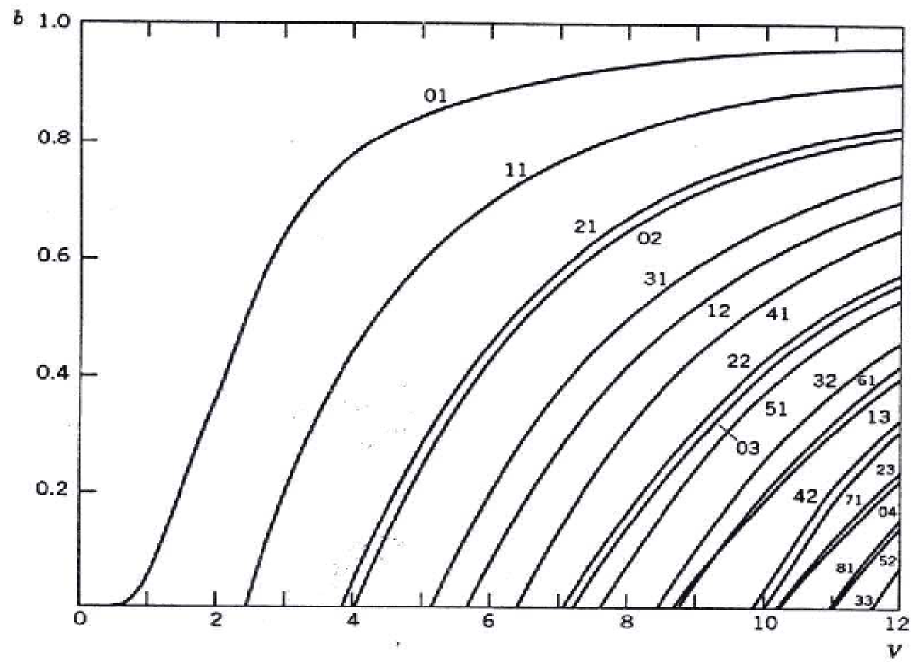
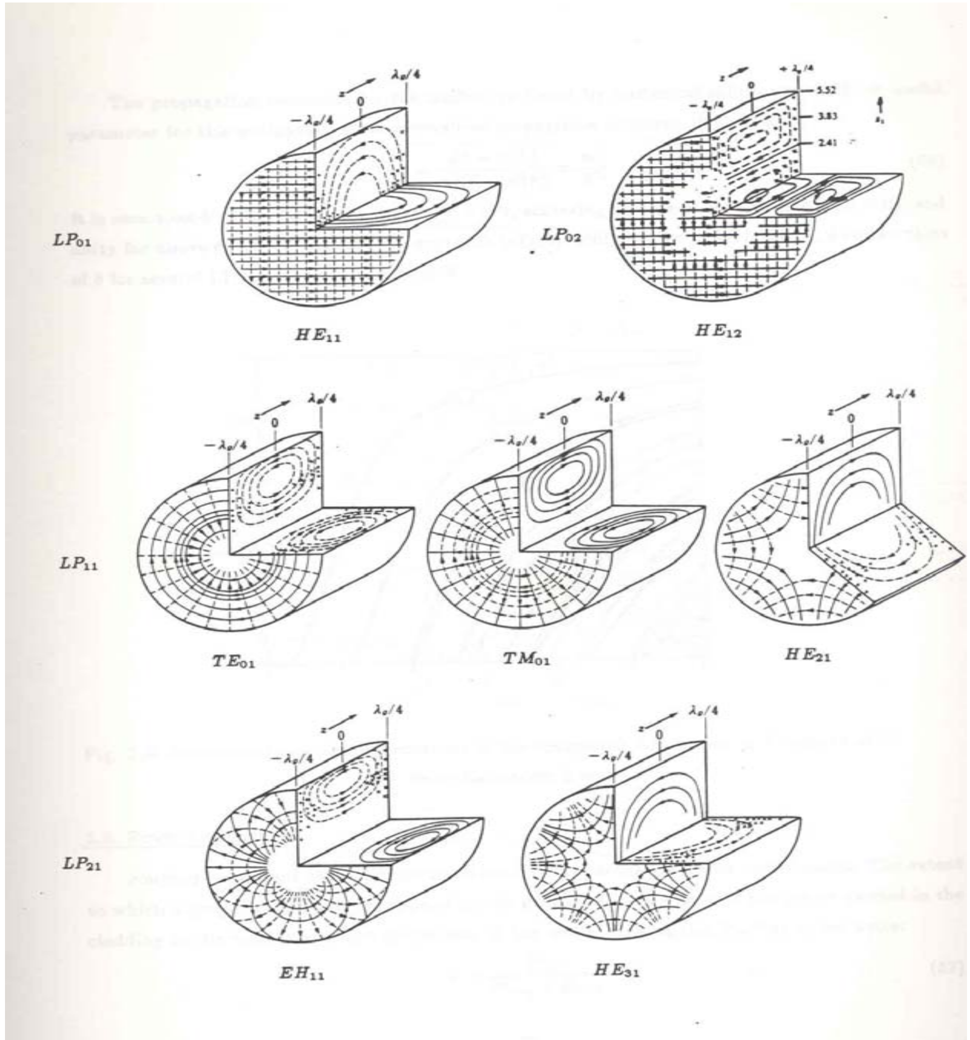
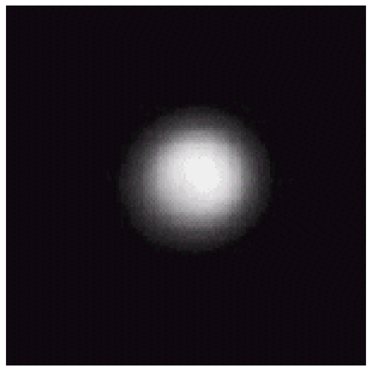


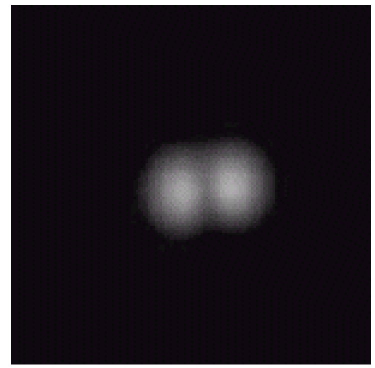
Figure 3.10. Normalized propagation constant, b, for designated LP modes as functions of V. (Adapted from ref. 5.)



Electric and magnetic fields for eight fundamental modes.



LP01



LP11

When the V number is less than 2.405 only the LP_{01} mode propagates. When the V number is greater than 2.405 the next linearly-polarized mode can be supported by the fiber, so that both the LP_{01} and LP_{11}, modes will propagate.

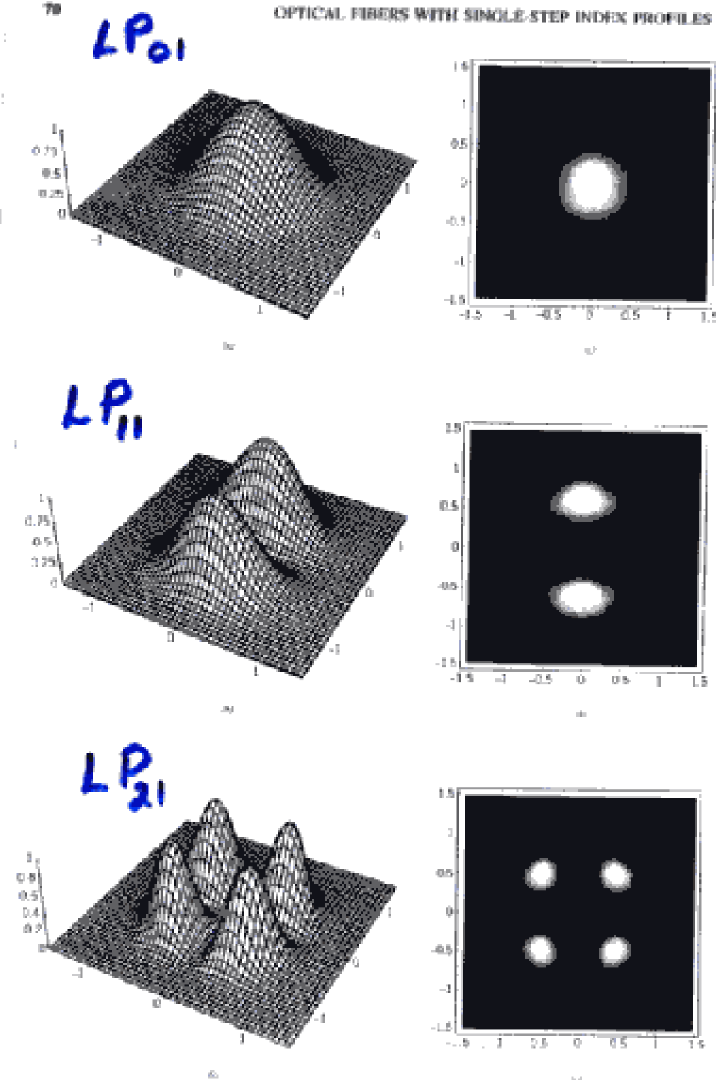


Figure 3.9. Intensity plots for the six LP modes, with $a = 1$. (a) LP_{01}: $v = 2$. (b) LP_{11}: $v = 3$. (c) LP_{21}: $v = 4.5$. (d) LP_{02}: $v = 4.5$. (e) LP_{31}: $v = 5.5$. (f) LP_{12}: $v = 6.3$.

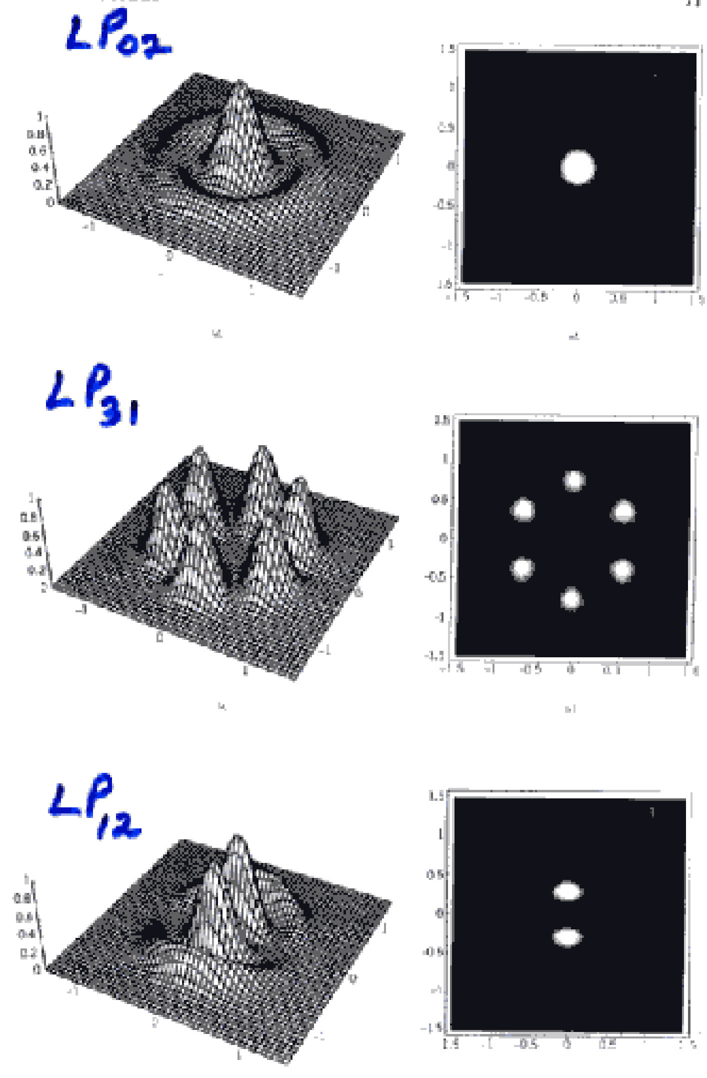
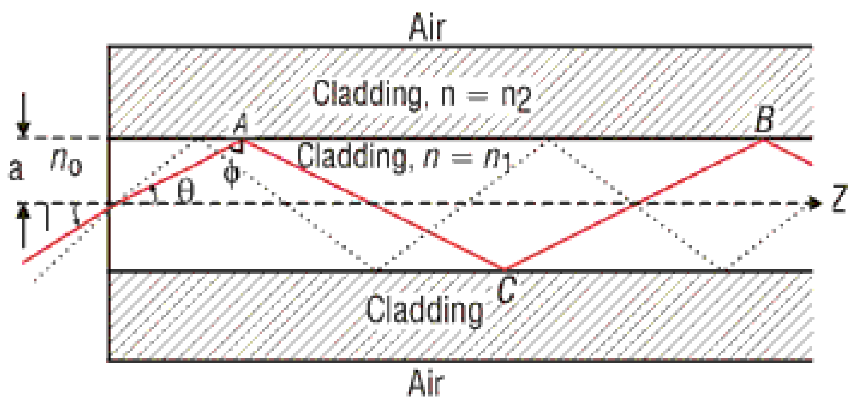


Figure 3.9. (*Continued*)

The SM fiber mentioned above is not truly single mode in that two modes with degenerate polarization states can propagate in the fiber. This can lead to signal interference and noise in the measurement. The degeneracy can be removed and a single mode polarization preserving fiber can be obtained by the use of an elliptical core fiber of very small size or with built in stress. In either case light launched along the major axis of the fiber is preserved in its state of polarization. It is also possible to make a polarizing fiber in which only one state of polarization is propagated. Polarimetric sensors make use of polarization preserving fibers. Thus, multimode fiber, single mode fiber and polarization preserving fiber are the three classes of fibers which are used in the intensity type, the interferometric type and the polarimetric type of sensors, respectively.

While discussing step-index fibers, we considered light propagation inside the fiber as a set of many rays bouncing back and forth at the core-cladding interface. There the angle θ could take a continuum of values lying between 0 and $\cos^{-1}(n_2/n_1)$, i.e.,



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$$0 < \theta < \cos^{-1}(n_2/n_1)$$

For $n_2 = 1.5$ and $\Delta \approx \frac{n_1 - n_2}{n_1} = 0.01$, we would get $n_2/n_1 \simeq$ and $\cos^{-1}\left(\frac{n_1}{n_2}\right) = 8.1°$, so

$$0 < \theta < 8.1°$$

Now, when the core radius (or the quantity Δ) becomes very small, ray optics does not remain valid and one has to use the more accurate wave theory based on Maxwell's equations.

In wave theory, one introduces the parameter

$$V = \frac{2\pi}{\lambda_0} a\sqrt{n_1^2 - n_2^2} = \frac{2\pi}{\lambda_0} a n_1 \sqrt{2\Delta} \approx \frac{2\pi}{\lambda_0} a n_2 \sqrt{2\Delta}$$

where Δ has been defined earlier and $n_1 \simeq n_2$. The quantity V is often referred to as the "*V-number*" or the "*waveguide parameter*" of the fiber. It can be shown that, if

$$V < 2.4045$$

only one guided mode (as if there is only one discrete value of θ) is possible and the fiber is known as a *single-mode fiber*. Further, for a step-index single-mode fiber, the corresponding (discrete) value of θ is approximately given by the following empirical formula

$$\cos\theta \approx 1 - \Delta\left[1 - \left(1.1428 - \frac{0.996}{V}\right)^2\right]$$

We may mention here that because of practical considerations the value Δ of ranges from about 0.002 to about 0.008

Assignment

Consider a step-index fiber (operating at 1300 nm) with $n_2 = 1.447$, $\Delta = 0.003$, and $a = 4.2$ μm. Thus,

$$V = \frac{2\pi}{\lambda_0(\mu m)} \times 4.2(\mu m) \times 1.447 \times \sqrt{0.006} \approx \frac{2.958}{\lambda_0(\mu m)}$$

Thus the fiber will be single moded and the corresponding value of θ—will be about θ = 3.1º. It may be mentioned that for the given fiber we may write

$$V = \frac{2\pi}{1.3(\mu m)} \times 4.2(\mu m) \times 1.447 \times \sqrt{0.006} \approx 2.275$$

Thus, for $\lambda_0 > 2.958/2.4045 = 1.23$ μm

which guarantees that $V < 2.4045$, the fiber will be single moded. The wavelength for which $V = 2.4045$ is known as the *cutoff wavelength* and is denoted by λ_c. In this example, $\lambda_c = 1.23$ μm and the fiber will be single moded for $\lambda_0 > 1.23$ μm.

Assignment

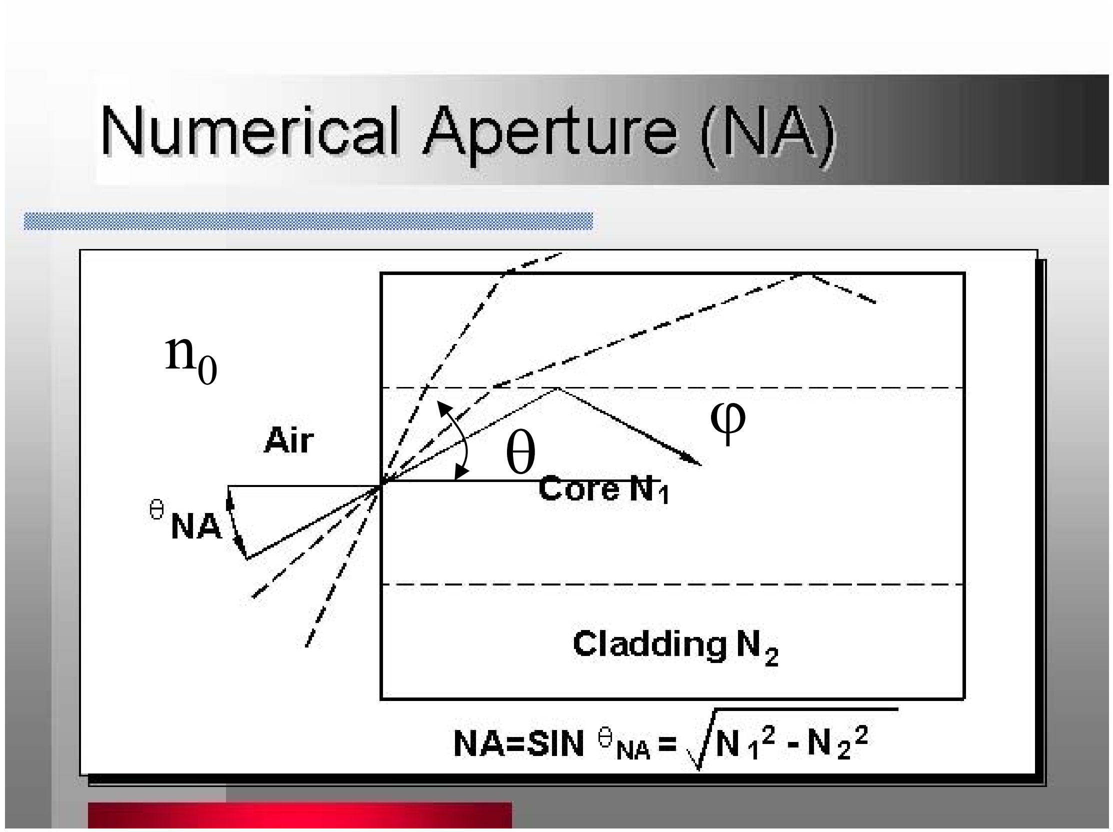
For reasons that will be discussed later, the fibers used in current optical communication systems (operating at 1.55 μm) have a small value of core radius and a large value of Δ. A typical fiber (operating at $\lambda_0 \approx 1.55$ μm) has $n_2 = 1.444$, $\Delta = 0.0075$, and $a = 2.3$ μm. Thus, at $\lambda_0 = 1.55$ μm, the V-number is,

$$V = \frac{2\pi}{1.55(\mu m)} \times 2.3(\mu m) \times 1.444 \times \sqrt{0.015} \approx 1.649$$

The fiber will be single moded (at 1.55 μm) with θ = 5.9°. Further, for the given fiber we may write

$$V = \frac{2\pi}{\lambda_0(\mu m)} \times 2.3(\mu m) \times 1.444 \times \sqrt{0.015} \approx \frac{2.556}{\lambda_0(\mu m)}$$

and therefore the cutoff wavelength will be $\lambda_c = 2.556/2.4045 = 1.06$ μm.



- The Numerical Aperture (NA) of a fiber is the measure of the maximum angle (θ_{NA}) of the light entering the end that will propagate within the core of the fiber
- Acceptance Cone = 20NA
- Light rays entering the fiber that exceed the angle θ_{NA} will enter the cladding and be lost
- For the best performance the NA of the transmitter should match the NA of the fiber

NA derivation

We know $\frac{\sin i}{\sin\theta} = \frac{n_1}{n_0}$ and $\sin\phi(=\cos\theta) > \frac{n_2}{n_1}$

Since $\sin\theta = \sqrt{1-\cos^2\theta}$ we get $\sin\theta < \left[1-\left(\frac{n_2}{n_1}\right)^2\right]^{1/2}$

Assume the θ_{NA} is the half angle of the acceptance cone,

$$\sin\theta_{NA}=(n_1^2-n_2^2)^{1/2} = n_1 \mathrm{sqrt}(2\Delta)$$

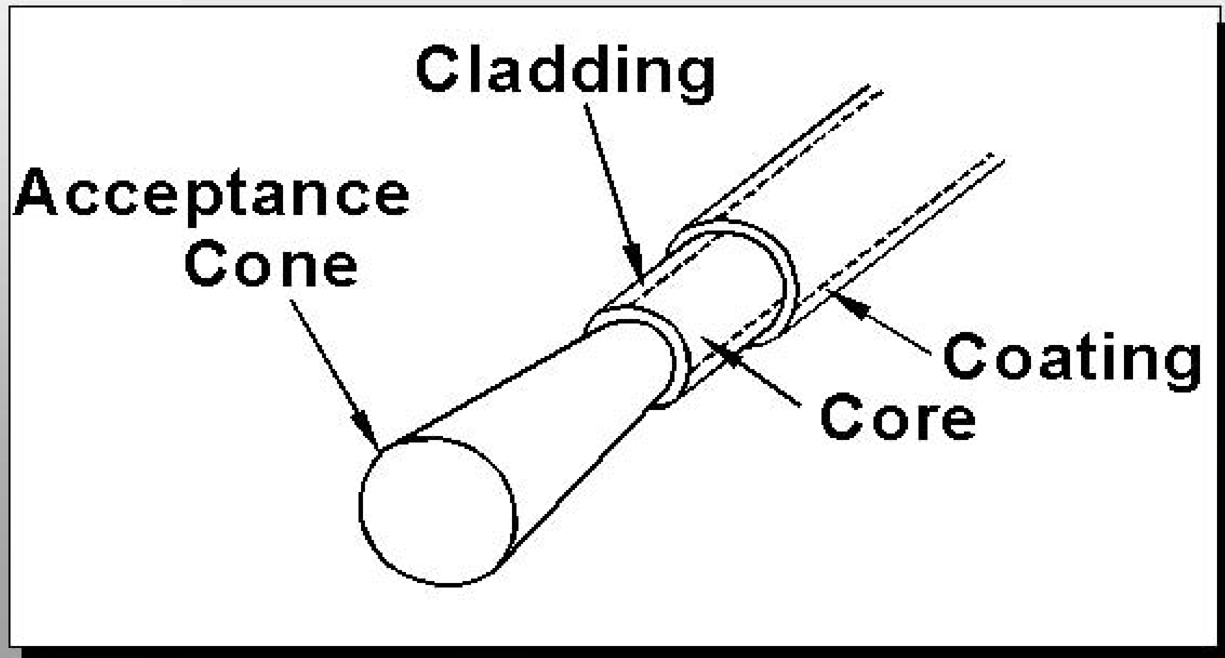
We define a parameter Δ through the following equations.

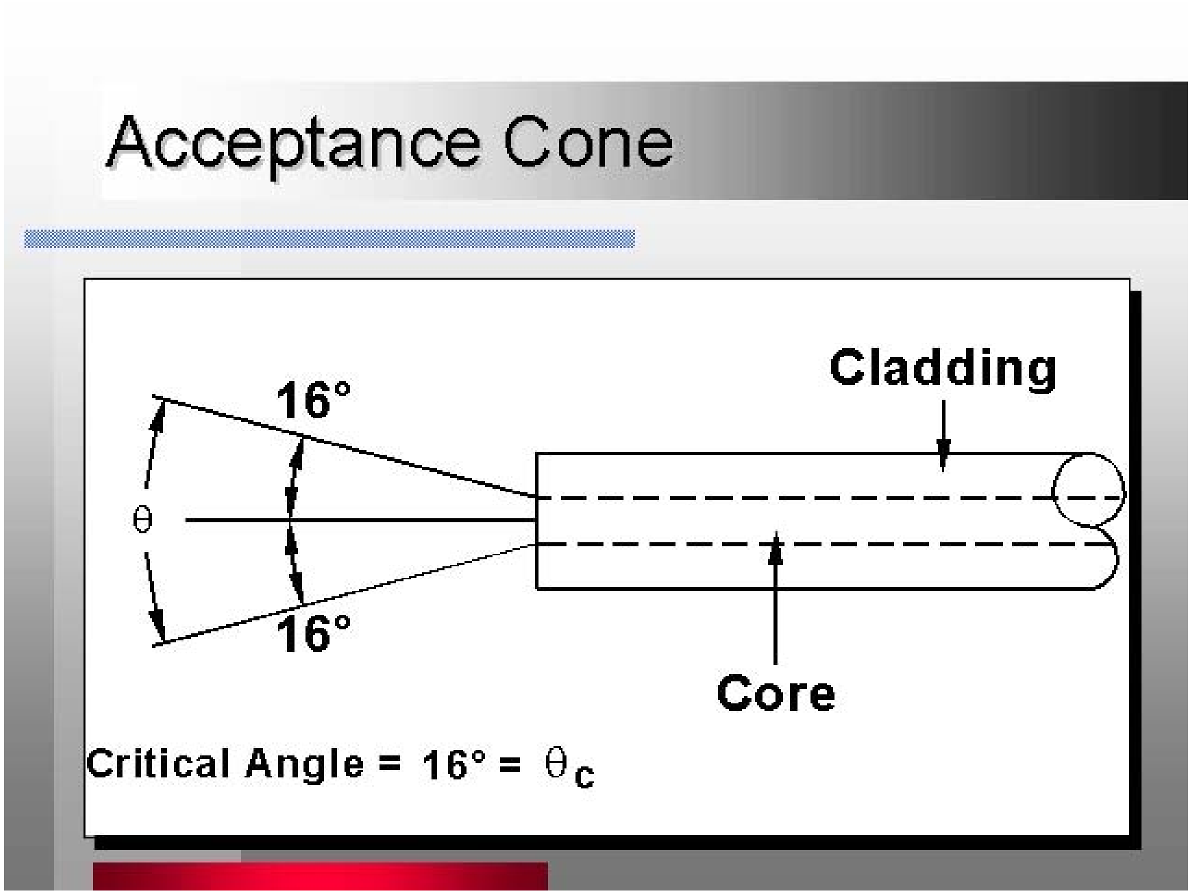
$$\Delta \equiv \frac{n_1^2 - n_2^2}{2n_2^2}$$

When Δ << 1 (as is indeed true for silica fibers where n_1 is very nearly equal to n_2) we may write

$$\Delta = \frac{(n_1 + n_2)(n_1 - n_2)}{2n_1^2} \approx \frac{(n_1 - n_2)}{n_1} \approx \frac{(n_1 - n_2)}{n_2}$$

Acceptance Cone





Single mode fiber critical angle <20°

Multimode fiber critical angle <60°

Example

For a typical step-index (multimode) fiber with $n_1 \approx 1.45$ and $\Delta \approx 0.01$, we get

$$\sin i_m = n_1 \sqrt{2\Delta} = 1.45\sqrt{2 \times (0.01)} = 0.205$$

so that $i_m \approx 12°$. Thus, all light entering the fiber must be within a cone of half-angle 12°.

In a short length of an optical fiber, if all rays between $i = 0$ and i_m are launched, the light coming out of the fiber will also appear as a cone of half-angle i_m emanating from the fiber end. If we now allow this beam to fall normally on a white paper and measure its diameter, we can easily calculate the *NA* of the fiber.