Problem 1.3.14

Solve The I.V.P. $y^3 y' + x^3 = 0 \quad y(0) = 1$

Separate $y^3 \frac{dy}{dx} = -x^3 \Rightarrow y^3 dy = -x^3 dx$

integrate $\int y^3 dy = -\int x^3 dx$

$$\frac{y^4}{4} = -\frac{x^4}{4} + C$$

Apply I.C. $\frac{1^2}{4} = -\frac{(0)^4}{4} + C \Rightarrow C = \frac{1}{4}$

Solution is $\boxed{y^4 + x^4 = 1}$

Problem 1.3.17: Solve The I.V.P.

$\frac{dy}{dx} \cosh^2(x) - \sin^2(y) = 0 \qquad y(0) = \pi/2$

Separate variables: $\frac{dy}{\sin^2(y)} = \frac{dx}{\cosh^2(x)}$

integrate $\int \frac{dy}{\sin^2 y} = \int \frac{dx}{\cosh^2 x} \Rightarrow$ Lookup in Tables! or use Math Program

$$-\cot y = \tanh x + C$$

Apply initial Condition

$$-\cot(\pi/2) = \tanh(0) + C$$

$$-0 = 0 + C \Rightarrow C = 0$$

Solution $\boxed{-\cot(y) = \tanh(x)}$

Problem 1.4.5

Find Fraction of ${}_6C^{14}$ in a Tree 3000 years old.

From Example 1; The model for decay is

$$y' = ky \quad \text{With Solution} \quad y(t) = y_0 e^{kt}$$

Further, from the example we see that $K = -0.000121$

So $y(3000\text{ years}) = y_0 e^{-0.000121(3000)} = y_0 e^{-0.363}$

$$\frac{y}{y_0} = 0.6955 \quad \text{or} \quad 69.55\% \text{ of the original concentration.}$$

Problem 1.5.1: Given $u(x,y)$ find the exact differential eqn $du = 0$ and plot curves $u(x,y) = \text{const}$ for

$$u = x^2 + 4y^2$$

$$du = \frac{du}{dx}dx + \frac{du}{dy}dy = 2x\,dx + 8y\,dy = 0$$

Problem 1.5.13

Test the equation for exactness and solve

$$3y^2\,dx + x\,dy = 0 \qquad y(1) = \tfrac{1}{2}$$

identify parts $\quad M = 3y^2$
$\quad N = x$

Test $\quad \dfrac{\partial M}{\partial y} \overset{?}{=} \dfrac{\partial N}{\partial x} \Rightarrow \dfrac{\partial}{\partial y}(3y^2) = \dfrac{\partial}{\partial x}(x)$

$$6y \overset{?}{=} 1 \Rightarrow \text{No!}$$
Not Exact

Do not need integrating factors, eqn is separable

$$x\,dy = -3y^2\,dx$$

Separate

$$\frac{dy}{-3y^2} = \frac{dx}{x}$$

Integrate $\quad \displaystyle\int \frac{dy}{-3y^2} = \int \frac{dx}{x} \Rightarrow \frac{1}{3y} = \ln x + C_1$

Apply initial condition ; Let $C_2 = 3C_1$

$$\frac{1}{\cancel{3}(1/2)} = 3\ln(1) + C_2 \Rightarrow C_2 = 2$$

$$\frac{1}{y} = 3\ln x + 2 \Rightarrow \boxed{y = \frac{1}{3\ln x + 2}}$$

1.5.32: Find an integrating factor by inspection or by Theorem, and solve

$$2xy\,dx + 3x^2 dy = 0$$

$$\begin{aligned} M(x,y) &= 2xy \\ N(x,y) &= 3x^2 \end{aligned} \quad \text{Testing: } \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = \frac{\partial}{\partial y}(2xy) - \frac{\partial}{\partial x}(3x^2) \Rightarrow$$

$2x - 6x = 4x \neq 0$ NOT exact.
but x is common factor, so there is a factor

Multiply through by F and apply test

$$2xyF\,dx + 3x^2 F\,dy = 0$$

$$\frac{\partial}{\partial y}\left[2xyF\right] = \frac{\partial}{\partial x}\left[3x^2 F\right]$$

$$2xF + 2xy\frac{\partial F}{\partial x} = 3x^2\frac{\partial F}{\partial x} + 6xF$$

Let $F = F(y)$; $\frac{\partial F}{\partial x} = 0$ + Rearrange

$$2\cancel{x}y\frac{\partial F}{\partial x} = (6\cancel{x} - 2\cancel{x})F$$

integrate $\displaystyle\int \frac{dF}{F} = \int \frac{4}{2y}\,dy \Rightarrow \ln F = 2\ln y$

$\boxed{F = y^2}$ is an integrating factor

Check: $\frac{\partial}{\partial y}\left[2xy^3\right] - \frac{\partial}{\partial x}\left[3x^2y^2\right] = 6xy^2 - 6xy^2 = 0$ ✓

Now integrate to get general solution
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1.5.32 Continued. Solving $2xy^3dx + 3x^2y^2dy = 0$

$$u = \int 2xy^3\,dx + f(y) = x^2y^3 + f(y)$$

$$\frac{du}{dy} = 3x^2y^2 + \frac{\partial f(y)}{dx}$$

Comparing to $3x^2y^2$ shows that $f(y) = 0$ so....

$$\boxed{0 = x^2y^3 + C}$$

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Problem 1.5.33 Find the integrating factor for:

$$(2\cos y + 4x^2)\,dx - x\sin y\,dy = 0$$ and solve

identify parts $P = 2\cos y + 4x^2$

$Q = -x\sin y$

Skipping straight to Theorem, and assuming the factor depends only on x:

$$F(x) = \exp\int \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)dx$$ doing this in parts

$$\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} = \left[\frac{\partial}{\partial y}\left(2\cos y + 4x^2\right) - \frac{\partial}{\partial x}\left(x\sin y\right)\right]$$

$$= -2\sin y + \sin y = -\sin y$$

$$\int \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)dx = \int \frac{-\cancel{\sin y}}{-x\cancel{\sin y}}\,dx = \ln x$$

$$F(x) = \exp(\ln x) = x$$

So the equation is

$$(2x\cos y + 4x^3)\,dx - x^2\sin y\,dy = 0$$

and we can integrate out to get the General Solution.

$$u = \int (2x\cos y + 4x^3)\,dx + C(y) = x^2\cos y + \frac{4x^4}{4} + C(y)$$

where $C(y)$ is some function of y.

Diffentiation with y will now give us $C(y)$

$$\frac{du}{dy} = -x^2\sin y + \frac{\partial C}{\partial y}\ ; \text{ So clearly } C(y) = 0$$

and the solution is

$$\boxed{u = x^2\cos y + x^4 + D}$$ where $D \equiv$ constant.

1.5.34: Find integrating Factor and Solve

$$2\cos y\,dx - \sin y\,dy = 0$$

$$P = 2\cos y \qquad \frac{\partial P}{\partial y} = -2\sin y \neq \frac{\partial Q}{\partial x} = 0$$

$$Q = -\sin y$$

NOT EXACT.

Apply $F(x)$ to find integrating factor

$$2F\cos y\,dx - F\sin y\,dy = 0$$

$$\frac{\partial}{\partial y}\left[2F\cos y\right] = -\frac{\partial}{\partial x}\left[F\sin y\right]$$

$$-2F\cancel{\sin y} = -\frac{\partial F}{\partial x}\cancel{\sin y}$$

$$\frac{dF}{F} = 2\,dx \Rightarrow \ln F = 2x \Rightarrow \underline{F = e^{2x}}$$

Substitute into original eqn and Solve.

$$2e^{2x}\cos y\,dx - e^{2x}\sin y\,dy = 0$$

$$u = \int 2e^{2x}\cos y\,dx + f(y)$$

$$u = e^{2x}\cos y\,dx + f(y)$$

Now differentiate with respect to y to find $f(y)$

$$\frac{du}{dy} = -e^{2x}\sin y + \frac{\partial f}{\partial y} \quad ; \quad \text{clearly } f(y) = 0$$

So $$\boxed{= e^{2x}\cos y + \text{Const}}$$