$\boxed{7.1.5}$ Find The Spectrum and Eigenvectors of The Following matrix

$A = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}$; form $\underline{\underline{A}}\,\underline{x} = \lambda \underline{x} \Rightarrow (A - \lambda I)x = 0 \Rightarrow$

$\begin{bmatrix} 3-\lambda & 4 \\ 4 & -3-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$ ie find λ_1, λ_2 ST. $\mathrm{Det}(A - \lambda I) = 0$

$(3-\lambda)(-3-\lambda) - 16 = 0$

$\lambda^2 - 9 - 16 = 0 \Rightarrow \lambda^2 - 25 = 0 \qquad \lambda = \pm\sqrt{25} = \pm 5$

$\boxed{\lambda_1 = 5, \lambda_2 = -5 \text{ is The Spectrum}}$

Now Find The associated ~~vectors~~ Eigenvectors.

For $\lambda = 5$

$\begin{bmatrix} 3-5 & 4 \\ 4 & -3-5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2 & 4 \\ 4 & -8 \end{bmatrix}\begin{matrix} x_1 \\ x_2 \end{matrix} = 0$

$-2x_1 + 4x_2 = 0 \Rightarrow x_1 = 2x_2$;

Let $x_2 = 1$, Then $x_1 = 2$; So One Eigenvector is

$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

For $\lambda = -5$: $\begin{bmatrix} 3-(-5) & 4 \\ 4 & -3-(-5) \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 8 & 4 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

$8x_1 + 4x_2 = 0 \Rightarrow 2x_1 = -x_2$

Let $x_1 = 1$, $x_2 = -2 \Rightarrow$ So another Vector is $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$

So The vectors for $\lambda = 5, -5$ are $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ Respectivly.

$\boxed{7.1.8}$ Find the spectrum and eigenvectors, of the following matrix.

$$\left| \begin{bmatrix} 0.8 & -0.6 \\ 0.6 & 0.8 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = 0$$

$$\begin{vmatrix} 0.8-\lambda & -0.6 \\ 0.6 & 0.8-\lambda \end{vmatrix} = 0 \Rightarrow \left(\frac{4}{5}-\lambda\right)\left(\frac{4}{5}-\lambda\right) + \left(\frac{3}{5}\right)\left(\frac{3}{5}\right) = 0$$

$$\frac{16}{25} - \frac{8}{5}\lambda + \lambda^2 + \frac{9}{25} = 0 \Rightarrow$$

$$5\lambda^2 - 8\lambda + 5 = 0$$

$$\lambda = \frac{8 \pm \sqrt{64 - 4(5)(5)}}{2(5)} = \frac{8}{10} \pm \frac{6i}{10}$$

So the "spectrum" ie, eigenvalues are --
$\lambda_1 = 0.8 + 0.6i \qquad \lambda_2 = 0.8 - 0.6i$

Now go after the eigenvectors

for λ_1: $(A - \lambda_1 I) X_1 = 0 \Rightarrow \begin{bmatrix} 0.8-(0.8+0.6i) & -0.6 \\ 0.6 & 0.8-(0.8+0.6i) \end{bmatrix} X_1 = 0 \Rightarrow$

$$\begin{bmatrix} -0.6i & -0.6 \\ 0.6 & -0.6i \end{bmatrix} \underline{X_1} = 0$$

Multiply row 1 by i and subtract from row 2

$$\begin{bmatrix} -0.6i & -0.6 \\ (-0.6i)(i) - 0.6 & (-0.6)(i) - 0.6i \end{bmatrix} \underline{X_1} = 0 \Rightarrow \begin{bmatrix} 0.6i & -0.6 \\ 0 & 0 \end{bmatrix} X_1 = 0$$

$$-0.6i\,X_1 - 0.6X_2 = 0 \Rightarrow X_1 i + X_2 = 0$$

So let $X_1 = 1$, $X_2 = -i$; $\boxed{\underline{X_1} = \begin{bmatrix} 1 \\ -i \end{bmatrix}}$

for λ_2

$$\begin{bmatrix} 0.8-(0.8-0.6i) & -0.6 \\ 0.6 & 0.8-(0.8-0.6i) \end{bmatrix} \underline{X_2} = 0$$

$$\begin{bmatrix} 0.6i & -0.6 \\ 0.6 & 0.6i \end{bmatrix} \underline{X_2} = 0 \Rightarrow 0.6i\,X_1 - 0.6X_2 = 0$$

$$X_1 i - X_2 = 0$$

Let $X_1 = 1$, $X_2 = i$ so $\boxed{\underline{X_2} = \begin{bmatrix} 1 \\ i \end{bmatrix}}$

7.1.9 Find the spectrum & eigenvectors of $\begin{bmatrix} 3 & 0 & 0 \\ 0 & -8 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

$$\begin{vmatrix} 3-\lambda & 0 & 0 \\ 0 & -8-\lambda & 0 \\ 0 & 0 & 4-\lambda \end{vmatrix} = (3-\lambda)(-8-\lambda)(4-\lambda) = 0$$

So $\boxed{\lambda = 3, -8, 4}$ (Diagonal)

$$\lambda_1: \begin{bmatrix} 3-3 & 0 & 0 \\ 0 & -8-3 & 0 \\ 0 & 0 & 4-3 \end{bmatrix} \underset{\sim}{X_1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -11 & 0 \\ 0 & 0 & -7 \end{bmatrix} \underset{\sim}{X_1} \Rightarrow \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \underset{\sim}{X_1}$$

$$\lambda_2: \begin{bmatrix} 3+8 & 0 & 0 \\ 0 & -8+8 & 0 \\ 0 & 0 & 4+8 \end{bmatrix} \underset{\sim}{X_2} = \begin{bmatrix} 11 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 12 \end{bmatrix} X_2 \Rightarrow \underset{\sim}{X_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\lambda_3: \begin{bmatrix} 3-4 & 0 & 0 \\ 0 & -8-4 & 0 \\ 0 & 0 & 4-4 \end{bmatrix} \underset{\sim}{X_3} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -12 & 0 \\ 0 & 0 & 0 \end{bmatrix} X_3 \Rightarrow \underset{\sim}{X_3} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

7.1.11 Find the spectrum and eigenvectors of $\begin{bmatrix} 3 & 5 & 3 \\ 0 & 4 & 6 \\ 0 & 0 & 1 \end{bmatrix}$

$$\left|\begin{bmatrix} 3-\lambda & 5 & 3 \\ 0 & 4-\lambda & 6 \\ 0 & 0 & 1-\lambda \end{bmatrix}\right| = 0$$

$$(3-\lambda)(4-\lambda)(1-\lambda) = 0 \qquad \boxed{\lambda = 3, 4, 1}$$

$$\lambda_1 = 3: \begin{bmatrix} 3-3 & 5 & 3 \\ 0 & 4-3 & 6 \\ 0 & 0 & 1-3 \end{bmatrix} \underset{\sim}{X_1} = 0 \Rightarrow$$

$$\begin{bmatrix} 0 & 5 & 3 \\ 0 & 1 & 6 \\ 0 & 0 & -2 \end{bmatrix} \underset{\sim}{X_1} = 0 \Rightarrow \boxed{\underset{\sim}{X_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}$$

Continued on next page

7.1.11 continued

$\lambda_2 = 4$:

$$\begin{bmatrix} 3-4 & 5 & 3 \\ 0 & 4-4 & 6 \\ 0 & 0 & 1-4 \end{bmatrix} \underset{\sim}{X_1} = \begin{bmatrix} -1 & 5 & 3 \\ 0 & 0 & 6 \\ 0 & 0 & -3 \end{bmatrix} X_2 = 0 \Rightarrow \begin{array}{l} X_3 = 0 \\ -X_1 + 5X_2 = 0 \\ X_1 = 5X_2 \end{array}$$

Let $X_2 = 1$, $X_1 = 5$; $\boxed{\underset{\sim}{X_2} = \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}}$

$\lambda_3 = 1$:

$$\begin{bmatrix} 3-1 & 5 & 3 \\ 0 & 4-1 & 6 \\ 0 & 0 & 1-1 \end{bmatrix} \underset{\sim}{X_3} = \begin{bmatrix} 2 & 5 & 3 \\ 0 & 3 & 6 \\ 0 & 0 & 0 \end{bmatrix} \underset{\sim}{X_3} = 0 \Rightarrow \begin{array}{l} (1)\ 2X_1 + 5X_2 + 3X_3 = 0 \\ (2)\ 3X_2 + 6X_3 = 0 \end{array}$$

by eqn 2, $X_2 = -2X_3$; $X_1 = -\dfrac{(5X_2 + 3X_3)}{2}$

Let $X_3 = 2$; Then $X_2 = -4$ $X_1 = 7$ and

$\boxed{\underset{\sim}{X_3} = \begin{bmatrix} 7 \\ -4 \\ 2 \end{bmatrix}}$

7.1.12 Find the spectrum and Eigenvectors for

$$\begin{bmatrix} a & 1 & 0 \\ 1 & a & 1 \\ 0 & 1 & a \end{bmatrix}; \quad [A - \lambda I] X = 0 \Rightarrow \left| \begin{bmatrix} a-\lambda & 1 & 0 \\ 1 & a-\lambda & 1 \\ 0 & 1 & a-\lambda \end{bmatrix} \right| = 0$$

$$(a-\lambda) \begin{vmatrix} a-\lambda & 1 \\ 1 & a-\lambda \end{vmatrix} - 1 \begin{vmatrix} 1 & 0 \\ 1 & a-\lambda \end{vmatrix} = 0$$

$$(a-\lambda)\left[(a-\lambda)^2 - 1\right] - (a-\lambda) = 0$$

$$(a-\lambda)\left((a-\lambda)^2 - 2\right) = 0$$

$$\lambda = a; \text{ and } (a-\lambda)^2 - 2 = 0 \Rightarrow \lambda = a \pm \sqrt{2}$$

$$\boxed{\lambda_1 = a, \quad \lambda_2 = a - \sqrt{2} \quad \lambda_3 = a + \sqrt{2}}$$

$\boxed{7.1.12}$ continued.

$$\lambda_1 = a: \quad \begin{bmatrix} a-a & 1 & 0 \\ 1 & a-a & 1 \\ 0 & 1 & a-a \end{bmatrix} \underset{\sim}{X_1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \underset{\sim}{X_1} = 0 \quad \begin{matrix} x_2 = 0 \\ x_1 + x_3 = 0 \end{matrix}$$

Let $x_1 = 1$, $x_3 = -1$; $\boxed{\underset{\sim}{X_1} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}}$

$$\lambda_2 = a+\sqrt{2}: \quad \begin{bmatrix} a-(a+\sqrt{2}) & 1 & 0 \\ 1 & a-(a+\sqrt{2}) & 1 \\ 0 & 1 & a-(a+\sqrt{2}) \end{bmatrix} \underset{\sim}{X_2} = 0$$

$$\begin{bmatrix} -\sqrt{2} & 1 & 0 \\ 1 & -\sqrt{2} & 1 \\ 0 & 1 & -\sqrt{2} \end{bmatrix} X_3 = 0 \quad \text{subtracting } R_3 \text{ from Row 1}$$

$$\begin{bmatrix} -\sqrt{2} & 0 & \sqrt{2} \\ 1 & -\sqrt{2} & 1 \\ 0 & 1 & -\sqrt{2} \end{bmatrix} X_2 = 0 \quad -x_1 + x_3 = 0; \ x_1 = x_3$$

$$x_2 = \frac{x_1 + x_3}{\sqrt{2}}$$

Choose $x_3 = 1$; $x_1 = 1$, $x_2 = \frac{2}{\sqrt{2}} = \sqrt{2}$

$$\boxed{\underset{\sim}{X_2} = \begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix} \quad \text{for } \lambda_2 = a+\sqrt{2}}$$

$\lambda_3 = a - \sqrt{2} \Rightarrow$ substituting in $\rightarrow$ $\begin{bmatrix} \sqrt{2} & 1 & 0 \\ 1 & \sqrt{2} & 1 \\ 0 & 1 & \sqrt{2} \end{bmatrix} \underset{\sim}{X_3} = 0$

Subtracting Row 3 from Row 1 as with λ_2

$$\begin{bmatrix} \sqrt{2} & 0 & -\sqrt{2} \\ 1 & \sqrt{2} & 1 \\ 0 & 1 & \sqrt{2} \end{bmatrix} \underset{\sim}{X_3} = 0 \quad \text{so and} \begin{cases} x_1 - x_3 = 0 \\ x_1 + \sqrt{2}x_2 + x_3 = 0 \end{cases}$$

Choose $x_3 = 1$, Then $x_1 = 1$, $x_2 = \frac{-2}{\sqrt{2}} = -\sqrt{2}$

$$\boxed{\underset{\sim}{X_3} = \begin{bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{bmatrix}}$$

$\boxed{7.2.1}$ Find the principal directions and corresponding factors of extension or contraction of the elastic deformation $y = Ax$ given A:

$$A = \begin{bmatrix} 4 & \sqrt{8} \\ \sqrt{8} & 6 \end{bmatrix}$$

"Principal Directions" $\Rightarrow$ Eigenvectors
"Factors of Extension" $\Rightarrow$ Eigenvalues.

Basically find where $Ax = y = \lambda x \Rightarrow Ax - \lambda x = 0$

$\Rightarrow [A - \lambda I]x = 0 \Rightarrow$ an Eigenvalue Problem.

$$\begin{vmatrix} 4-\lambda & \sqrt{8} \\ \sqrt{8} & 6-\lambda \end{vmatrix} = 0 \Rightarrow (4-\lambda)(6-\lambda) - 8 = 0$$

$$\lambda^2 - 10\lambda + 24 - 8 = 0$$

$$\lambda^2 - 10\lambda + 16 = 0$$

$$(\lambda - 8)(\lambda - 2) = 0 \quad \boxed{\lambda = 2, 8}$$ are the factors of extension

Now, get Principal Directions: for $\lambda = 8$

$$\begin{bmatrix} 4-8 & \sqrt{8} \\ \sqrt{8} & 6-8 \end{bmatrix} \underset{\sim}{x_1} = \begin{bmatrix} -4 & \sqrt{8} \\ \sqrt{8} & -2 \end{bmatrix} x_1 = 0$$; multiply Row 1 by $\sqrt{8}/4$ and add it to Row 2

$$\begin{bmatrix} -4 & \sqrt{8} \\ -4\frac{\sqrt{8}}{4} + \sqrt{8} & \frac{\sqrt{8}}{4}\sqrt{8} - 2 \end{bmatrix} \underset{\sim}{x_1} = \begin{bmatrix} -4 & \sqrt{8} \\ -\sqrt{8}+\sqrt{8} & 8/4 - 2 \end{bmatrix} \underset{\sim}{x_1}$$

$$= \begin{bmatrix} -4 & \sqrt{8} \\ 0 & 0 \end{bmatrix} \underset{\sim}{x_1} = 0 \; ; \quad -4X_1 + \sqrt{8}X_2 = 0 \Rightarrow$$

$$\Rightarrow \sqrt{2}X_1 = X_2$$

Let $X_1 = 1, \Rightarrow X_2 = \sqrt{2}$

for $\lambda = 8$ Extension Factor, $\underset{\sim}{X} = \begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix}$ is the Principal Direction for $\lambda = 2$:

$$\begin{bmatrix} 4-2 & \sqrt{8} \\ \sqrt{8} & 6-2 \end{bmatrix} \underset{\sim}{X_2} = \begin{bmatrix} 2 & \sqrt{8} \\ \sqrt{8} & 4 \end{bmatrix} X_2 = 0$$

again The second Row can be eliminated by $\frac{\sqrt{8}}{2} R_1 - R_2 \rightarrow R_2'$

The top Line shows $2X_1 + \sqrt{8}\,X_2 = 0$

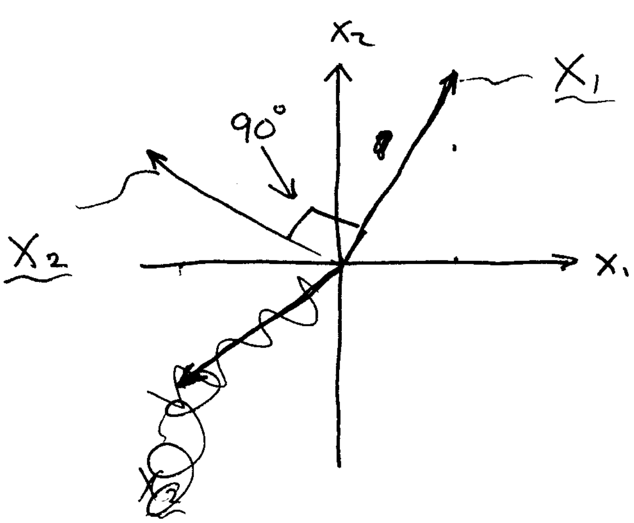
or

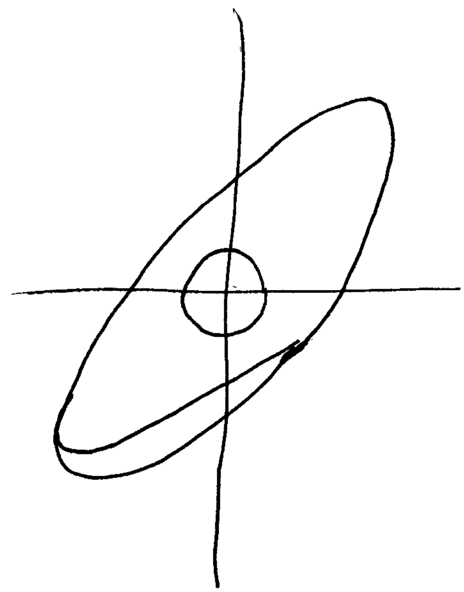
$$X_1 = -\sqrt{2}\,X_2$$

Letting $x_2 = 1$, $x_1 = -\sqrt{2}$

So $\underset{\sim}{X_2} = \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix}$ is The other Principal direction

This works out Like so





Something Like That

$\boxed{7.2.10}$ Find The Growth rate of The Leslie model With The matrix Given; Also Pick an age distribution Vector and Compute how The distribution Vector Changes over Several Generations. Estimate The Growth factor and steady-state age distribution from your Results.

$$\begin{bmatrix} 0 & 9 & 5 \\ 0.4 & 0 & 0 \\ 0 & 0.4 & 0 \end{bmatrix}$$; Basically, Find The Eigenvalue That is Largest

$$[A - \lambda I]x = 0 \Rightarrow \text{Det}[A - \lambda I] = 0 \Rightarrow \begin{vmatrix} -\lambda & 9 & 5 \\ 0.4 & -\lambda & 0 \\ 0 & 0.4 & -\lambda \end{vmatrix} = 0$$

$$-\lambda \begin{vmatrix} -\lambda & 0 \\ 0.4 & -\lambda \end{vmatrix} - 0.4 \begin{vmatrix} 9 & 5 \\ 0.4 & -\lambda \end{vmatrix} = 0$$

$$-\lambda^3 + 3.6\lambda + 0.8 = 0$$

$$\lambda^3 - 3.6\lambda - 0.8 = 0$$

using Mathematica; $\lambda = 2, -1.77, -0.22$.

$\boxed{\text{So } \lambda = 2 \text{ is The Growth Rate}}$

See MatLab output for additional Part of Problems

```
>> A=[0 9 5;.4 0 0;0 .4 0]

A =

                   0   9.00000000000000   5.00000000000000
   0.40000000000000                  0                  0
                   0   0.40000000000000                  0

>> x=[1,0,0]'

x =

     1
     0
     0

>> x = A*x

x =

                   0
   0.40000000000000
                   0

>> x = A*x

x =

   3.60000000000000
                   0
   0.16000000000000

>> x = A*x

x =

   0.80000000000000
   1.44000000000000
                   0

>> x = A*x

x =

  12.96000000000000
   0.32000000000000
   0.57600000000000

>> x = A*x

x =

   5.76000000000000
   5.18400000000000
   0.12800000000000

>> x = A*x
```

```
x =

  47.29600000000001
   2.30400000000000
   2.07360000000000

>> x = A*x

x =

  31.10400000000001
  18.91840000000000
   0.92160000000000

>> x = A*x

x =

  1.0e+002 *

   1.74873600000000
   0.12441600000000
   0.07567360000000

>> x = A*x

x =

  1.0e+002 *

   1.49811200000000
   0.69949440000000
   0.04976640000000

>> x = A*x

x =

  1.0e+002 *

   6.54428160000000
   0.59924480000000
   0.27979776000000

>> x = A*x

x =

  1.0e+002 *

   6.79219200000000
   2.61771264000000
   0.23969792000000

>> x = A*x

x =
```

```
  1.0e+003 *

   2.47579033600000
   0.27168768000000
   0.10470850560000

>> x = A*x

x =

  1.0e+003 *

   2.96873164800000
   0.99031613440000
   0.10867507200000

>> x = A*x

x =

  1.0e+003 *

   9.45622056960000
   1.18749265920000
   0.39612645376000

>> x = A*x

x =

  1.0e+004 *

   1.26680662016000
   0.37824882278400
   0.04749970636800

>> x = A*x

x =

  1.0e+004 *

   3.64173793689600
   0.50672264806400
   0.15129952911360

>> xnext = A*x

xnext =

  1.0e+004 *

   5.31700147814400
   1.45669517475840
   0.20268905922560

>> direction = xnext/xnext(3)
```

```
direction =

  26.23230626486846
   7.18684659312098
   1.00000000000000

>> growth_rate = norm(xnext)/norm(x)

growth_rate =

   1.49911952574375

>>
```

```
direction =

  25.01721753272173
   5.03055417622054
   1.00000000000000


growth_rate =

   1.99009495619540

>> After 50 iterations
```

$\boxed{7.3.1}\boxed{4,5}$ Are the following matrices symmetric, skew symmetric or orthogonal? Find eigenvalues to verify.

$\boxed{7.3.1}$ $A = \begin{bmatrix} 0.96 & -0.28 \\ 0.28 & 0.96 \end{bmatrix}$

$$A^{-1} = \frac{\begin{bmatrix} 0.96 & 0.28 \\ -0.28 & 0.96 \end{bmatrix}}{\begin{vmatrix} 0.96 & -0.28 \\ 0.28 & 0.96 \end{vmatrix}} = \frac{\begin{bmatrix} 0.96 & 0.28 \\ -0.28 & 0.96 \end{bmatrix}}{0.96^2 + 0.28^2} \Rightarrow$$

$$A^{-1} = \frac{\begin{bmatrix} 0.96 & 0.28 \\ -0.28 & 0.96 \end{bmatrix}}{1} = A^T$$ orthogonal.

$$\begin{vmatrix} 0.96-\lambda & -0.28 \\ 0.28 & 0.96-\lambda \end{vmatrix} = 0$$

$$(0.96 - \lambda)^2 + 0.28^2 = 0$$

$$\lambda^2 - 1.92\lambda + 0.9216 + 0.0784 = 0$$

$$\lambda^2 - 1.92\lambda + 1 = 0$$

With roots $\lambda = 0.96 \pm 0.28i$

Theorem 5

$\boxed{7.3.4}$ $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$

Same instructions as for prior problem

Let $C = \cos\theta$, $S = \sin\theta$

Row reducing to check orthogonality

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & c & -s & 0 & 1 & 0 \\ 0 & s & c & 0 & 0 & 1 \end{array}\right] \xrightarrow{S R_2 - C R_3 \to R_3}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & c & -s & 0 & 1 & 0 \\ 0 & 0 & -(s^2+c^2) & 0 & s & -c \end{array}\right] \xrightarrow[\text{Multiply } R_3(-1) \to R_3']{\text{Note } S^2+C^2=1}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & c & -s & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -s & c \end{array}\right] \xrightarrow{s(R_3)+R_2 \to R_2'}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & c & 0 & 0 & 1-s^2 & sc \\ 0 & 0 & 1 & 0 & -s & c \end{array}\right] \xrightarrow[R_2/\cos\theta \to R_2']{\text{Note That}} \quad 1-\sin^2\theta = \cos^2\theta$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & c & s \\ 0 & 0 & 1 & 0 & -s & c \end{array}\right]; \quad \boxed{A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} = A^T}$$

$\boxed{\text{Orthogonal}}$

Finding Eigenvalues: $\left|\begin{bmatrix} 1-\lambda & 0 & 0 \\ 0 & \cos\theta-\lambda & -\sin\theta \\ 0 & \sin\theta & \cos\theta-\lambda \end{bmatrix}\right| = 0$

$$(1-\lambda)\left[(\cos\theta-\lambda)(\cos\theta-\lambda) + \sin^2\theta\right] = 0$$

$$(1-\lambda)\left[\cos^2\theta - 2\lambda\cos\theta + \lambda^2 + \sin^2\theta\right] = 0$$

$$\underbrace{(1-\lambda)}_{\lambda = 1 \text{ is one Root:}}\left[\lambda^2 - 2\lambda\cos\theta + 1\right] = 0$$

from $\lambda^2 - 2\lambda\cos\theta + 1 = 0$; we have

$$\lambda = \frac{+2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2} = \cos\theta \pm \frac{\sqrt{4}}{2}\sqrt{\cos^2\theta - 1}$$

$1-\cos^2\theta = \sin^2\theta \Rightarrow$

$\cos^2\theta - 1 = -\sin^2\theta \Rightarrow$

$$\boxed{\lambda_{2,3} = \cos\theta \pm i\sin\theta}$$

$\boxed{7.3.5}$ Find Symmetry, check with Eigenvalues.

$$A = \begin{bmatrix} 0 & 9 & -12 \\ -9 & 0 & 20 \\ 12 & -20 & 0 \end{bmatrix} \qquad -A = \begin{bmatrix} 0 & -9 & 12 \\ 9 & 0 & -20 \\ -12 & 20 & 0 \end{bmatrix} = A^T$$

<u>Skew Symmetric</u>

Checking Eigenvalues.

$$|A - \lambda I| = \begin{vmatrix} -\lambda & 9 & -12 \\ -9 & -\lambda & 20 \\ 12 & -20 & -\lambda \end{vmatrix} = 0$$

$$-\lambda \begin{vmatrix} -\lambda & 20 \\ -20 & -\lambda \end{vmatrix} + 9 \begin{vmatrix} 9 & -12 \\ -20 & -\lambda \end{vmatrix} + 12 \begin{vmatrix} 9 & -12 \\ -\lambda & 20 \end{vmatrix} = 0$$

$$-\lambda[\lambda^2 + 400] + 9[-9\lambda - 240] + 12[180 - 12\lambda] = 0$$

$$-\lambda^3 - 400\lambda - 81\lambda - \cancel{2160} + \cancel{2160} - 144\lambda = 0$$

$$-\lambda^3 - 625\lambda = 0$$

$$\lambda(\lambda^2 + 625) = 0 \quad \Rightarrow \quad \boxed{\lambda = 0, \ \lambda = \pm 25i}$$

$\boxed{\text{Theorem 1 (b)}}$

$\boxed{7.4.9 | 10}$ Are The Matrices Hermitian, Skew Hermitian, or Unitary? Verify with EigenValues and Eigenvectors.

⑨ : $A = \begin{bmatrix} i & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{bmatrix}$ $\quad \bar{A}^T = \begin{bmatrix} -i & 0 & 0 \\ 0 & 0 & -i \\ 0 & -i & 0 \end{bmatrix} \Rightarrow$ Skew Hermitian

Check for unitary!

$$\left[\begin{matrix} i & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{matrix} \middle| \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}\right] \xrightarrow{\text{Swap Rows 2\&3}}$$

Multiply all Rows by $-i$

$$\left[\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} \middle| \begin{matrix} -i & 0 & 0 \\ 0 & 0 & -i \\ 0 & -i & 0 \end{matrix}\right] \Rightarrow A^{-1} = \begin{bmatrix} -i & 0 & 0 \\ 0 & 0 & -i \\ 0 & -i & 0 \end{bmatrix} = \bar{A}^T = -A$$

$\boxed{\text{Skew Hermitian and unitary.}}$

Finding Eigenvalues

$$|A - \lambda I| = \begin{vmatrix} -(i+\lambda) & 0 & 0 \\ 0 & -\lambda & -i \\ 0 & -i & -\lambda \end{vmatrix} = 0$$

$$-(i+\lambda)\begin{vmatrix} -\lambda & -i \\ -i & -\lambda \end{vmatrix} = -(i+\lambda)[\lambda^2 + 1] = 0$$

$\boxed{\lambda = -i \text{ Repeated, and } \lambda = i\text{; Pure Imaginary, AS stated in Theorem 1}}$

Finding Eigenvectors

For $\lambda = i$

$$\begin{bmatrix} -2i & 0 & 0 \\ 0 & -i & -i \\ 0 & -i & -i \end{bmatrix} \underset{\sim}{X}_1 = 0 \Rightarrow$$

$x_1 = 0$

$x_2 + x_3 = 0 \qquad x_2 = -x_3$

Let $x_3 = 1$, $x_2 = -1$;

$$\boxed{\underset{\sim}{X}_1 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}}$$

See Next Page for The Rest of The Vectors.

For $\lambda = -i$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & i & -i \\ 0 & -i & i \end{bmatrix} \underset{\sim}{X} = 0 \Rightarrow \begin{array}{l} X_1 \to \text{arbitrary}; \\ X_2 - X_3 = 0; \quad X_2 = X_3 \end{array}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & i & -i \\ 0 & -i & i \end{bmatrix} \underset{\sim}{X} \Rightarrow \boxed{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \ \& \ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}}$$

$\boxed{7.4.10}$ identify Symmetry, Find Eigenvalues and Vectors.

$$\begin{bmatrix} 0 & 1+i & 0 \\ 1-i & 0 & 1+i \\ 0 & 1-i & 0 \end{bmatrix}$$

$$\bar{A}^T = \begin{bmatrix} 0 & 1-i & 0 \\ 1+i & 0 & 1-i \\ 0 & 1+i & 0 \end{bmatrix}^T = \begin{bmatrix} 0 & 1+i & 0 \\ 1-i & 0 & 1+i \\ 0 & 1-i & 0 \end{bmatrix} = A$$

Matrix is Hermitian

Determine Eigenvectors:

$$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} -\lambda & 1+i & 0 \\ 1-i & -\lambda & 1+i \\ 0 & 1-i & -\lambda \end{vmatrix} = 0$$

$$-\lambda^3 + \lambda(1-i^2) + \lambda(1-i^2) = 0$$

$$-\lambda^3 + 4\lambda = 0$$

$$\lambda(4-\lambda^2) = 0 \; ; \quad \boxed{\begin{array}{c} \text{Eigenvalues are} \\ \lambda_1 = 0, \ \lambda_2 = 2, \ \lambda_3 = -2 \end{array}}$$

All Real, as Expected, Now Get Vectors.

for $\lambda = 0$:

$$\begin{bmatrix} 0 & 1+i & 0 \\ 1-i & 0 & 1+i \\ 0 & 1-i & 0 \end{bmatrix} \underset{\sim}{V_1} = 0 \qquad \begin{array}{l} (1+i)X_2 = 0 \Rightarrow X_2 = 0 \\ (1-i)X_1 + (1+i)X_3 = 0 \\ X_3 = \dfrac{(i-1)}{1+i} X_1 = -iX_1 \end{array}$$

$$\boxed{V_1 = \begin{bmatrix} 1 \\ 0 \\ -i \end{bmatrix}}$$

Continued on Next Page

For Eigenvector V_2 corresponding to $\lambda = 2$

$$\begin{bmatrix} -2 & 1+i & 0 \\ 1-i & -2 & 1+i \\ 0 & 1-i & -2 \end{bmatrix} \underline{V_2} = 0$$

$$R_1 + (1+i) R_2 \to R_2 \begin{bmatrix} -2 & 1+i & 0 \\ 0 & -(1+i) & (1+i)^2 \\ 0 & 1-i & -2 \end{bmatrix} \underline{V_2} = 0$$

$$(1-i) R_2 + (1+i) R_3 \to \begin{bmatrix} -2 & 1+i & 0 \\ 0 & -(1+i) & 2i \\ 0 & 0 & 0 \end{bmatrix} \underline{V_2} = 0$$

$-2x_1 + 1+i\,x_2 = 0 \quad \Rightarrow \quad x_2 = \frac{2}{1+i} x_1 = (1-i)x_1$

$-(1+i)x_2 + 2i\,x_3 = 0$

$$X_3 = \frac{1+i}{2i} x_2 = \frac{(1+i)(1-i)}{2i} X_1$$

$$X_3 = -iX_1$$

Let $X_1 = i$; Then $x_2 = 1+i$; $x_3 = 1$; $\boxed{\underline{V_2} = \begin{bmatrix} i \\ 1+i \\ 1 \end{bmatrix}}$

For The Third Eigenvalue. $\lambda_3 = -2$

$$\begin{bmatrix} 2 & 1+i & 0 \\ 1-i & 2 & 1+i \\ 0 & 1-i & 2 \end{bmatrix} \underline{V_3} = 0 \Rightarrow \quad \begin{array}{ll} 2X_1 + 1+i\,x_2 = 0 & (1) \\ 1-i\,x_2 + 2x_3 = 0 & (2) \end{array}$$

From (1) $\quad X_2 = \frac{-2}{1+i} X_1 = (i-1) X_1$

From (2) $\quad X_3 = \frac{i-1}{2} x_2 = \frac{(1-i)^2}{2} x_1 = -iX_1$

Choosing $x_1 = i$ Gives $x_2 = -1-i$, $x_3 = 1$

$$\boxed{V_3 = \begin{bmatrix} i \\ -1-i \\ 1 \end{bmatrix}}$$

$\boxed{7.5.13}$ Find a basis of Eigenvectors and diagonalize

$$A = \begin{bmatrix} 16 & 0 & 0 \\ 48 & -8 & 0 \\ 84 & -24 & 4 \end{bmatrix}; \quad |A - \lambda I| = \begin{vmatrix} 16-\lambda & 0 & 0 \\ 48 & -8-\lambda & 0 \\ 84 & -24 & 4-\lambda \end{vmatrix} = 0$$

$$(16-\lambda)(8+\lambda)(4-\lambda) = 0 \qquad \underline{\lambda = 4, -8, 16.}$$

Now Find The Eigenvectors.

for $\lambda_1 = 16$, $\underset{\sim}{V_1}$;

$$\begin{bmatrix} 16-16 & 0 & 0 \\ 48 & -8-16 & 0 \\ 84 & -24 & 4-16 \end{bmatrix} \underset{\sim}{V_1} = \begin{bmatrix} 0 & 0 & 0 \\ 48 & -24 & 0 \\ 84 & -24 & -12 \end{bmatrix}$$

Gives The equations

(1) $2x_1 = x_2$

(2) $7x_1 - 2x_2 = x_3$

Choose $x_1 = 1$; Gives $x_2 = 2$, $x_3 = 3$

$\underline{V_1} = [1 \ 2 \ 3]^T$

For $\lambda_2 = -8$, $\underline{V_2}$:

$$\begin{bmatrix} 16+8 & 0 & 0 \\ 48 & -8+8 & 0 \\ 84 & -24 & 4+8 \end{bmatrix} \underline{V_2} = \begin{bmatrix} 24 & 0 & 0 \\ 48 & 0 & 0 \\ 84 & -24 & 12 \end{bmatrix} V_2 \Rightarrow \begin{matrix} x_1 = 0 \\ x_3 = 2x_2 \end{matrix}$$

Choose $x_2 = 1 \Rightarrow x_3 = 2$; $\underline{V_2} = [0 \ 1 \ 2]$

For $\lambda_3 = 4$; $\underline{V_3}$

$$\begin{bmatrix} 16-4 & 0 & 0 \\ 48 & -8-4 & 0 \\ 84 & -24 & 4-4 \end{bmatrix} V_3 = \begin{bmatrix} 12 & 0 & 0 \\ 48 & -12 & 0 \\ 84 & -24 & 0 \end{bmatrix} V_3 = 0$$

$x_1 = 0$
$4x_1 = x_2 \Rightarrow x_2 = 0$
x_3 = arbitrary

$\underset{\sim}{V_3} = [0 \ 0 \ 1]^T$

Forming The vector X, I Get $X = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$

$D = X^{-1} A X$ Needs X^{-1} (See Next Page)

$\boxed{7.5.13}$ Continued.

$$[X \,\vdots\, I] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[③ - 3①]{② - 2①} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 2 & 1 & -3 & 0 & 1 \end{array}\right] \xrightarrow{③ - 2②}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right] \Rightarrow X^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix}$$

$$D = X^{-1} A X = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 16 & 0 & 0 \\ 48 & -8 & 0 \\ 84 & -24 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$$

$$\boxed{D = \begin{bmatrix} 16 & 0 & 0 \\ 16 & -8 & 0 \\ 4 & -8 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 16 & 0 & 0 \\ 0 & -8 & 0 \\ 0 & 0 & 4 \end{bmatrix}}$$

$\boxed{7.5.14}$ Find a basis of Eigenvectors and diagonalize.

$$A=\begin{bmatrix}-2.5 & -3 & 3\\ -4.5 & -4 & 6\\ -6 & -6 & 8\end{bmatrix};\quad |A-\lambda I|=0 \Rightarrow$$

$$\begin{vmatrix}-2.5-\lambda & -3 & 3\\ -4.5 & -4-\lambda & 6\\ -6 & -6 & 8-\lambda\end{vmatrix}=0$$

$$(-2.5-\lambda)(-4-\lambda)(8-\lambda)+(-3)(6)(-6)+(3)(-4.5)(-6)\ \dots$$
$$\dots-(-6)(-4-\lambda)(3)-(-6)(6)(-2.5-\lambda)-(8-\lambda)(-4.5)(-3)=0$$

Collecting together and multiplying out

$$(-2.5-\lambda)(-4-\lambda)(8-\lambda)+108+4.5(18)+18(-4-\lambda)+\ \dots$$
$$\dots-35(-2.5-\lambda)-3(4.5)(8-\lambda)=0$$

$$(-2.5-\lambda)(\lambda^2-4\lambda-32+36)+18(-4-\lambda)-13.5(8-\lambda)+189=0$$

$$(-2.5-\lambda)(\lambda^2-4\lambda+4)-72-18\lambda-108+13.5\lambda+189=0$$

$$(-2.5-\lambda)(\lambda-2)^2-4.5\lambda+9=0$$

$$(-2.5-\lambda)(\lambda-2)(\lambda-2)-4.5(\lambda-2)=0$$

$$(\lambda-2)[(\lambda-2)(-2.5-\lambda)-4.5)=0$$

$$(\lambda-2)(\lambda^2+0.5\lambda-0.3)=0$$

$$(\lambda-2)(\lambda+1)(\lambda-0.5)=0$$

So $\lambda = 2, -1, 0.5$ are Eigenvalues.

Now find Eigenvectors.

For $\lambda=0.5$:

$$\begin{bmatrix}-3 & -3 & 3\\ -4.5 & -4.5 & 6\\ -6 & -6 & -7.5\end{bmatrix}\xrightarrow[R_3-2R_1\to R_3]{R_2-1.5R_1\to R_2}\begin{bmatrix}-3 & -3 & 3\\ -0 & -0 & 1.5\\ 0 & 0 & 1.5\end{bmatrix}$$

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$\boxed{7.5.14}$ Continued.

$X_3 = 0$; $X_2 = -X_1$; choose $X_1 = 1$, $X_2 = -1$ $\quad \underset{\sim}{V_1} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

Next Eigenvector: $\lambda = 2$

$$\begin{bmatrix} -4.5 & -3 & -3 \\ -4.5 & -6 & 6 \\ -6 & -6 & 6 \end{bmatrix} V_2 = 0 \qquad \begin{matrix} R_2 - R_1 \to R_2' \\ R_3 - 2R_1 \to R_3' \end{matrix}$$

$$\begin{bmatrix} -4.5 & -3 & -3 \\ 0 & -3 & 3 \\ 3 & 0 & 0 \end{bmatrix} \quad X_1 = 0 \quad \begin{matrix} X_2 - X_3 = 0 \\ X_2 = X_3 \end{matrix}$$

Choose $\cancel{1} = X_2$, $X_3 = 1$

$$V_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

For $\lambda_3 = -1$

$$\begin{bmatrix} -1.5 & -3 & 3 \\ -4.5 & -3 & 6 \\ -6 & -6 & 9 \end{bmatrix} \underset{\sim}{V_3} = 0 \longrightarrow \begin{matrix} R_2 - 3R_1 \to R_2 \\ R_3 - 4R_1 \to R_3 \end{matrix}$$

$$\begin{bmatrix} -1.5 & -3 & 3 \\ 0 & 6 & -3 \\ 0 & 6 & -3 \end{bmatrix} \underset{\sim}{U_3} = 0; \Rightarrow -1.5X_1 - 3X_2 + 3X_3 = 0 \Rightarrow X_1 = -2X_2 + 2X_3$$

$2X_2 = X_3$; $\quad$ Let $X_3 = 2$, $\Rightarrow X_2 = 1$

Then $X_1 = -3(1) + 3(2) = 3$

$$\underset{\sim}{V_3} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$$

Now we have $X = [V_1\ V_2 V_3] = \begin{bmatrix} 1 & 0 & 3 \\ -1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$

Now Get X^{-1}

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 + R_2 \to R_2'}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & 1 & 3 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 & 1 & 1 \end{array}\right] \xrightarrow{R_3 - R_2 \to R_3'}$$

$\boxed{7.5.14}$ Continued.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 3 & 1 & 1 & 0 \\ 0 & 0 & -1 & -1 & 0 & 1 \end{array}\right] \xrightarrow[R_1 + 2R_3]{(-1)R_3 \to R_3' \quad R_2 + 3R_3}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & -2 & 2 \\ 0 & 1 & 0 & -2 & -2 & 3 \\ 0 & 0 & 1 & 1 & 1 & -1 \end{array}\right] \therefore X^{-1} = \begin{bmatrix} -1 & -2 & 2 \\ -2 & -2 & 3 \\ 1 & 1 & -1 \end{bmatrix}$$

$$D = X^{-1} A X = \begin{bmatrix} -1 & -2 & 2 \\ -2 & -2 & 3 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} -2.5 & -3 & 3 \\ -4.5 & -4 & 6 \\ -6 & -6 & 8 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ -1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

$$D = \begin{bmatrix} -0.5 & -1 & 1 \\ -4 & -4 & 6 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ -1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

$$\boxed{D = \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}}$$