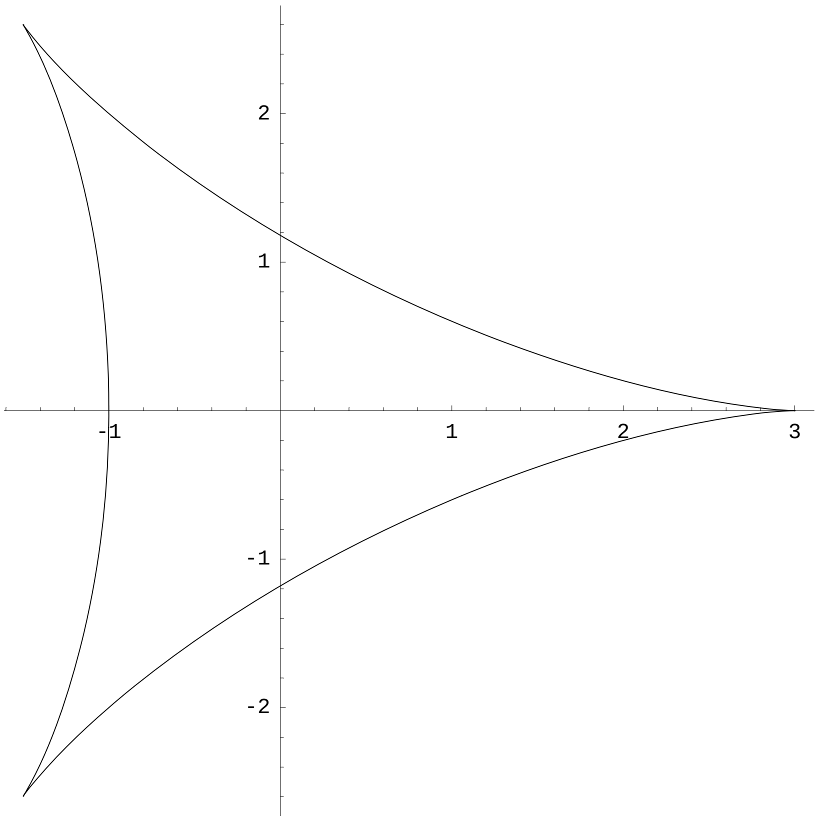
Problem 8.6.8 part a.

```
In[1]:= r[t_] := {2 Cos[t] + Cos[2 t], 2 Sin[t] - Sin[2 t]}

In[17]:= ParametricPlot[r[t], {t, 0, 2*Pi}, AspectRatio → 1, PlotPoints → 100]
```



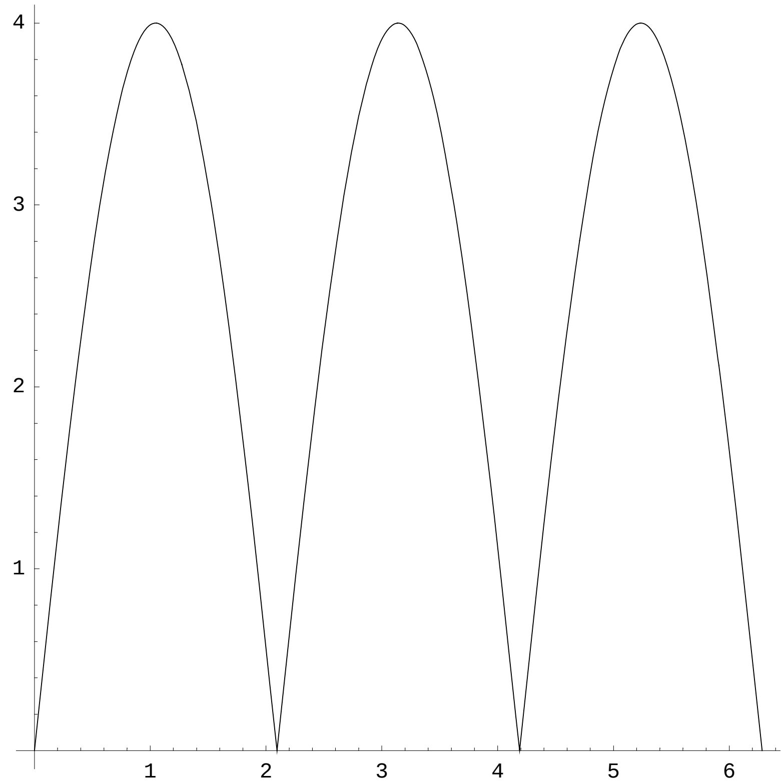

```
Out[17]= - Graphics -

In[3]:= v[t_] := D[r[t], t]
```

In[4]:= speed[t_] := $\sqrt{v[t] . v[t]}$

In the next line, I use the Evaluate[speed[t]] in the Plot function because otherwise, plot will not evaluate the function, it will just give errors that "the function is not a number at...". Evaluate takes the functions in, and return numerical values.

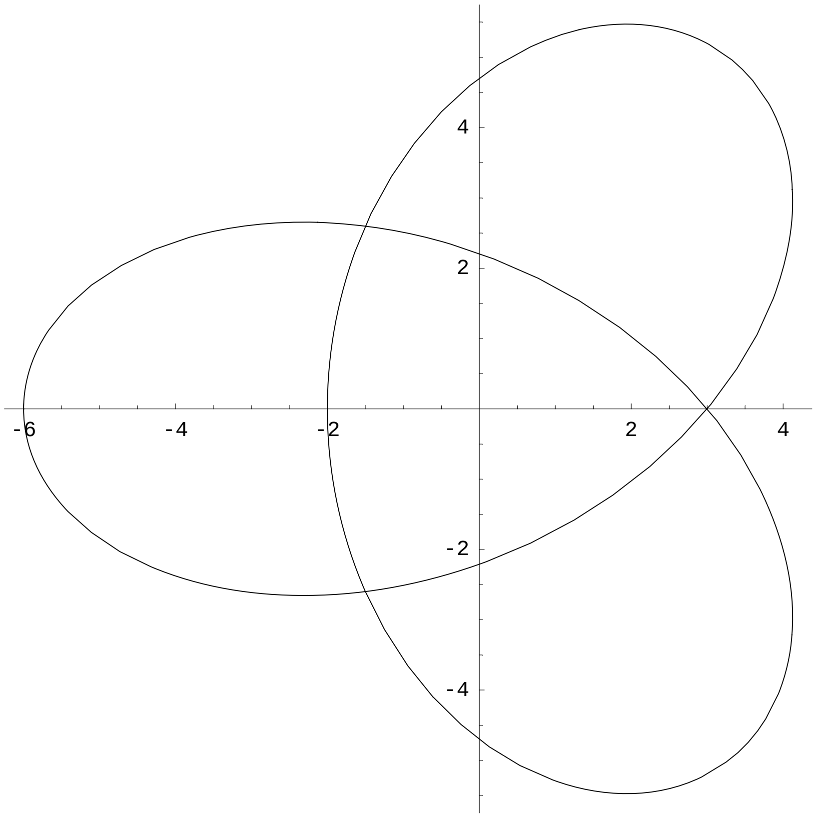
```
In[18]:= Plot[Evaluate[speed[t]], {t, 0, 2*Pi}, AspectRatio → 1, PlotPoints → 100]
```




```
Out[18]= - Graphics -

In[6]:= a[t_] := D[v[t], t]
```

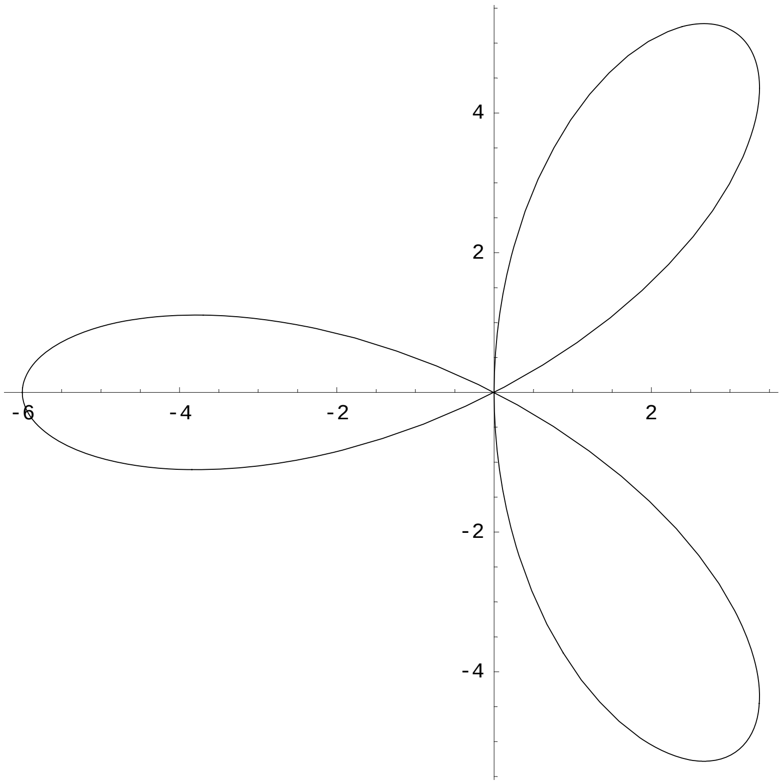
```
In[19]:= ParametricPlot[Evaluate[a[t]], {t, 0, 2*Pi}, AspectRatio → 1, PlotPoints → 100]
```




```
Out[19]= - Graphics -
```

In[8]:= **tangentacceleration[t_] :=** $\frac{a[t].v[t]}{v[t].v[t]}$ **v[t]**

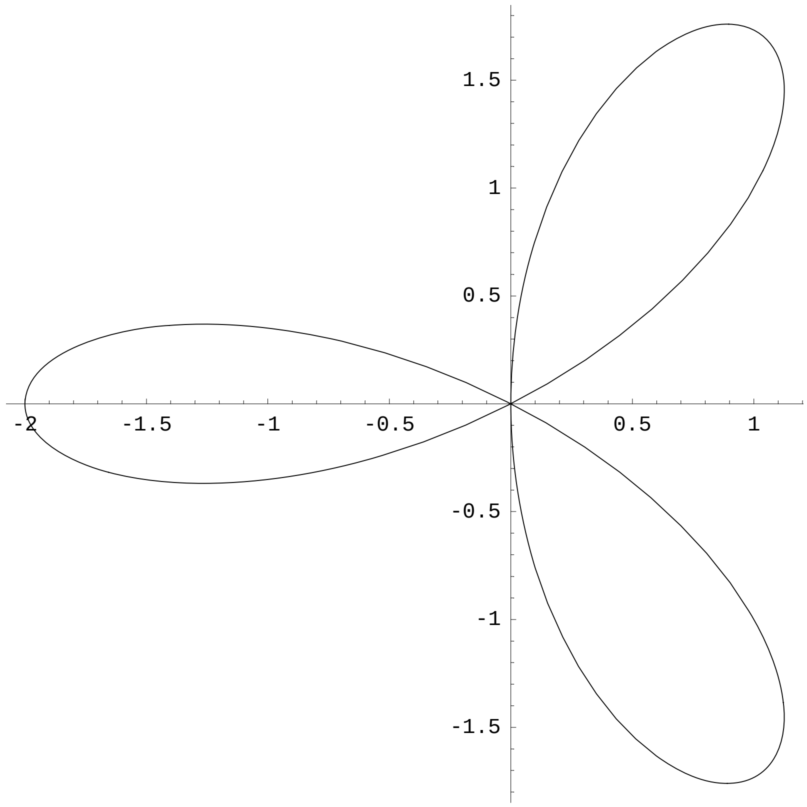
```
In[20]:= ParametricPlot[Evaluate[tangentacceleration[t]],
          {t, 0, 2*Pi}, AspectRatio → 1, PlotPoints → 100]
```




```
Out[20]= - Graphics -
```

```
In[10]:= normalacceleration[t_] := a[t] - tangentacceleration[t]
```

```
In[21]:= ParametricPlot[Evaluate[normalacceleration[t]],
          {t, 0, 2*Pi}, AspectRatio→1, PlotPoints → 100]
```




```
Out[21]= - Graphics -
```

For those who actually solved for the velocity, speed and acceleration, I'll go ahead and have mathematica evaluate them, so that you can check your results. To do this, I'll use Simplify[] which often makes the expressions more readable.

```
In[12]:= Simplify[v[t]]
```

Out[12]= $\{-2\,(\mathrm{Sin}[t]+\mathrm{Sin}[2\,t]),\ 2\,(\mathrm{Cos}[t]-\mathrm{Cos}[2\,t])\}$

```
In[13]:= Simplify[speed[t]]
```

Out[13]= $4\sqrt{\mathrm{Sin}\left[\frac{3\,t}{2}\right]^2}$

```
In[14]:= Simplify[a[t]]
```

Out[14]= $\{-2\,(\mathrm{Cos}[t]+2\,\mathrm{Cos}[2\,t]),\ -2\,\mathrm{Sin}[t]+4\,\mathrm{Sin}[2\,t]\}$

```
In[15]:= Simplify[tangentacceleration[t]]
```

Out[15]= $\left\{-6\,\mathrm{Cos}\left[\frac{t}{2}\right]^2(-1+2\,\mathrm{Cos}[t]),\ -3\,\mathrm{Sin}[t]+3\,\mathrm{Sin}[2\,t]\right\}$

```
In[16]:= Simplify[normalacceleration[t]]
```

Out[16]= $\left\{2\,(1+2\,\mathrm{Cos}[t])\,\mathrm{Sin}\left[\frac{t}{2}\right]^2,\ (1+2\,\mathrm{Cos}[t])\,\mathrm{Sin}[t]\right\}$

8.9.7 Heat Flows in The direction of maximum decrease in Temperature. Find This direction in general and at The Given Point

$$T = \frac{z}{x^2+y^2} \; ; \; P = (0, 1, 2)$$

$$q \propto -\nabla T = -\left(\frac{\partial T}{\partial x}\hat{i}, \frac{\partial T}{\partial y}\hat{j}, \frac{\partial T}{\partial z}k \right)$$

$$= -\left[\frac{\partial}{\partial x}\left(\frac{z}{x^2+y^2}\right)\hat{i}, \frac{\partial}{\partial y}\left(\frac{z}{x^2+y^2}\right)\hat{j}, \frac{\partial}{\partial z}\left(\frac{z}{x^2+y^2}\right)\hat{k} \right]$$

$$= -\left[\frac{-2zx}{(x^2+y^2)^2}\hat{i}, \frac{-2yz}{(x^2+y^2)^2}\hat{j}, \frac{1}{x^2+y^2}\hat{k} \right]$$

← (minus)

$$\boxed{-\nabla T = \left[\frac{+2xz}{(x^2+y^2)^2}\hat{i}, \frac{+2yz}{(x^2+y^2)^2}\hat{j}, \frac{-1}{x^2+y^2}\hat{k} \right]}$$

at P: (0, 1, 2)

$$-\nabla T = \left[\frac{+2(0)(2)}{(0^2+1^2)^2}, \frac{+2(1)(2)}{(0^2+1^2)^2}, \frac{-1}{(0^2+1^2)} \right]$$

$$\boxed{-\nabla T = [0, 4, -1]}$$

8.9.14 If on a mountain, the elevation above sea level is $z(x,y) = 1500 - 3x^2 - 5y^2$. What is the direction of steepest ~~desc~~ ascent at $P: [-0.2, 0.1]$?

i.e. what is the direction of the gradient at P:

$$\nabla z = \left[\frac{\partial z}{\partial x} \hat{i}, \ \frac{\partial z}{\partial y} \hat{j}\right]$$

$$\nabla z = \left[-6x \hat{i}, -10y \hat{j}\right]$$

at P: $\nabla z(P) = \left[-6(-0.2)\hat{i}, -10(0.1)\hat{j}\right]$

$$\boxed{\nabla z(P) = [1.2, -1]}$$

8.9.18 Using gradients, find unit normal vectors for the surface:

$$f = ax + by + cz + d = 0 \quad \text{at any point } P:$$

By Thm 2: $n = \frac{\nabla f}{|\nabla f|} \Rightarrow$

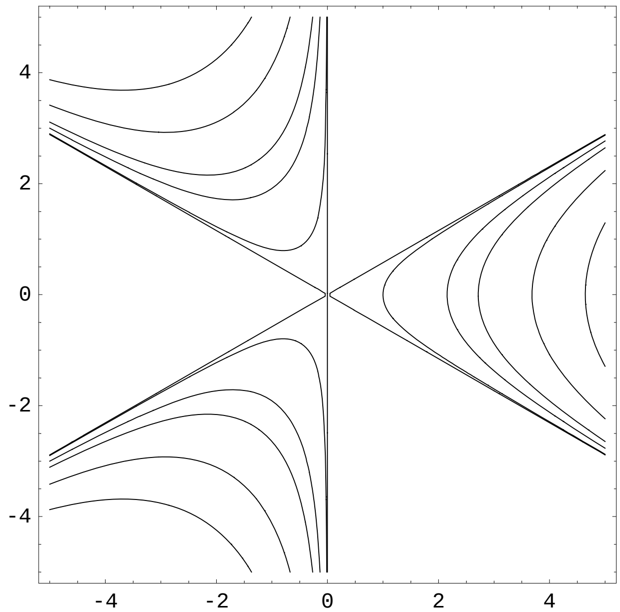
$$\nabla f = \left[\frac{\partial f}{\partial x}\hat{i}, \frac{\partial f}{\partial y}\hat{j}, \frac{\partial f}{\partial z}\hat{k}\right] = \left[a\hat{i} + b\hat{j} + c\hat{k}\right]$$

$$\boxed{n = \frac{\nabla f}{\sqrt{\nabla f \cdot \nabla f}} = \frac{[a\hat{i} + b\hat{j} + c\hat{k}]}{\sqrt{a^2 + b^2 + c^2}} = \frac{[a, b, c]}{\sqrt{a^2+b^2+c^2}}}$$

Problem 8.9.27 Part a:

```
t[x_, y_] := x^3 - 3 x y^2
```

```
isotherms = ContourPlot[t[x, y], {x, -5, 5}, {y, -5, 5}, ContourShading → False,
  Contours → {0, 1, 10, 20, 50, 100}, PlotPoints → 200, AspectRatio → 1]
```



- ContourGraphics -

The options that I have chosen here are to do the following: ContourShading → False turns off shading the contours, which is annoying and does not give you Isotherms. The option PlotPoints → 200 just makes the plot smoother so that you can tell what is happening. The option Contours → {1,10,20,50,100} tells it which temperature isotherms I want drawn, in this case for T=1, 10, 20 50 and 100 degrees. The innermost curve is for 1, the outermost 100. Note that the arrows requested are not drawn, but would point from the furthest out isotherm toward the inner isotherms, with adiabats (lines of no heat transfer occuring at the T=0 lines, as they are axes of symmetry.

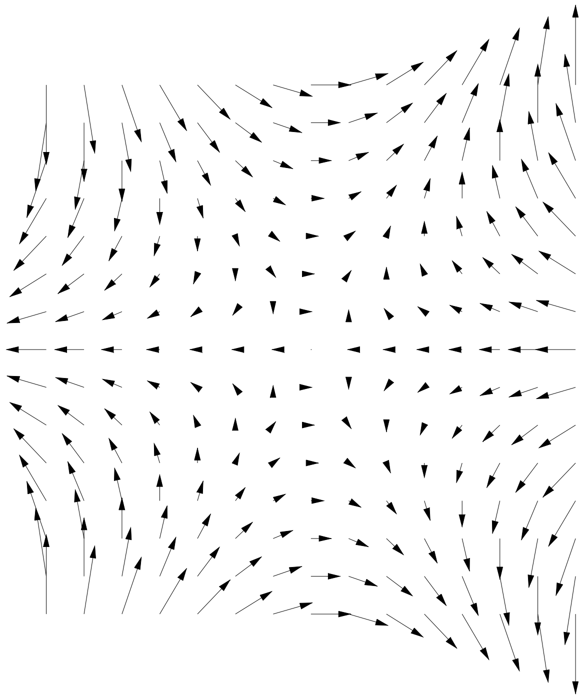
Now lets go after a plot that will hopefully give us some arrows for this. First we need to load another package.

```
<< Graphics`PlotField`
```

Now I get a function that defines the direction of heat flow. I'll call that heatdirection. I then evaluate that function in PlotVectorField, a function from the Graphics'PlotField' package, to generate a plot of the direction of heat flow. This we'll call heatflow.

```
heatdirection[x_, y_] := -{D[t[x, y], x], D[t[x, y], y]}
```

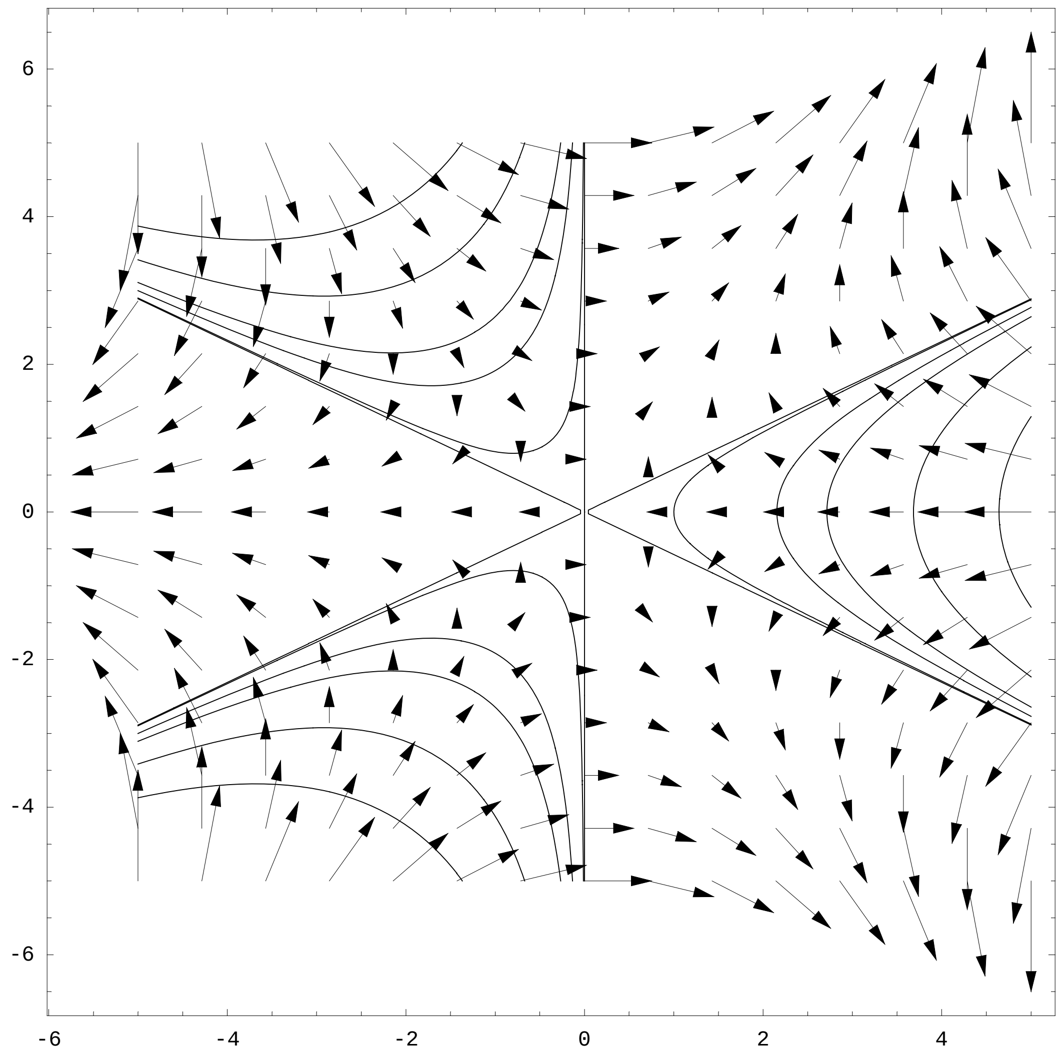
```
heatflow = PlotVectorField[
  Evaluate[heatdirection[x, y]], {x, -5, 5}, {y, -5, 5}, ScaleFactor → 1.5]
```



```
- Graphics -
```

Now I just combine the graphics using the Show function, and the names of the two plots I generated.

Show[{**isotherms, heatflow**}]



- Graphics -

8.9.31 Find the directional Derivative of 'f' at 'P' in the direction $\underline{a}$

$$f = \frac{1}{\sqrt{x^2+y^2+z^2}} \qquad P: = [3, 0, 4] \qquad a = [\hat{i}, \hat{j}, \hat{k}]$$

$$D_a f = \frac{a}{|a|} \cdot \nabla f \Big|_P$$ by eqn 6' Page 448

first, lets get ∇f:

$$\nabla f = \left[\frac{\partial}{\partial x}\left[\frac{1}{\sqrt{x^2+y^2+z^2}}\right]\hat{i}, \frac{\partial}{\partial y}\left[\frac{1}{\sqrt{x^2+y^2+z^2}}\right]\hat{j}, \frac{\partial}{\partial z}\left[\frac{1}{\sqrt{x^2+y^2+z^2}}\right]\hat{k}\right]$$

$$= \left[\frac{-x}{(x^2+y^2+z^2)^{3/2}}\hat{i}, \frac{-y}{(x^2+y^2+z^2)^{3/2}}\hat{j}, \frac{-z}{(x^2+y^2+z^2)^{3/2}}\hat{k}\right]$$

So at the point P:

$$\nabla f\Big|_P = \left[\frac{-3}{(3^2+0^2+4^2)^{3/2}} i + 0\hat{j} + \frac{-4}{(3^2+4^2)^{3/2}}\hat{k}\right]$$

$$\nabla f\Big|_P = \left[\frac{-3}{25^{3/2}}\hat{i}, 0\hat{j}, \frac{-4}{25^{3/2}}\hat{k}\right]$$

$$D_{\underline{a}} f = \frac{a}{|\underline{a}|} \nabla f\Big|_P = \frac{[i, j, k]}{\sqrt{1+1+1}} \begin{bmatrix} -3/25^{3/2} \\ 0 \\ -4/25^{3/2} \end{bmatrix}$$

NOTE $(25^{3/2} = 125)$

$$D_a f = \frac{(1)\left(\frac{-3}{125}\right) + 0 + (1)\left(\frac{-4}{125}\right)}{\sqrt{3}}$$

$$\boxed{D_{\underline{a}} f = \frac{-7}{125\sqrt{3}}}$$

8.10.11 Consider flow with $\underset{\sim}{V} = y i$. Show that this flow is incompressible. Show that the particles that at $t=0$ are in a cube bounded by the planes $x=0, x=1, y=0, y=1, z=0, z=1$ occupy volume $=1$ at time $t=1$

$$\underset{\sim}{V} = V_1 i + V_2 j + V_3 k = y i \qquad \nabla \cdot \underset{\sim}{V} = 0 \Rightarrow \text{incompressible condition (eq 7)}$$

$$\nabla \cdot V = \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z} = \frac{\partial y}{\partial x} + 0 + 0 = 0 \ \checkmark \text{ incompressible}$$

$$\underset{\sim}{V} = \frac{dx}{dt} i + \frac{dy}{dt} j + \frac{dz}{dt} k = y i :$$

Therefore we have the equations

$$(1)\ \frac{dx}{dt} = y \qquad (2)\ \frac{dy}{dt} = 0 \qquad (3)\ \frac{dz}{dt} = 0$$

integrating (3) gives $\underline{z = C_3}$

integrating (2) gives $\underline{y = C_2}$

and integrating (1) noting that $y = C_2$ gives

$$\underline{x = C_2 t + C_1}$$

for the faces on z: $C_3 = 0$ for the faces on $z=0$
$C_3 = 1$ for the face on $z = 1$

for the faces on y:
$C_2 = 0$ for $y=0$ face
$C_2 = 1$ for $y=1$ face.

C_2 and C_3 are clearly independent of time.

Lastly, for the faces on x; we have the following
for the $x=0$ face; we have $x(0) = 0$, $x(1) = C_2$

for the $x=1$ face, we have $x(0) = C_1 = 1$
and $x(1) = C_2 + 1$

So the distance between the two planes is still 1, though shifted.

$t=$

8.10.12 Consider the flow having the velocity $\underline{V} = x\hat{i}$. Show that individual particles have position vectors $\underline{r}(\tau) = C_1 e^{\tau}\hat{i} + C_2\hat{j} + C_3\hat{k}$, where C_1, C_2, C_3 are constants; the flow is compressible, and e is the volume occupied at $\tau = 1$ by the cube defined in 8.10.11.

$\underline{V} = V_1 i + V_2 j + V_3 k$; for incompressible flow $\text{Div}\,\underline{V} = 0$

$$\nabla \cdot \underline{V} = \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z} = \frac{\partial}{\partial x}(x) = 1 \neq 0 \therefore \text{Compressible.}$$

Since $\underline{V} = \frac{\partial x}{\partial \tau} \hat{i} + \frac{\partial y}{\partial \tau}\hat{j} + \frac{\partial z}{\partial \tau}\hat{k}$; we have the equations

$$\frac{\partial x}{\partial \tau} = x \Rightarrow \text{integrating} \Rightarrow \frac{dx}{x} = d\tau \Rightarrow$$

$$\ln x = \tau + \ln C_1 \Rightarrow \underline{C_1 e^{\tau} = x}$$

$$\frac{dy}{d\tau} = 0 \Rightarrow \underline{y = C_2} \quad ; \quad \frac{dz}{\partial \tau} = 0 \Rightarrow \underline{z = C_3}$$

So $\underline{r} = x\hat{i} + y\hat{j} + z\hat{k} = C_1 e^{\tau}\hat{i} + C_2\hat{j} + C_3\hat{k}$

Therefore, $r(0) = C_1 i + C_2 j + C_3 k$

and $r(1) = C_1 e\, i + C_2 j + C_3 k$.

This shows that the boundaries at $y=0, y=1, z=0, z=1$ will not change, nor will $x=0$: however, for $x=1$, we get $x = e$ at $\tau = 1$, so the volume has stretched to have

Volume $= (1)(1)(e) = \boxed{e}$

8.11.12 Given $\underset{\sim}{V}$ is the flow irrotational?, incompressible? Find the particle paths:

$$V = [-\tfrac{1}{4}y,\ 4x,\ 0]$$

for irrotational flow $\nabla \times \underset{\sim}{V} = 0$

$$\nabla \times \underset{\sim}{V} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \frac{-y}{4} & 4x & 0 \end{vmatrix} = -\frac{\partial 4x}{\partial z}\, i - \left(-\frac{\partial}{\partial z}\left(\frac{-y}{4}\right)\right)\hat{j} + \left[\frac{\partial(4x)}{\partial x} + \frac{\partial -y/4}{\partial y}\right] k$$

$$\nabla \times \underset{\sim}{V} = \left(4 + \frac{1}{4}\right) k = \boxed{\frac{17}{4} k \Rightarrow \text{Rotational}}$$

is it incompressible?

$$\nabla \cdot \underset{\sim}{V} = \frac{\partial}{\partial x}\left(\frac{-y}{4}\right) + \frac{\partial}{\partial y}\left(4x\right) + \frac{\partial}{\partial z}(0) = 0 \Rightarrow \text{incompressible}$$

Since $\underset{\sim}{V} = \frac{\partial x}{d\tau}\hat{i} + \frac{\partial y}{d\tau}\hat{j} + \frac{\partial z}{\partial \tau}\hat{k} = \frac{-y}{4} i + 4x j$

So we have the equations

a) $\frac{dx}{d\tau} = \frac{-y}{4}$ b) $\frac{dy}{d\tau} = 4x \Rightarrow x = \frac{1}{4}\frac{dy}{d\tau} \Rightarrow b'$

multiplying a and b' together

$$x\frac{dx}{d\tau} = \left(-\frac{y}{4}\right)\left(\frac{1}{4}\frac{dy}{d\tau}\right) \Rightarrow x\,dx = -\frac{y\,dy}{16} \Rightarrow \text{integrate}$$

$$\frac{x^2}{2} = -\frac{y^2}{32} + \frac{c}{2} \Rightarrow \boxed{x^2 + \frac{y^2}{4^2} = c}$$

elipses!

8.11.13 Given $\underset{\sim}{V}$; is the flow rotational? incompressible? find streamlines:

$$\underset{\sim}{V} = x\hat{i} + y\hat{j} - z\hat{k};$$

Checking Rotation:

$$\nabla \times \underset{\sim}{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & -z \end{vmatrix}$$

$$\nabla \times \underset{\sim}{V} = \left(\frac{\partial(-z)}{\partial y} - \frac{\partial y}{\partial z}\right)\hat{i} - \left(\frac{\partial(-z)}{\partial x} - \frac{\partial x}{\partial z}\right)\hat{j} + \left(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y}\right)\hat{k}$$

$$\nabla \times \underset{\sim}{V} = 0 \Rightarrow \text{irrotational:}$$

Checking incompressibility:

$$\nabla \cdot \underset{\sim}{V} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} - \frac{\partial z}{\partial z} = 1 + 1 - 1 = 1 \Rightarrow \text{Compressible.}$$

Finding ~~Streamlines~~ Particle Paths:

$$\underset{\sim}{V} = \frac{dx}{d\tau}\hat{i} + \frac{dy}{d\tau}\hat{j} + \frac{dz}{d\tau}\hat{k} = x\hat{i} + y\hat{j} + (-z)\hat{k}$$

$$\frac{dx}{d\tau} = x \Rightarrow x = C_1 e^{\tau}$$

$$\frac{dy}{d\tau} = y \Rightarrow y = C_2 e^{\tau}$$

$$\frac{dz}{d\tau} = -z \Rightarrow z = C_3 e^{-\tau}$$

So $r(\tau) = x\hat{i} + y\hat{j} + z\hat{k}$

$$\boxed{r(\tau) = C_1 e^{\tau}\hat{i} + C_2 e^{\tau}\hat{j} + C_3 e^{-\tau}\hat{k}}$$