ME564 | HW #6 | Solutions

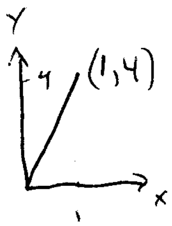
$\boxed{9.1.1}$ Calculate $\int_C F(r)\cdot dr$ for the case:

$F(r) = y^2\hat{i} - x^2\hat{j}$

C is a straight line from $[0,0]$ to $[1,4]$

Parameterizing path as $r(\tau) = \tau\hat{i} + 4\tau\hat{j}$

Gives $x=\tau$, $y=4\tau$; Now the vector field is



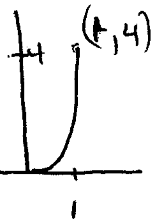
$$F(r(\tau)) = y^2 i - x^2 j = 16\tau^2\hat{i} - \tau^2\hat{j}$$

$$\int_C [16\tau^2\hat{i} - \tau^2\hat{j}]\cdot d[\tau i + 4\tau\hat{j}] = \int_C [16\tau^2 i - \tau^2\hat{j}]\cdot[i+4j]\,d\tau$$

$r(0) = [0,0]$ $r(1) = [1,4]$, So limits of integral are $\tau: 0\to 1$

$$\int_0^1 [16\tau^2 i - \tau^2 j]\cdot[\hat{i}+4\hat{j}]\,d\tau = \int_0^1 [16\tau^2 - 4\tau^2]\,d\tau$$

$$= \int_0^1 12\tau^2\,d\tau = 4\tau^3\Big|_0^1 = \boxed{4}$$



$\boxed{9.1.2}$ Repeat for $\underline{F}$ same as in 9.1.1, but new contour

$$y = 4x^2 \text{ from } (0,0) \text{ to } (1,4)$$

This path is described by

$r(\tau) = \tau\hat{i} + 4\tau^2\hat{j}$; Giving $x=\tau$, $y=4\tau^2$

$F(r(\tau)) = 16\tau^4\hat{i} - \tau^2\hat{j}$

$$\int_0^1 [16\tau^4 i - \tau^2\hat{j}]\cdot d[\tau\hat{i} + 4\tau^2\hat{j}] = \int_0^1 [16\tau^4 i - \tau^2\hat{j}]\cdot[i + 8\tau j]\,d\tau$$

$$= \int_0^1 16\tau^4 - 8\tau^3\,d\tau = \left[\frac{16}{5}\tau^5 - 2\tau^4\right]\Big|_0^1 = \boxed{\frac{6}{5}}$$

$\boxed{9.2.11}$ Check for Path independance; if independant, integrate from $[0,0,0]$ to $[a,b,c]$

$$2xy^2\,dx + 2x^2y\,dy + dz = F_1 dx + F_2 dy + F_3 dz$$

where $\underline{F} = [F_1, F_2, F_3]$

Check with $\nabla \times F = 0$

$$\nabla \times F = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2xy^2 & 2x^2y & 1 \end{vmatrix} = \begin{bmatrix} \left(\frac{\partial(1)}{\partial y} - \frac{\partial(2x^2y)}{\partial z}\right) i \\ -\left(\frac{\partial(1)}{\partial x} - \frac{\partial[2xy^2]}{\partial z}\right) \\ \left(\frac{\partial(2x^2y)}{\partial x} - \frac{\partial(2xy^2)}{\partial y}\right) k \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \checkmark$$

is Path independant.

By inspection $\boxed{f = x^2y^2 + z}$ Gives

$$\nabla f = \underline{F}$$

$\boxed{9.2.12}$ Same instructions as 9.2.11, except

$$y\,dx - zx\,dy + z\,dz = F_1 dx + F_2 dy + F_3 dz = \underline{F}$$

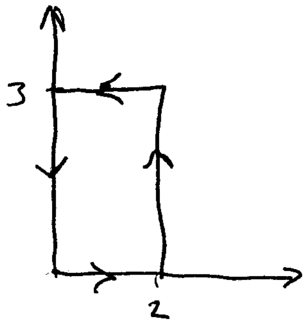
Check $\nabla \times \underline{F}$

$$\nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & -zx & z \end{vmatrix} = \begin{bmatrix} \left(\frac{\partial z}{\partial y} - \frac{\partial(-zx)}{\partial z}\right) i \\ -\left(\frac{\partial z}{\partial x} - \frac{\partial y}{\partial z}\right) j \\ \left(\frac{\partial(-zx)}{\partial x} - \frac{\partial y}{\partial y}\right) k \end{bmatrix} = \begin{bmatrix} x \\ 0 \\ -z-y \end{bmatrix} \neq \underset{\sim}{0}$$

$\therefore$ Path dependant

9.4.1 & 5 Using Green's Theorem, evaluate the line integral $\oint_C F(r)\cdot dr$ counter clockwise around the boundary C of the Region R.

$\underset{\sim}{F} = [x^2e^y, y^2e^x]$, C the Rectangle shown.



from $\underset{\sim}{F}$ we see that

$$F_1 = x^2e^y \;;\; \text{so} \quad \frac{\partial F_1}{\partial y} = x^2e^y$$

$$F_2 = y^2e^x \qquad \frac{\partial F_2}{\partial x} = y^2e^x$$

By Green's Theorem

$$\oint_C (F_1\,dx + F_2\,dy) = \iint_R \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) dx\,dy$$

$$= \iint (y^2e^x - x^2e^y)\,dx\,dy$$

$$= \int_{y=0}^{3}\int_{x=0}^{2} (y^2e^x - x^2e^y)\,dx\,dy$$

$$= \int_0^3 \left[y^2e^x - \frac{x^3}{3}e^y\right]\Bigg|_0^2 dy$$

$$= \int_0^3 \left[y^2(e^2-1) - \frac{8}{3}e^y\right] dy$$

$$= \left[\frac{1}{3}y^3(e^2-1) - \frac{8}{3}e^y\right]_0^3$$

$$\boxed{\oint_C (F_1dx + F_2dy) = 9(e^2-1) - \frac{8}{3}(e^3-1)}$$

9.4.5 Same instructions as 9.4.1

$$F = \nabla(\sin x \cos y) = \cos x \cos y \,\hat{i} - \sin x \sin y \,\hat{j}$$

on The Contour C: is The ellipse $25x^2 + 9y^2 = 225 = 15^2$

$$F_1 = \cos x \cos y \Rightarrow \frac{\partial f_1}{\partial y} = -\cos x \sin y$$

$$F_2 = -\sin x \sin y \Rightarrow \frac{\partial F_2}{\partial y} = -\cos x \sin y$$

$$\oint_C [F_1 dx + F_2 dy] = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) dx\,dy$$

$$= \iint_R (-\cos x \sin y + \cos x \sin y)\, dx\,dy$$

$$\oint_C [F_1 dx + F_2 dy] = \iint_R 0 \, dx\,dy = \boxed{0}$$

9.4.19 Laplace's equation.

Show That For a Solution $W(x,y)$ of Laplace's equation $\nabla^2 W = 0$ in a Region R with boundary Curve C and outer unit Normal Vector $\vec{n}$ That

$$\iint_R \left[\left(\frac{\partial W}{\partial x}\right)^2 + \left(\frac{\partial W}{\partial y}\right)^2\right] dx\,dy = \oint_C w \frac{\partial w}{\partial n} ds$$

Work on The Right hand Side.

$$RHS = \oint_C w \frac{\partial w}{\partial n} ds$$

By eqn (8) on Page 489; with $\nabla w = \frac{\partial w}{\partial x} i + \frac{\partial w}{\partial y} \hat{j}$

and $\hat{n} = \frac{dy}{ds} i - \frac{\partial x}{\partial s} \hat{j}$

We have

$$\frac{\partial w}{\partial n} = \nabla w \cdot \hat{n} = -\frac{\partial w}{\partial y}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial x}\frac{dy}{ds}$$

$\boxed{9.4.19}$ Continued.

With This Result we have

$$\omega \frac{\partial \omega}{\partial n} = -\omega \frac{\partial \omega}{\partial y}\frac{\partial x}{\partial s} + \omega \frac{\partial \omega}{\partial x}\frac{\partial y}{\partial s}$$

$$\omega \frac{\partial \omega}{\partial n} = -\frac{\partial(\omega^2/2)}{\partial y}\frac{dx}{ds} + \frac{\partial[\omega^2/2]}{\partial x}\frac{dy}{ds}$$

Letting $F_1 = -\frac{\partial}{\partial y}\left[\frac{\omega^2}{2}\right]$ and $F_2 = \frac{\partial}{\partial x}\left[\frac{\omega^2}{2}\right]$

Leaves us with

$$RHS = \oint_C \left[F_1 \frac{dx}{ds} + F_2 \frac{dx}{ds}\right] ds$$

$$= \oint_C \left[F_1\, dx + F_2\, dy\right]$$

By Green's Theorem, This is equivelant TO

$$= \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy$$

Substituting F_1, and F_2 in

$$= \iint_R \left\{\left[\frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}\left[\frac{\omega^2}{2}\right]\right]\right] - \frac{\partial}{\partial y}\left[-\frac{\partial}{\partial y}\left[\frac{\omega^2}{2}\right]\right]\right\} dx\,dy$$

$$= \iint_R \left\{\frac{\partial}{\partial x}\left[\omega \frac{\partial \omega}{\partial x}\right] + \frac{\partial}{\partial y}\left[\omega \frac{\partial \omega}{\partial y}\right]\right\} dx\,dy$$

$$= \iint_R \left\{\left(\frac{\partial \omega}{\partial x}\right)^2 + \omega \frac{\partial^2 \omega}{\partial x^2} + \left(\frac{\partial \omega}{\partial y}\right)^2 + \omega \frac{\partial^2 \omega}{\partial y^2}\right\} dx\,dy$$

Collecting Terms

$$= \iint_R \left\{\left(\frac{\partial \omega}{\partial x}\right)^2 + \left(\frac{\partial \omega}{\partial y}\right)^2 + \omega\left[\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2}\right]\right\} dx\,dy$$

9.4.19 Continued.

The term in the Right side is $\nabla^2 w$, which is Given ; $\nabla^2 w = 0$ for Laplaces equation

$$\iint_R \left\{ \left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 + w \cancel{\nabla^2 w} \right\} dx\,dy$$

$\to 0$ for Laplace's equation

$$\boxed{= \iint_R \left[\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 \right] dx\,dy = \underline{\underline{LHS}}}$$

9.7.16 Applications of Divergence Theorem

evaluate $\iint_S \underline{F} \cdot \underline{n}\, dA$ by the divergence Theorem.

for $\underline{F} = [\cos y, \sin x, \cos z]$ with 'S' the Surface $x^2+y^2 \le 9$, $0 \le z \le 2$

$F_1 = \cos y \quad F_2 = \sin x \quad F_3 = \cos z$

$$\iint_S \vec{F} \cdot \vec{n}\, dA = \iint_S F_1\, dy\,dz + F_2\, dz\,dx + F_3\, dx\,dy$$

Apply Divergence Theorem $= \iiint_T \nabla \cdot \vec{F}\, dx\,dy\,dz$

First, Lets Get $\nabla \cdot \underline{F}$:

$$\nabla \cdot F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = 0 + 0 - \sin z = -\sin z$$

Now we have

$$\iiint -\sin z\, dx\,dy\,dz$$

But how do we get Limits?

Nifty trick; $x^2+y^2 \le 9$ is a sphere with radius $\sqrt{9} = 3$

So Lets use Polar Coordinates.

$\boxed{9.7.16}$ Continued.

Letting $x = r\cos\theta$, $y = r\sin\theta$, $z = z$

Then $dV = r\,d\theta\,dr\,dz$; Now Setup Limits for The volume integration.

$$= \int_{z=0}^{2}\int_{r=0}^{3}\int_{\theta=0}^{2\pi} -\sin z \; r \, d\theta \, dr \, dz$$

$$= -\int_0^2\int_0^3 -\sin z \cdot r \, \theta\Big|_0^{2\pi} dr\,dz$$

$$= -2\pi \int_0^2\int_0^3 \sin z \cdot r \, dr \, dz$$

$$= -2\pi \int_0^2 \sin z \left[\frac{r^2}{2}\right]\Bigg|_0^3 dz$$

$$= -\pi \int_0^2 9 \sin z \, dz = \boxed{9\pi(\cos 2 - 1)}$$

$\boxed{9.7.19}$ Same Problem as 16, but with

$F = [x^3, y^3, z^3]$ and S is The sphere $x^2+y^2+z^2=9$. Which is a sphere of Radius 3.

$f_1 = x^3 \quad f_2 = y^3, \quad F_3 = z^3$

$$\iint_S F \cdot n \, dA = \iiint_T \nabla \cdot F \, dV = \iiint_T \left[\frac{\partial F_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial F_3}{\partial z}\right] dV$$

$$= \iiint 3\left[x^2+y^2+z^2\right] dV$$

Since we know its a sphere, we know we should be Thinking about spherical Coordinates:

$\boxed{9.7.19}$ Continued NOTE $x^2 + y^2 + z^2 = r^2$ and with $dV = r^2 \sin\phi \, d\theta \, dr \, d\phi$

We have

$$\iint_S \underline{F} \cdot \underline{n} \, dA = \int_{\phi=0}^{\pi} \int_{\theta=0}^{2\pi} \int_{r=0}^{3} 3(r^2) r^2 \sin\phi \, dr \, d\theta \, d\phi$$

$$= 3 \int_{\phi=0}^{\pi} \int_{\theta=0}^{2\pi} \left[\frac{r^5}{5} \right] \Big|_0^3 \sin\phi \, d\theta \, d\phi$$

$$= \frac{3^6}{5} \int_0^{\pi} \left[\theta \right] \Big|_0^{2\pi} \sin\theta \, d\phi$$

$$= \frac{3^6}{5} (2\pi) \left(-\cos\phi \right) \Big|_0^{\pi} =$$

$$= \frac{3^6}{5} 2\pi \left(1 - \cos\pi \right) =$$

$$\boxed{= \frac{3^6 4\pi}{5} = \frac{2916}{5} \pi}$$

$\boxed{9.8.3}$ Evaluation of Surface integrals by divergence Theorem

Use The divergence Theorem to evaluate

$\iint_S F \cdot n \, dA$ for $\underline{F} = [x, z, y]$, $F_1 = x$, $F_2 = z$, $F_3 = y$; $S \Rightarrow$ hemisphere $x^2 + y^2 + z^2 = 4$, $z \geq 0$

$$\iint_S F \cdot n \, dA = \iiint \nabla \cdot F \, dV = \iiint \left[\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right] dV$$

$$= \iiint (1 + 0 + 0) \, dV = \iiint dV$$ $\Rightarrow$ volume of hemisphere with $R = 2$

$$= \frac{4/3 \, \pi R^3}{2} = \frac{2}{3} \pi (2)^3 = \boxed{\frac{16}{3} \pi}$$

$\boxed{9.8.4}$ evaluate $\iint \underline{F}\cdot\underline{n}\, dA$ For

$F = [x, 3y, 6z]$ S is The Surface of The cone

$$\sqrt{x^2+y^2} \le z \quad 0 \le z \le 3$$

$$\iint_S \underline{F}\cdot\underline{n}\, dA = \iiint_T \nabla\cdot F\, dV = \iiint_T \frac{\partial}{\partial x}[x] + \frac{\partial}{\partial y}[3y] + \frac{\partial}{\partial z}[6z]\, dV$$

$$= \iiint_T (1+3+6)\, dV = \iiint 10\, dV = 10 \iiint dV$$

Which is 10 Times The volume of That Cone.

$V_{cone} = \frac{\pi r^2 h}{3}$; So

$$\iiint \underline{F}\cdot\underline{n}\, dA = 10\left[\frac{\pi (3)^2 3}{3}\right] = \boxed{90\pi}$$

$\boxed{9.9.1}$ Direct integration of The Surface integral

integrate both Directly and by Line integral

$F = [z^2, 5x, 0]$ S The square $0 \le x \le 1$, $0 \le y \le 1$, $z = 1$

(1) Using The Curl

$$\nabla\times F = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \frac{\partial}{\partial z} \\ z^2 & 5x & 0 \end{vmatrix} = 0\hat{i} + 2z\hat{j} + 5\hat{k}$$

For This Plane, clearly $\bar{n} = \hat{k}$

$$\iint_S [0i + 2zj + 5k]\,[k]\, dA = \iint_S 5\, dA = 5\iint dA = \boxed{5}$$

Line integrals on The Next Page

$\boxed{9.9.1}$ Continued. by Line integrals

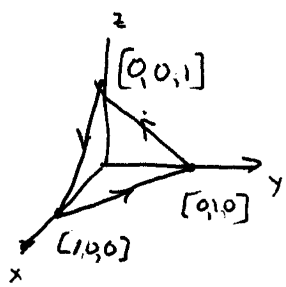
$$\oint F_1\,dx + F_2\,dy = F_1(1-0) + F_2\Big|_{x=1}(1-0) + F_1(0-1) + F_2\Big|_{x=0}(0-1)$$

$$= 1^2(1) + 5(1)(1) + 1(-1) + 5(0)(0-1)$$

$$\boxed{= 5}$$

$\boxed{9.9.11}$ Evaluate $\oint_C F\cdot r'(s)\,ds$ by STOKe's Theorem.

$F = [0, xyz, 0]$ C The boundary of The Triangle shown.



$$\oint F\cdot r(s)\,ds = \iint (\nabla\times F)\cdot n\; dA$$

by STOKES Theorem

First Get $(\nabla\times F)$

$$\nabla\times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & xyz & 0 \end{vmatrix} = -xy\,\hat{i} + 0\hat{j} + yz\,\hat{k}$$

And The Unit Normal Vector Looks Like

$$\hat{n} = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\cancel{\text{so} \Rightarrow \int_C F\cdot r(s)\,ds = \iint \left[-xy\,i + yz\,k\right]\frac{1}{\sqrt{3}}\begin{bmatrix} i \\ j \\ k \end{bmatrix} dA}$$

$$\cancel{= \iint \frac{[-xy + yz]}{\sqrt{3}}\; dA}$$

$$\oint_C F \cdot r(s) = \iint_S \nabla \times F \cdot \hat{n}\, ds$$

$$= \iint \left[-xy\hat{i} + yz\hat{k} \right] \cdot \frac{[i + j + k]}{\sqrt{3}}\, dS$$

Now we project the plane onto the x-y axis, letting $dS = \sqrt{1+1+1}\, dA = \sqrt{3}\, dx\,dy$

$$= \int_0^1 \int_0^{1-x} (-xy + yz) \frac{\sqrt{3}}{\sqrt{3}}\, dy\, dx$$

The plane is defined by $z = 1 - x - y$, so substitute this in for z to get

$$= \int_{x=0}^{1} \int_{y=0}^{1-x} \left[-xy + y(1-x-y) \right] dy\, dx$$

$$= \int_0^1 \left[\frac{y^2}{2} - xy^2 - \frac{y^3}{3} \right]_0^{y=1-x} dx$$

$$= \int_0^1 \left[\frac{y^2}{2}\left[(1-x) - x\right] - \frac{y^3}{3} \right] dx$$

$$= \int_0^1 \left[\frac{1}{6}(1-x)^3 - \frac{x(1-x)^2}{2} \right] dx$$

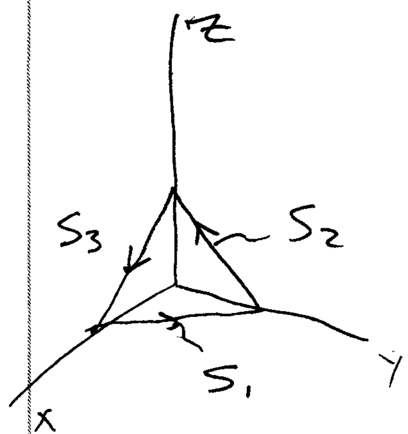
$$= \left[\frac{(1-x)^4}{24} - x^2\left[\frac{1}{4} - \frac{x}{3} + \frac{x^4}{8} \right] \right]_0^1$$

$$= -\left[\frac{1}{4} - \frac{1}{3} + \frac{1}{8} \right] - \frac{1}{24} = \underline{0}$$

Problem 9.9.11 Continued

Now, The easy way.

Lets define the Segments To integrate



integrating using Paths S_1, S_2 and S_3 Looks Like This:

$$\oint_C F\cdot r\, ds = \int_{S_1} F\cdot r\, dS_1 + \int_{S_2} F\cdot r\, dS_2 + \int_{S_3} F\cdot r\, dS_3$$

$$= \int_{S_1} [0i + xy(0)\hat{j} + 0\hat{k}]\cdot r\, dS_1 \; +$$

$$\int_{S_2} [0i + (0)(yx)\hat{j} + 0\hat{k}]\cdot r\, dS_2 +$$

$$\int_{S_3} [0i + xz(0)\hat{k} + 0k]\cdot r\, dS_3$$

$$= \int_{S_1} [0i + 0j + 0k]\cdot r\, dS_1 + \int_{S_2} [0i + 0j + 0k]\cdot r\, dS_2$$

$$+ \int_{S_3} [0i + 0\hat{j} + 0\hat{k}]\cdot r\, dS_3 = \boxed{0}$$

$\boxed{9.9.12}$ Evaluate the integral $\int_c F \cdot r \, ds$ by Stokes Theorem for the function on ~~[illegible]~~. Contour

$F = [x^4, y^4, z^4]$ C is the intersection of $x^2+y^2+z^2 = a^2$ and $y^2 = z$

$$\int_c F \cdot r' \, ds = \iint_A (\nabla \times F) \cdot \hat{n} \, dA$$

$$\nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^4 & y^4 & z^4 \end{vmatrix} = \left(\frac{\partial z^4}{\partial y} - \frac{\partial y^4}{\partial z}\right) i - \left(\frac{\partial z}{\partial x} - \frac{\partial x}{\partial z}\right) j + \left(\frac{\partial x}{\partial y} - \frac{\partial y}{\partial x}\right) k = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

So $$\iint [0 i + 0 j + 0 k] \cdot \hat{n} \, dA = \boxed{0}$$