

P2.1.5

use The methods in example 7 or in Problems 1 and 2 TO Reduce TO first order and solve:

$$y y'' = 2y'^2$$

Let $z = y'$; ① Then

$$y'' = \frac{d}{dx} \left[\frac{dy}{dx} \right] = \frac{d}{dx} [z] = \frac{dz}{dy} \frac{dy}{dx} = z \frac{dz}{dy} \quad \text{②}$$

Substitute ① and ② into The ODE:

$$y z \frac{dz}{dy} = 2z^2 \quad \text{and solve}$$

$$\frac{dz}{z} = \frac{2}{y} dy \Rightarrow \int \frac{dz}{z} = \int \frac{2}{y} dy \Rightarrow \ln z = 2 \ln y + C$$

$$\ln z = 2 \ln y + C = \ln y^2 + C \Rightarrow z = A y^2$$

$$z = A y^2 \quad (\text{where } A = \ln C); \text{ but } z = y'$$

So $y' = A y^2$ Now integrate

$$\frac{dy}{y^2} = A dx \Rightarrow \frac{-1}{y} = Ax + B$$

$$y = -\frac{1}{Ax + B}$$

2.1.9 Reduce to first order and solve

$$x^2 y'' - 5xy' + 9y = 0 \quad ; \quad y_1 = x^3$$

Let $y_2 = uy_1$; Then $y_2' = u'y_1 + y_1'u$

$$y_2'' = u''y_1 + u'y_1' + u'y_1' + u''y_1''$$

$$y_2'' = u''y_1 + 2u'y_1' + uy_1''$$

Plug into the ODE:

$$x^2(u''y_1 + 2u'y_1' + uy_1'') - 5x(u'y_1 + uy_1') + 9y = 0$$

$$(x^2y_1)(u'') + (2x^2y_1' - 5xy_1)u' + (x^2y_1'' - 5xy_1' + 9y)u = 0$$

$$x^2y_1 u'' + (2x^2y_1' - 5xy_1')u' = 0$$

Now Reduce order by $U = u'$

$$x^2y_1 U' + (2x^2y_1' - 5xy_1')U = 0$$

$$x^2(x^3)U' + [(2(x^4)(3x^2)) - 5(x)(3x^3)]U$$

$$x^5 U' + x^4 U = 0$$

$$U' + \frac{U}{x} = 0 \Rightarrow \text{Solve}$$

$$\frac{dU}{U} = -\frac{dx}{x} \Rightarrow U = \frac{1}{x}$$

Now Get u from U ;

$$u' = U = \frac{1}{x} \Rightarrow du = \frac{dx}{x} \Rightarrow$$

$$u = \ln x$$

So $y_2 = uy_1 = x^3 \ln x$

$$\boxed{y_2 = x^3 \ln x}$$

2.2.16 Solve The I.V.P. Check That your Solution works.

$$y'' - ky = 0; \quad k \neq 0 \quad y(0) = 1 \quad y'(0) = 1$$

Guess $y = e^{\lambda t}$; $y' = \lambda e^{\lambda t}$ $y'' = \lambda^2 e^{\lambda t}$
 substitute into the ODE to get

$$\lambda^2 e^{\lambda t} - k e^{\lambda t} = 0 \Rightarrow \lambda^2 - k = 0; \quad \lambda = \pm k$$

So the General Solution is $y = C_1 e^{kt} + C_2 e^{-kt}$

The first derivative is $y' = C_1 k e^{kt} - C_2 k e^{-kt}$

Now apply initial conditions:

$$y(0) = 1 = C_1 + C_2 \quad (1)$$

$$y'(0) = 1 = C_1 k - C_2 k \quad (2)$$

From (1) $C_1 = 1 - C_2$
 Plugging into (2)

$$1 = (1 - C_2)k - C_2 k = (1 - 2C_2)k$$

$$C_2 = \left(\frac{1}{k} - 1\right) \frac{1}{2} = -\left[\frac{1-k}{2k}\right] = \frac{k-1}{2k}$$

Now substitute back into (1)

$$C_1 = 1 + \frac{1-k}{2k} = \frac{2k + 1 - k}{2k} = \frac{k+1}{2k}$$

and Plug into general Solution

$$y = \left[(k+1)e^{kt} + (k-1)e^{-kt} \right] \frac{1}{2k}$$

I'll leave showing this to work to the reader

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ME 564

HW #7

Solutions

P 2.2.17 solve the I.V.P.

$$4y'' - 4y' - 3y = 0 \quad y(-2) = e \quad y'(-2) = -\frac{e}{2}$$

Plug in $y = e^{\lambda x}$

$$4\lambda^2 e^{\lambda x} - 4\lambda e^{\lambda x} - 3e^{\lambda x} = 0 \Rightarrow 4\lambda^2 - 4\lambda - 3 = 0$$

Roots are $\lambda = -\frac{1}{2}, \frac{3}{2}$; So the General Solution is

$$y = C_1 e^{-\frac{1}{2}x} + C_2 e^{\frac{3}{2}x} \quad \text{Taking the first derivative gives}$$

$$y' = -\frac{C_1}{2} e^{-\frac{x}{2}} + C_2 e^{\frac{3x}{2}}$$

$$y(-2) = e = C_1 e^1 + C_2 e^{-2} = C_1 e + C_2 e^{-2} \quad (1)$$

$$y'(-2) = -\frac{e}{2} = -\frac{C_1}{2} e^1 + \frac{3C_2}{2} e^{-2} \quad (2)$$

Solving for C_1 in eq (1)

$$C_1 = 1 - C_2 e^{-3} \quad (3)$$

Substituting (3) into (2) gives

$$-\frac{e}{2} = \frac{-[1 - C_2 e^{-3}]}{2} e + \frac{3}{2} C_2 e^{-2}$$

$$-\frac{e}{2} = -\frac{e}{2} + \frac{C_2 e^{-2}}{2} + \frac{3}{2} C_2 e^{-2}$$

$$0 = 2C_2 e^{-2} \Rightarrow \underline{C_2 = 0}$$

From equation (3), $C_1 = 1$; So solution is

$$y = e^{-\frac{x}{2}}$$

2.3.16 Solve The IVP

$$y'' + 0.4y' + 0.29y = 0 \quad \text{for } y(0) = 1 \quad y'(0) = -1.2$$

Plug in $y = e^{\lambda x}$

$$\lambda^2 e^{\lambda x} + 0.4 \lambda e^{\lambda x} + 0.29 e^{\lambda x} = 0$$

$$\lambda^2 + 0.4\lambda + 0.29 = 0 \quad \rightarrow \text{Characteristic equation}$$

$$\lambda = \frac{-0.4 \pm \sqrt{0.4^2 - 4(1)(0.29)}}{2} = -0.2 \pm 0.5i$$

So general solution is

$$y = e^{-0.2x} [A \cos 0.5x + B \sin 0.5x]$$

$$y' = e^{-0.2x} \left[-\frac{A}{2} \sin \frac{x}{2} + \frac{B}{2} \cos \frac{x}{2} \right] - 0.2 e^{-0.2x} [A \cos \frac{x}{2} + B \sin \frac{x}{2}]$$

Apply initial conditions.

$$y(0) = 1 = A \cos 0 + B \sin 0 \Rightarrow \boxed{A = 1}$$

$$y'(0) = -1.2 = -\frac{A}{2} \sin 0 + \frac{B}{2} \cos 0 - 0.2 [A \cos 0 + B \sin 0]$$

$$-1.2 = \frac{B}{2} - 0.2 \Rightarrow \boxed{B = -2}$$

So solution is

$$\boxed{y = e^{-0.2x} \left[\cos \frac{x}{2} - 2 \sin \frac{x}{2} \right]}$$

2.7.1 Find The Wronskians of The Given Bases, and Verify Theorem 2

$$e^{\lambda_1 x}, e^{\lambda_2 x}$$

$$W = \begin{vmatrix} e^{\lambda_1 x} & e^{\lambda_2 x} \\ \lambda_1 e^{\lambda_1 x} & \lambda_2 e^{\lambda_1 x} \end{vmatrix} = \lambda_2 e^{\lambda_1 x} e^{\lambda_2 x} - \lambda_1 e^{\lambda_1 x} e^{\lambda_2 x}$$

$$= (\lambda_2 - \lambda_1) e^{(\lambda_1 + \lambda_2)2x} \neq 0$$

unless $\lambda_1 = \lambda_2$
Therefore, The Bases are independent if $\lambda_1 \neq \lambda_2$

2.9.19 initial value Problems for Non homogeneous Equations
Solve The I.V.P., indicate The Rules used.

$$y'' + 1.2y' + 0.36y = 4e^{-0.6x} \quad y(0) = 0 \quad y'(0) = 1$$

① Solve y_h :

$$\lambda^2 + 1.2\lambda + 0.36 = 0 \rightarrow (\lambda + 0.6)^2 = 0 \quad \text{Repeated Root } \lambda = -0.6$$

So by Rule B.

$$y_h = (C_1 + C_2 x) e^{-0.6x}$$

Now Get y_p : From table 2.1 on Page 105
we choose

$$y_p = Kx^2 e^{-0.6x}$$

$$y_p' = 2Kx e^{-0.6x} - 0.6Kx^2 e^{-0.6x}$$

$$y_p'' = 2Ke^{-0.6x} - 1.2Kx e^{-0.6x} - 1.2Kx e^{-0.6x} + 0.36Kx^2 e^{-0.6x}$$

$$y_p'' = 2Ke^{-0.6x} - 2.4Kx e^{-0.6x} + 0.36Kx^2 e^{-0.6x}$$

Substitute y_p, y_p' and y_p'' into The ODE to get K

$$[2K - 2.4Kx + 0.36Kx^2] e^{-0.6x} + 1.2[2Kx - 0.6Kx^2] e^{-0.6x} + 0.36Kx^2 e^{-0.6x} = 4e^{-0.6x}$$

See next Page

2.9.19 Continued.

$$2ke^{-0.6x} = 4e^{-0.6x}$$

$$2k = 4 \Rightarrow k = 2$$

$$y_p = 2x^2e^{0.6x}$$

So general solution is

$$y = y_p + y_h$$

$$y = (C_1 + C_2x)e^{-0.6x} + 2x^2e^{-0.6x}$$

With The first derivative

$$y' = (C_2 + 4x)e^{-0.6x} - 0.6e^{-0.6x}(C_1 + C_2x + 2x^2)$$

Now Apply ~~B~~ initial conditions

$$y(0) = 0 = C_1 \Rightarrow C_1 = 0$$

$$y'(0) = 1 = C_2 - 0.6C_1 \Rightarrow C_2 = 1$$

So The ~~general~~ solution is

$$y = (x + 2x^2)e^{-0.6x}$$

2.10.16 General Solutions of Non homogeneous equations
Find The Solution For

$$(D^2 + 2D + 2)y = 4e^{-x} \sec^3 x$$

① Find y_h ; $(D^2 + 2D + 2)y = y'' + 2y' + 2y \Rightarrow$

Characteristic equation is Then

$$\lambda^2 + 2\lambda + 2 = 0 \Rightarrow \lambda = -1 \pm i; \text{ This gives solutions}$$

$$y_h = e^{-x} [C_1 \cos x + C_2 \sin x]$$

Now, To use eq. 2 on Page 108, we need The Wronskian.

$$W = \begin{vmatrix} e^{-x} \cos x & e^{-x} \sin x \\ -e^{-x} \sin x - e^{-x} \cos x & e^{-x} \cos x - e^{-x} \sin x \end{vmatrix}$$

$$= e^{-2x} \cos x [\cos x - \sin x] + e^{-2x} \sin x [\sin x + \cos x]$$

$$W = e^{-2x} [\cos^2 x - \cos x \sin x + \sin^2 x + \cos x \sin x]$$

$$W = e^{-2x} \quad \text{Cos}^2 + \text{sin}^2 = 1; \text{ so}$$

Now use The equation.

$$y_p = -e^{-x} \cos x \int \frac{e^{+x} \sin x e^{+x} \sec^3 x}{e^{-2x}} dx + e^{-x} \sin x \int \frac{e^{+x} \cos x e^{+x} \sec^3 x}{e^{-2x}} dx$$

$$= -e^{-x} \cos x \int \frac{\tan x}{\cos^2 x} dx + e^{-x} \sin x \int \frac{dx}{\cos^2 x}$$

$$= e^{-x} \cos x \left(\frac{2}{\cos^2 x} \right) + e^{-x} \sin x (4 \tan x)$$

$$y_p = e^{-x} \left[\frac{2}{\cos x} + \frac{4 \sin^2 x}{\cos x} \right] = e^{-x} \left[\frac{-2 \cos 2x}{\cos x} \right]$$

So The solution is

$$y = e^{-x} \left[C_1 \cos x + C_2 \sin x - \frac{2 \cos 2x}{\cos x} \right]$$

2.13.15 Linear independent or dependent?

$$(x-1)^2, (x+1)^2, x^*$$

Find Their Wronskian

$$W = \begin{vmatrix} (x-1)^2 & (x+1)^2 & x \\ 2(x-1) & 2(x+1) & 1 \\ 2 & 2 & 0 \end{vmatrix}$$

using Laplace
expansion
on the last
row

$$= 2[(x+1)^2 - 2x(x+1)] - 2[(x-1)^2 - 2x(x-1)] + 0$$

$$= 2[x^2 + 2x + 1 - 2x^2 - 2x - x^2 + 2x + 1 + 2x^2 - 2x]$$

$$= \underline{2[0] = 0}$$

So They are **Linearly dependent.**

3.3.9

General Solution: Find a Real General Solution of The Following System.

$$\underline{Y}' = \begin{bmatrix} -1 & -4 & 2 \\ 2 & 5 & -1 \\ 2 & 2 & 2 \end{bmatrix} \underline{Y}; \text{ Try } \underline{Y} = \underline{X} e^{\lambda T}; \underline{Y}' = \lambda \underline{X} e^{\lambda T} = A \underline{Y}$$

$\therefore A \underline{X} = \lambda \underline{X} \Rightarrow$ SETUP Eigenvalue Problem.

$$\begin{vmatrix} -1-\lambda & -4 & 2 \\ 2 & 5-\lambda & -1 \\ 2 & 2 & 2-\lambda \end{vmatrix} = 0$$

Doing the diagonal multiplication thing for 3x3 Determinants

$$(-1-\lambda)(5-\lambda)(2-\lambda) + (-4)(-1)(2) + (2)(2)(2) - (2)(2)(5-\lambda) - (-1)(2)(-1-\lambda) - (2-\lambda)(-4)(2) = 0 \Rightarrow$$

$$\Rightarrow - (1+\lambda)(10-7\lambda+\lambda^2) + 8+8 - 4(5-\lambda) - 2(1+\lambda) + 8(2-\lambda) = 0$$

$$-10 + 7\lambda - \lambda^2 - 10\lambda + 7\lambda^2 - \lambda^3 + 16 - 20 + 4\lambda - 2 - 2\lambda + 16 - 8\lambda = 0$$

$$-\lambda^3 + 6\lambda^2 - 9\lambda = 0$$

$$\lambda(\lambda^2 - 6\lambda + 9) = 0$$

$$\lambda(\lambda-3)^2 = 0 \Rightarrow \lambda = 0, 3, 3$$

So we have a Repeated Root...

First, find $X^{(1)}$ corresponding to $\lambda_1 = 0$

$$(-1-\lambda)X_1 - 4X_2 + 2X_3 = 0 \Rightarrow X_1 = -4X_2 + 2X_3$$

Letting $X_3 = C_3, X_2 = C_2$ we have $X^{(1)}$

For $\lambda_1 = 0$; The System is below, now Row Reduce.

$$\begin{bmatrix} -1 & -4 & 2 \\ 2 & 5 & -1 \\ 2 & 2 & 2 \end{bmatrix} \begin{array}{l} R_2 + 2R_1 \\ R_3 + 2R_1 \end{array} \rightarrow \begin{bmatrix} -1 & -4 & 2 \\ 0 & -3 & 3 \\ 0 & -6 & 6 \end{bmatrix} \begin{array}{l} R_3 + 2R_2 \end{array} \rightarrow \begin{bmatrix} -1 & -4 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

So we have $x_2 = x_3$; $x_1 = -4x_2 + 2x_3$ x_3 arbitrary

$$X^{(1)} = c_1 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \text{ is The Eigenvector for } \lambda_1 = 0$$

Now, lets Go after The Repeated Root
for $\lambda = 3$:

$$\begin{bmatrix} -1-3 & -4 & 2 \\ 2 & 5-3 & -1 \\ 2 & 2 & 2-3 \end{bmatrix} = \begin{bmatrix} -4 & -4 & 2 \\ 2 & 2 & -1 \\ 2 & 2 & -1 \end{bmatrix} \xrightarrow{-R_1/2} \begin{bmatrix} 2 & 2 & -1 \\ 2 & 2 & -1 \\ 2 & 2 & -1 \end{bmatrix}$$

$$2x_1 + 2x_2 - x_3 = 0 \quad \text{Let } x_2 = 0, x_3 = 2c_2$$

$$x_1 = c_2$$

$$\text{So } X^{(2)} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} c_2$$

and finally: Letting $x_3 = 0, x_2 = c_3$; Gets $x_1 = -c_3$

$$X^{(3)} = c_3 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

For a General Solution

$$\underline{y} = c_1 \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} e^{0t} + c_2 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} e^{3t} + c_3 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} e^{3t}$$

$$\underline{y} = c_1 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} e^{3t} + c_3 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} e^{3t}$$

3.6.6 General Solution: Find a Real General Solution for of The following Nonhomogeneous Linear system

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} \tau$$

• Solve homogeneous

$$y' = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix} y \Rightarrow \text{Det} \begin{vmatrix} 4-\lambda & 1 \\ 2 & 3-\lambda \end{vmatrix} = 12 - 7\lambda + \lambda^2 - 2 \\ = \lambda^2 - 7\lambda + 10 = 0$$

$$\text{So } \lambda = 5, 2$$

Get Eigenvectors: for $\lambda = 5$; $(4-5)y_1 + y_2 = 0 \Rightarrow x^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

for $\lambda = 2$; $(4-2)y_1 + y_2 = 0 \Rightarrow x^{(2)} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

So The homogeneous solution is Then

$$y = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{5\tau} + c_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{2\tau}$$

Now Go after Particular Solution.

$$y_p = U + V\tau$$

$$y_p' = v = Au + Av\tau + g$$

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 4u_1 + u_2 \\ 2u_1 + 3u_2 \end{pmatrix} + \begin{pmatrix} 4v_1 + v_2 \\ 2v_1 + v_2 \end{pmatrix} \tau + \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

continued on next page.

τ -Terms:

$$\begin{cases} 4U_1 + V_2 + 1 = 0 \\ 2U_1 + 3V_2 - 1 = 0 \end{cases} \xrightarrow{\text{solved w/mathematica}} \begin{cases} V_1 = -0.4 \\ V_2 = 0.6 \end{cases}$$

Other Terms

$$\begin{cases} V_1 = 4U_1 + U_2 \\ V_2 = 2U_1 + 3U_2 \end{cases} \rightarrow \begin{cases} U_1 = -0.18 \\ U_2 = 0.32 \end{cases}$$

$$y_p = \begin{pmatrix} -0.18 \\ 0.32 \end{pmatrix} + \begin{pmatrix} -0.4 \\ 0.6 \end{pmatrix} \tau$$

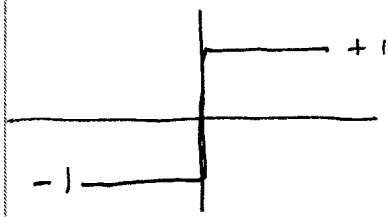
$$y = y_h + y_p$$

$$y = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{5\tau} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{2\tau} + \begin{pmatrix} -0.18 \\ 0.32 \end{pmatrix} + \begin{pmatrix} -0.4 \\ 0.6 \end{pmatrix} \tau$$

10.3.1 Find The Fourier Series for The Periodic Function $f(x)$ with $P = 2L$.

$$f(x) = -1 \quad (0 - 1L < x < 0) \quad \text{and} \quad f(x) = 1 \quad (0 < x < 1) \quad P = 2L = 2$$

So $L = 1$; and This function looks like



Use equations 1, and 2 from Page 537

eg 2.A

$$a_0 = \frac{1}{2L} \int_{-L}^L f(x) dx = \frac{1}{2} \int_{-1}^1 f(x) dx = \frac{1}{2} \left[\int_{-1}^0 -1 dx + \int_0^1 1 dx \right]$$

$$a_0 = \frac{1}{2} \left[[-x]_{-1}^0 + [x]_0^1 \right] = \frac{1}{2} \left[0 + (+(-1)) + 1 \right]$$

$$a_0 = 0$$

Since function is odd, we know $a_n = 0$

Now find b_n

eg 2.C

$$b_n = \frac{1}{L} \int_{-L}^L f(x) \sin \frac{n\pi x}{L} dx = \frac{1}{1} \int_{-1}^1 f(x) \sin \frac{n\pi x}{1} dx$$

$$b_n = \int_{-1}^0 -\sin \frac{n\pi x}{1} dx + \int_0^1 \sin \frac{n\pi x}{1} dx$$

$$= \frac{\cos n\pi x}{n\pi} \Big|_{-1}^0 + \frac{-\cos n\pi x}{n\pi} \Big|_0^1$$

$$= \left[\frac{1}{n\pi} - \frac{\cos n\pi}{n\pi} \right] - \left[\frac{\cos n\pi}{n\pi} - \frac{1}{n\pi} \right]$$

$$b_n = \frac{2}{n\pi} - \frac{2 \cos n\pi}{n\pi}$$

10.3.1 continued.

$$b_n = \frac{4}{n\pi} \text{ for } n = \text{odd}, \quad b_n = 0 \text{ for } n = \text{even}$$

$$\text{So } f(x) = \sum_{n=1}^{\infty} \frac{2}{n\pi} (1 - \cos n\pi) \sin \frac{n\pi x}{\cancel{\pi}}$$

$$\text{or } f(x) = \sum_{n=1}^{\infty} \frac{4}{n\pi} \sin n\pi x, \quad n = \text{odd}, 1, 3, 5, \dots$$

$$\text{or } f(x) = \frac{4}{\pi} \left[\sin \pi x + \frac{\sin 3\pi x}{3} + \frac{\sin 5\pi x}{5} \dots \right]$$

See next page for plot.

Note that I have redefined the way I write F so that I don't have to skip the even numbered n's. Basically, the way this is written makes it so that I am counting by odds, $2n-1$ for $n=1 \Rightarrow 2(1)-1=1$, for $n=2 \Rightarrow 2(2)-1=3$, for $n=3 \Rightarrow 2(3)-1=5$ etc.

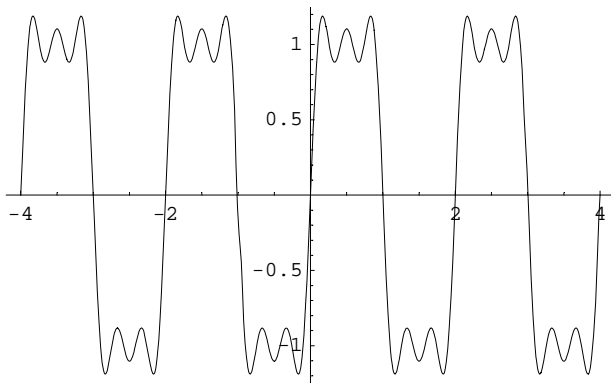
```
In[1]:= f[x_, nmax_] := Sum[ $\frac{4}{(2n-1)\pi} \text{Sin}[(2n-1)\pi x]$ , {n, 1, nmax}]
```

```
In[2]:= f[x, 3]
```

```
Out[2]=  $\frac{4 \text{Sin}[\pi x]}{\pi} + \frac{4 \text{Sin}[3\pi x]}{3\pi} + \frac{4 \text{Sin}[5\pi x]}{5\pi}$ 
```

Now I'll plot the first few terms of this thing.

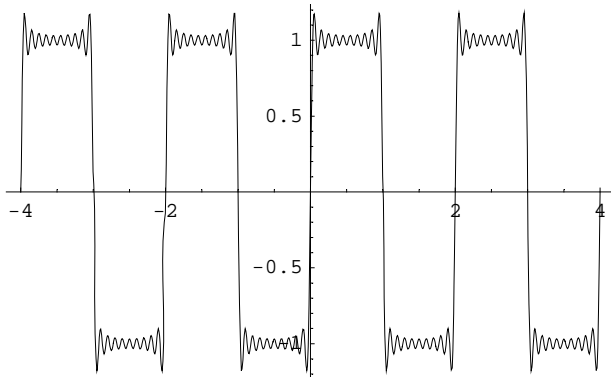
```
In[3]:= Plot[f[x, 3], {x, -4, 4}]
```



```
Out[3]= - Graphics -
```

Using more terms smooths this out.

```
In[4]:= Plot[f[x, 10], {x, -4, 4}]
```



```
Out[4]= - Graphics -
```

10.3.4 Find The Fourier Series of The Periodic function $f(x)$, of Period $P=2L$, and sketch $F(x)$ and The first 3 Partial Sums.

$$f(x) = |x| \text{ on } (-2 < x < 2) \quad P = 2L = 4; L = 2$$

Using equations (1) and (2) on Page 537:

$$2.a \quad a_0 = \frac{1}{2L} \int_{-L}^L f(x) dx = \frac{1}{2(2)} \int_{-2}^2 |x| dx = \frac{1}{4} \int_{-2}^0 -x dx + \frac{1}{4} \int_0^2 x dx$$

$$= \frac{1}{4} \left[-\frac{x^2}{2} \Big|_{-2}^0 + \frac{x^2}{2} \Big|_0^2 \right] = \frac{1}{4} \left[0 - (-4) + \frac{4}{2} + 0 \right]$$

$$a_0 = 1$$

$$2.b \quad a_n = \frac{1}{L} \int_{-L}^L f(x) \cos \frac{n\pi x}{L} dx = \frac{1}{2} \int_{-2}^2 |x| \cos \frac{n\pi x}{2} dx$$

$$a_n = \frac{1}{2} \int_0^2 x \cos \frac{n\pi x}{2} dx + \int_{-2}^0 -x \cos \frac{n\pi x}{2} dx$$

$$= \frac{1}{2} \left\{ 2 \left[\frac{2 \cos \frac{n\pi x}{2}}{n^2 \pi^2} + \frac{x \sin \frac{n\pi x}{2}}{n\pi} \right]_0^2 \right\}$$

$$\dots \left[2 \left[\frac{2 \cos \frac{n\pi x}{2}}{n^2 \pi^2} + \frac{x \sin \frac{n\pi x}{2}}{n\pi} \right]_0^2 \right\}$$

$$= \frac{1}{2} \left\{ 2 \left[\left(\frac{2 \cos n\pi}{n^2 \pi^2} + \frac{2 \sin n\pi}{n\pi} \right) - \left(\frac{2 \cos 0}{n^2 \pi^2} + 0 \right) \right] \right\}$$

$$- 2 \left[\left(\frac{2 \cos 0}{n^2 \pi^2} + 0 \right) - \left(\frac{2 \cos n\pi}{n^2 \pi^2} + \frac{(-2) \sin n\pi}{n\pi} \right) \right]$$

$$a_n = \frac{4 \cos n\pi}{n^2 \pi^2} - \frac{4 \sin n\pi}{n\pi}$$

See Next Page

10.3.4 continued

finally

$$a_n = \frac{4 \cos n\pi}{n^2 \pi^2} - \frac{4}{n^2 \pi^2} \Rightarrow \text{or}$$

$$a_n = \frac{8}{n^2 \pi^2} \text{ for } n = 1, 3, 5, \dots \text{ (odd)}, \quad a_n = 0 \text{ for } n = \text{even}$$

$b_n = 0$, because the function $|x|$ is even,
so the solution becomes

$$f(x) = 1 + \sum_{\substack{n=1 \\ \text{odd}}}^{\infty} \frac{-8}{n^2 \pi^2} \cos \frac{n\pi x}{2}$$

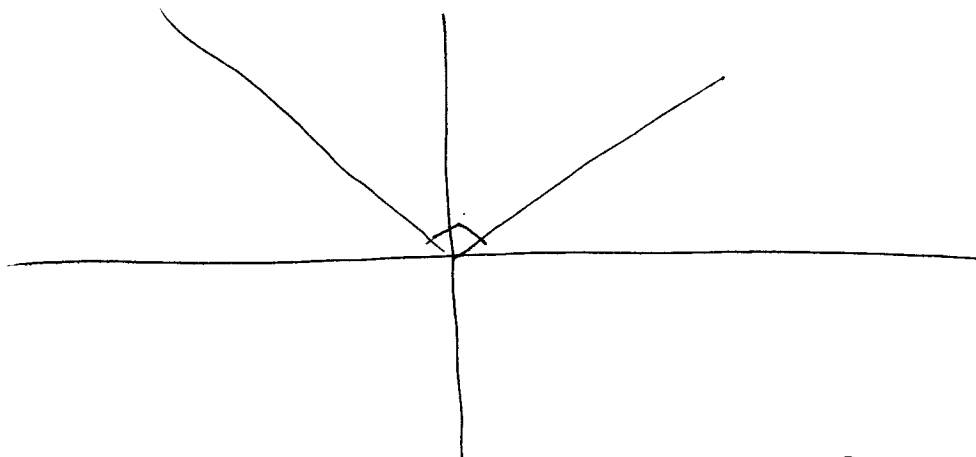
or

$$f(x) = 1 + \sum_{n=1}^{\infty} \frac{4}{n^2 \pi^2} (\cos n\pi - 1) \cos \frac{n\pi x}{2}$$

or

$$f(x) = 1 - \frac{8}{\pi^2} \left[\cos \frac{\pi x}{2} + \frac{1}{9} \cos \frac{3\pi x}{2} + \frac{1}{25} \cos \frac{5\pi x}{2} + \dots \right]$$

The original function looks like



see plot on next page
for the ~~plot~~ series version.

$$f[x_] := 1 + \sum_{n=1}^5 \left(\frac{4}{n^2 \pi^2} (\cos[n \pi] - 1) \cos\left[n \pi \frac{x}{2}\right] \right)$$

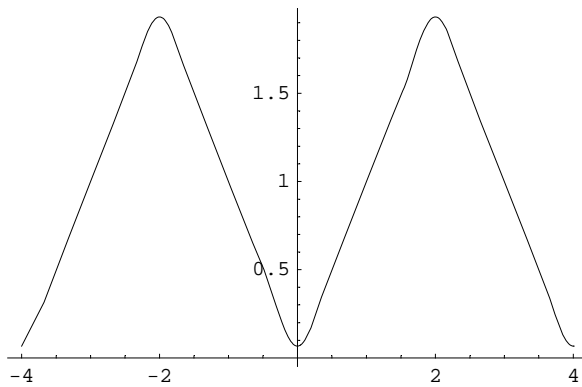
Now, Just to show what it looks like;

f[x]

$$1 - \frac{8 \cos\left[\frac{\pi x}{2}\right]}{\pi^2} - \frac{8 \cos\left[\frac{3 \pi x}{2}\right]}{9 \pi^2} - \frac{8 \cos\left[\frac{5 \pi x}{2}\right]}{25 \pi^2}$$

And the plot looks lie the following

Plot[f[x], {x, -4, 4}]



- Graphics -

10.4.1 Are the following functions odd, even, or neither

(A) $|x^3|$: $f(-x) = |(-x)^3| = |-x^3| = |x^3| \Rightarrow$ even

(B) $x \cos x = f(x)$: $f(-x) = -x \cos(-x) = -x \cos x \Rightarrow$ odd

(C) $f(x) = \cosh x = \frac{e^x + e^{-x}}{2}$; $f(-x) = \frac{e^{-x} + e^x}{2} = f(x) \Rightarrow$ even

(D) $x^2 \cos x$: $f(-x) = (-x)^2 \cos(-x) = x^2 \cos x \Rightarrow$ even

(E) $\sinh x = \frac{e^x - e^{-x}}{2}$; $f(-x) = \frac{e^{-x} - e^x}{2} = -\sinh x \Rightarrow$ odd

(F) $f(x) = \sin x + \cos x$

$f(-x) = \sin(-x) + \cos(x) = -\sin(x) + \cos x \Rightarrow$ neither

(G) $x|x| = f(x)$: $f(-x) = -x|-x| = -x|x| \Rightarrow$ odd

Switched
sorry

10.4.22 Find The Fourier Cosine Series as well as The Sine Series. ~~Plot~~ Sketch $f(x)$ and its two Periodic Extensions.

$$f(x) = x^2 \quad (0 \leq x < L)$$

Cosine series; using equations (3) and (4) Page ~~542~~ 542

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}$$

$$a_0 = \frac{1}{L} \int_0^L f(x) dx = \frac{1}{L} \int_0^L x^2 dx = \frac{1}{L} \left[\frac{x^3}{3} \right]_0^L = \frac{L^2}{3}$$

$$a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} dx = \frac{2}{L} \int_0^L x^2 \cos \frac{n\pi x}{L} dx$$

by using integration by parts

$$a_n = \frac{2}{L} \left[\frac{2L^2 x \cos \frac{n\pi x}{L}}{n^2 \pi^2} + \frac{n^2 \pi^2 x^2 - 2L^2}{n^3 \pi^3} \sin \frac{n\pi x}{L} \right] \Big|_0^L$$

0 for $x=0$, $n = \text{integer}$

$$a_n = \frac{4L^2}{n^2 \pi^2} \cos n\pi$$

So for The even Periodic extension we have

$$f(x) = \frac{L^2}{3} + \sum_{n=1}^{\infty} (-1)^n \frac{4L^2}{n^2 \pi^2} \cos \frac{n\pi x}{L}$$

$$\text{or } f(x) = \frac{L^2}{3} + \frac{4L^2}{\pi^2} \left[-\cos \frac{\pi x}{L} + \cos \frac{2\pi x}{L} - \cos \frac{3\pi x}{L} \right]$$

NOW for The odd Periodic Extension; use equations 5 and 6 from Page 542:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$$

Where

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx$$

$$b_n = \frac{2}{L} \int_0^L x^2 \sin \frac{n\pi x}{L} dx \quad \text{Using integration by parts to get}$$

$$b_n = \frac{2}{L} \left[\left(\frac{2L^3 - Ln^2\pi^2 x^2}{n^3\pi^3} \right) \cos \frac{n\pi x}{L} + \frac{2L^2 x \sin \frac{n\pi x}{L}}{n^2\pi^2} \right]_0^L$$

$$= \frac{2}{L} \left[\frac{2L^3}{n^3\pi^3} - \frac{L^2 x^2}{n\pi} \right]$$

$$b_n = \frac{2}{L} \left[\left(\frac{2L^3}{n^3\pi^3} - \frac{L^3}{n\pi} \right) \cos n\pi - \frac{2L^3}{n^3\pi^3} \right]$$

$$\cancel{b_n = \frac{2L^2}{\pi} \left[\frac{2}{n^3\pi} \cos n\pi - \frac{2L^2}{n^3\pi^2} (1 + \cos n\pi) \right]}$$

$$f(x) = \sum_{n=1}^{\infty} 2 \left[\left(\frac{2L^2}{n^3\pi^3} - \frac{L^2}{n\pi} \right) \cos n\pi - \frac{2L^2}{n^3\pi^3} \right] \sin \frac{n\pi x}{L}$$

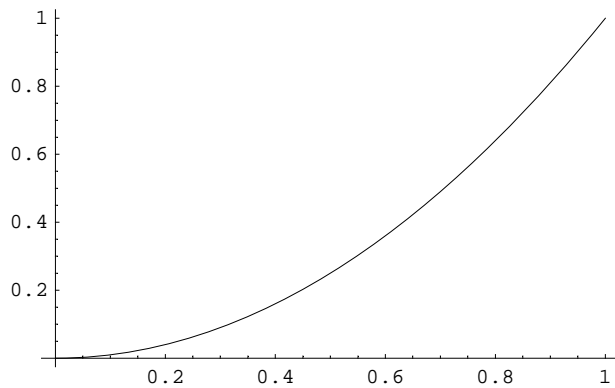
∴ The first several terms are

$$f(x) = \left(\frac{2L^2}{\pi} - \frac{8L^2}{\pi^3} \right) \sin \frac{\pi x}{L} - \frac{1}{2} \sin \frac{2\pi x}{L} + \left(\frac{2L^2}{3\pi} - \frac{8L^2}{3^3\pi^3} \right) \sin \frac{3\pi x}{L}$$

Plots are on the next page.

Okay, so the function that we are plotting is pretty easy.

```
In[24]:= Plot[x2, {x, 0, 1}]
```



```
Out[24]= - Graphics -
```

First I'll do the even half range expansion

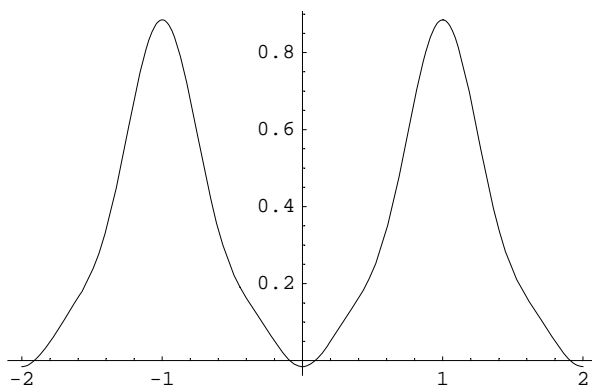
$$\text{feven}[x_, L_, nmax_] := \frac{L^2}{3} + \sum_{n=1}^{nmax} \left((-1)^n \frac{4 L^2}{n^2 \pi^2} \text{Cos}\left[n \pi \frac{x}{L}\right] \right)$$

```
feven[x, L, 5]
```

$$\frac{L^2}{3} - \frac{4 L^2 \text{Cos}\left[\frac{\pi x}{L}\right]}{\pi^2} + \frac{L^2 \text{Cos}\left[\frac{2\pi x}{L}\right]}{\pi^2} - \frac{4 L^2 \text{Cos}\left[\frac{3\pi x}{L}\right]}{9 \pi^2} + \frac{L^2 \text{Cos}\left[\frac{4\pi x}{L}\right]}{4 \pi^2} - \frac{4 L^2 \text{Cos}\left[\frac{5\pi x}{L}\right]}{25 \pi^2}$$

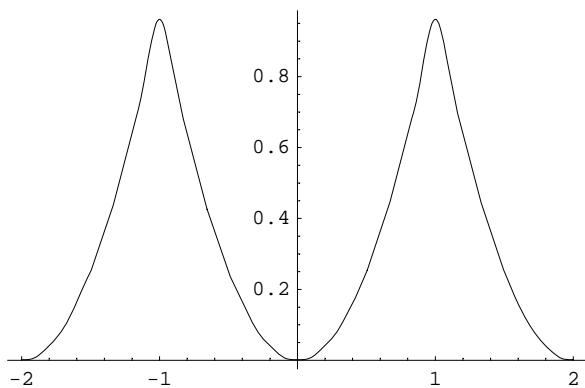
The first term is just the space coordinate, the second term just normalizes the length to length 1, and the last term tells how many terms are being evaluated. I plotted it from -2 to 2 so that you could see how the series looks beyond the domain that we evaluated the integral on. Note that it is a pretty good match from $x=0$ to L .

```
Plot[feven[x, 1, 3], {x, -2, 2}]
```



```
- Graphics -
```

Plot[feven[x, 1, 10], {x, -2, 2}]

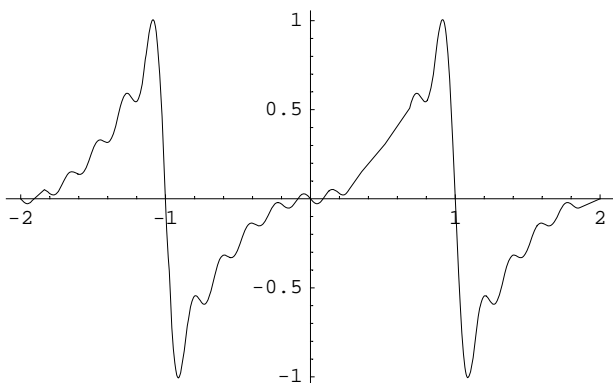


- Graphics -

Note that it doesn't really take a lot of terms to do this. Pretty nifty. Now lets try that with the sine series half range expansion.

$$In[30]:= \text{fodd}[x_, L_, nmax_] := \sum_{n=1}^{nmax} \left(\left(\frac{2 L^2 (-2 + (2 - n^2 \pi^2) \text{Cos}[n \pi] + 2 n \pi \text{Sin}[n \pi])}{n^3 \pi^3} \right) \text{Sin}\left[n \pi \frac{x}{L}\right] \right)$$

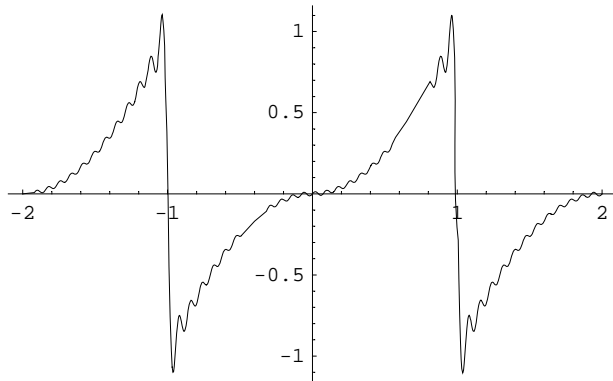
In[31]:= Plot[fodd[x, 1, 10], {x, -2, 2}]



Out[31]= - Graphics -

Blech, not so good. Lets try it with more terms And note that the expansion beyond the range that we were worried about, $0 < x < L$ is now an odd periodic function.

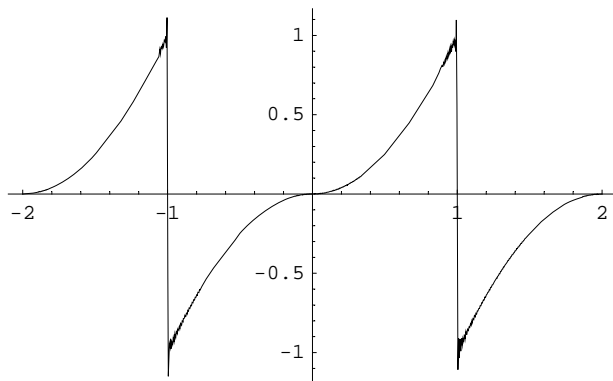
```
In[34]:= Plot[fodd[x, 1, 25], {x, -2, 2}]
```



```
Out[34]= - Graphics -
```

Better, but still not as good as the cosine series. Lets get ridiculous.

```
In[35]:= Plot[fodd[x, 1, 200], {x, -2, 2}]
```



```
Out[35]= - Graphics -
```

With 200 terms of the series, it finally looks pretty good, but note that at the edge of the range we were working on, things are still getting bad.

Find The General Solution of:

$$y'' + y = f(x) \quad \text{where } f(x) \text{ is The even periodic Extension from Problem 10.4.22 with } L=1$$

Homogeneous part

$$y'' + y = 0 \rightarrow \lambda^2 + 1 = 0 \rightarrow y = A \cos x + B \sin x$$

Particular

$$y'' + y = \frac{1}{3} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos n\pi x$$

Guess a y :

$$y = c_0 + \sum_{n=1}^{\infty} [c_n \cos n\pi x + d_n \sin n\pi x]$$

$$y' = \sum_{n=1}^{\infty} \left[\frac{-c_n}{n\pi} \sin n\pi x + \frac{d_n}{n\pi} \cos n\pi x \right]$$

$$y'' = \sum_{n=1}^{\infty} \left[\frac{-c_n}{n^2\pi^2} \cos n\pi x - \frac{d_n}{n^2\pi^2} \sin n\pi x \right]$$

and substitute The whole y and y'' mess back in to get c_0 , c_n and d_n

$$\begin{aligned} & \left(\sum_{n=1}^{\infty} \frac{-c_n}{n^2\pi^2} \cos n\pi x - \frac{d_n}{n^2\pi^2} \sin n\pi x \right) + \left(c_0 + \sum_{n=1}^{\infty} [c_n \cos n\pi x + d_n \sin n\pi x] \right) \\ & = \frac{1}{3} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos n\pi x \end{aligned}$$

and match up terms

$$c_0 = \frac{1}{3}; \quad d_n = 0 \quad (\text{No sine terms allowed})$$

and we get

$$c_n \left(1 - \frac{1}{n^2\pi^2} \right) = \frac{4(-1)^n}{n^2\pi^2}$$

Continued on next page.

Multiplying through by $n^2\pi^2$

$$C_n(n^2\pi^2 - 1) = 4(-1)^n$$

$$C_n = \frac{4(-1)^n}{n^2\pi^2 - 1}$$

So the ~~the~~ General Particular Solution is

$$y = A \cos x + B \sin x + \frac{1}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{(n^2\pi^2 - 1)} \cos n\pi x$$

To plot, you need initial conditions.

This is your choice; but the way I'll do it is to say $A=B=0$, ~~there~~ so that we are basically plotting the particular solution.

NOTE This does NOT correspond to the initial conditions $y(0)=0$, $y'(0)=0$. I've arbitrarily chosen these to make my life easier.

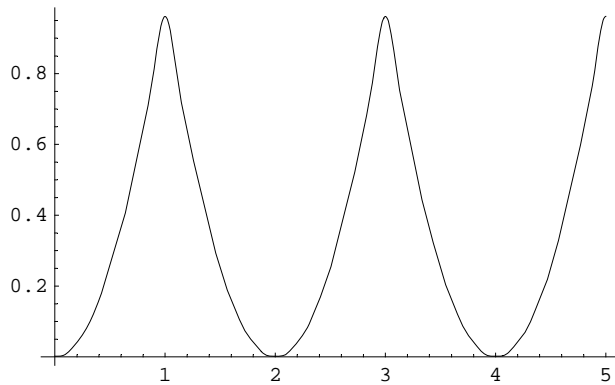
Define the input function.

$$\text{In}[3]:= \text{funcinput}[\mathbf{x}_-, \text{nmax}_-] := \frac{1}{3} + \sum_{n=1}^{\text{nmax}} \left(\frac{(-1)^n 4}{n^2 \pi^2} \text{Cos}[n \pi \mathbf{x}] \right)$$

Now setup the solution found for the problem

$$\text{In}[1]:= \mathbf{y}[\mathbf{x}_-, \text{nmax}_-, \mathbf{c1}_-, \mathbf{c2}_-] := \mathbf{c1} \text{Cos}[\mathbf{x}] + \mathbf{c2} \text{Sin}[\mathbf{x}] + \frac{1}{3} + \sum_{n=1}^{\text{nmax}} \left(\frac{(-1)^n 4}{n^2 \pi^2 - 1} \text{Cos}[n \pi \mathbf{x}] \right)$$

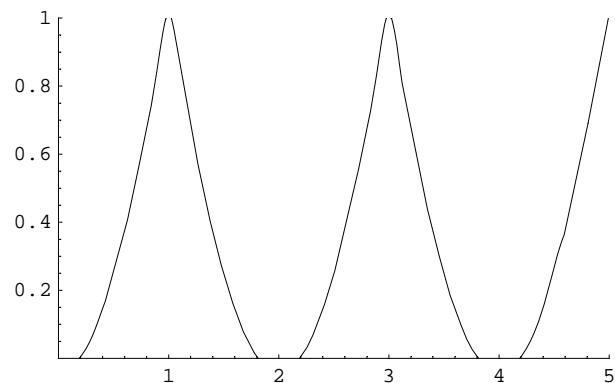
`In[14]:= inputfunctionplot = Plot[funcinput[x, 10], {x, 0, 5}]`



`Out[14]= - Graphics -`

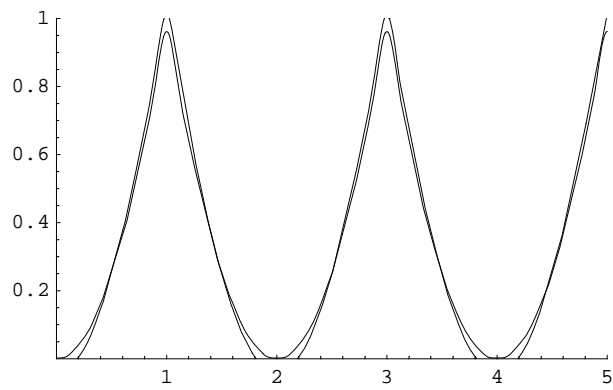
As I mentioned, I simply choose zero for the two constants, basically plotting the particular solution.

`In[22]:= outputplot = Plot[y[x, 10, 0, 0], {x, 0, 5}, PlotRange -> {{0, 5}, {0, 1}}]`



`Out[22]= - Graphics -`

```
In[18]:= Show[{outputplot, inputfunctionplot}]
```



```
Out[18]= - Graphics -
```