Introduction to Differential Forms in $\boldsymbol{R}^3$

1-form: $A_1\,dx^1 + A_2\,dx^2 + A_3\,dx^3$

2-form: $B_1\,dx^2 \wedge dx^3 + B_2\,dx^3 \wedge dx^1 + B_3\,dx^1 \wedge dx^2$

3-form: $C\,dx^1 \wedge dx^2 \wedge dx^3$

where A_i, B_i, and C are functions of x^1, x^2 and x^3.

The <u>wedge product</u> is antisymmetric: $dx^i \wedge dx^j = -dx^j \wedge dx^i \Rightarrow dx^i \wedge dx^i = 0$.

Example: Let α and β be 1-forms in $\boldsymbol{R}^2$; i.e. $\alpha = A_1\,dx^1 + A_2\,dx^2$, $\beta = B_1\,dx^1 + B_2\,dx^2$.

$$\begin{aligned}\alpha \wedge \beta &= A_1B_1\,dx^1 \wedge dx^1 + A_1B_2\,dx^1 \wedge dx^2 + A_2B_1\,dx^2 \wedge dx^1 + A_2B_2\,dx^2 \wedge dx^2\\ &= \left(A_1B_2 - A_2B_1\right)dx^1 \wedge dx^2\end{aligned}$$

If α is a *p*-form and β is a *q*-form then $\alpha \wedge \beta$ is a (*p*+*q*)-form.

Let ω be a *p*-form with typical term $A\,dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_p}$.

The <u>exterior derivative</u> $d\omega$ is the (*p*+1)-form with typical term $dA \wedge dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_p}$, where $dA = \partial_1 A\,dx^1 + \partial_2 A\,dx^2 + \partial_3 A\,dx^3$, $\left(\partial_i A = \partial A / \partial x^i\right)$. For example:

$$\begin{aligned}\omega &= B_1\,dx^1 + B_2\,dx^2 + B_3\,dx^3\\ d\omega &= dB_1 \wedge dx^1 + dB_2 \wedge dx^2 + dB_3 \wedge dx^3\\ &= \left(\partial_1 B_2 - \partial_2 B_1\right)dx^1 \wedge dx^2 + \left(\partial_2 B_3 - \partial_3 B_2\right)dx^2 \wedge dx^3 + \left(\partial_3 B_1 - \partial_1 B_3\right)dx^3 \wedge dx^1 \;(*)\end{aligned}$$

Generalized Integral Theorem

Let ω be a *p*-form with $d\omega$ as its (*p*+1)-form exterior derivative. Let *S* be a *p*+1 dimensional region (with ∂S as its *p*-dimensional boundary) on which the components of ω have continuous derivatives. Then:

$$\int_{\partial S} \omega = \int_S d\omega$$

e.g. with ω a 1-form and *S* a 2-dimensional region (a surface), then ∂S is a curve and $d\omega$ is given by (*) above. The resulting integral theorem can be written in vector notation as

$$\int_{\partial S} \vec{B} \cdot d\vec{l} = \int_S \nabla \times \vec{B} \cdot \hat{n}\,dA \quad \text{(Stokes' Theorem).}$$

Exercises: Let $\alpha = A_1\,dx^1 + A_2\,dx^2 + A_3\,dx^3$, $\beta = B_1\,dx^2 \wedge dx^3 + B_2\,dx^3 \wedge dx^1 + B_3\,dx^1 \wedge dx^2$.

1) Compute $\alpha \wedge \beta$.
2) Compute $d\beta$.
3) Show $d(d\alpha) = 0$ and $d(d\beta) = 0$.
4) Show $\int_{\partial S} \beta = \int_S d\beta$ gives the divergence theorem.
5) Restrict the discussion to $\boldsymbol{R}^2$ (no x^3 terms) and show that $\int_{\partial S} \alpha = \int_S d\alpha$ gives Green's theorem of the plane.