(* ME564,   HW 2 *)

(* 1– Pick 2 problems from HW#1 and use Mathematica to produce the solution.
2– Assign initial conditions as necessary,
and plot particular solutions corresponding to your results from problem #1.
3. For one of your examples,
solve again but this time obtain a numerical solution.(Look up NDSolve and use it.)
Make a plot of the numerical solution together with the analytic solution.What can you say about the accuracy of your numerical solution? *)

(* 1.1.13 *)

```
DSolve[y'[x] + 2*x*y[x] == 0, y[x], x]
```

Out[6]= $\{\{y[x] \to e^{-x^2} C[1]\}\}$

In[7]:=

```
psol1 = DSolve[{y'[x] + 2*x*y[x] == 0, y[1] == 1/e}, y, x]
```

Out[7]= $\{\{y \to \text{Function}[\{x\}, e^{-x^2}]\}\}$

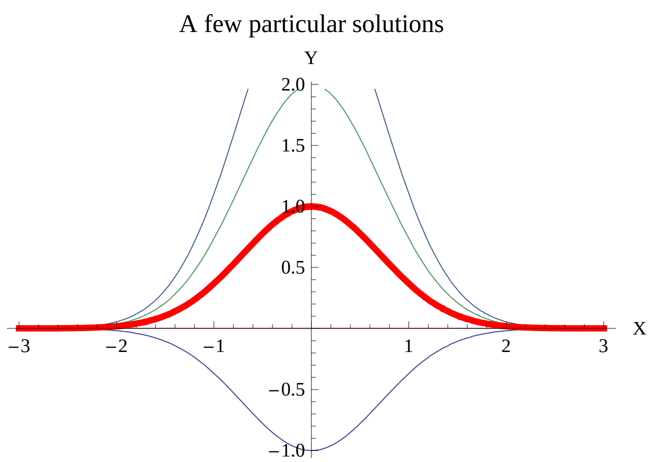
In[8]:= `sol1[x_] := Table[e^(-x^2) c, {c, -1, 3, 1}]`

In[9]:= `sol1plot = Plot[Evaluate[sol1[x]], {x, -3, 3}];`

In[10]:=

```
psol1plot =
  Plot[y[x] /. psol1, {x, -3, 3}, AxesLabel -> {x, y}, PlotStyle -> {Red, AbsoluteThickness[3]}];
```

In[68]:= `Show[sol1plot, psol1plot, AxesLabel -> {"X", "Y"}, PlotLabel -> "A few particular solutions"]`



In[12]:= `nsol1 = NDSolve[{y'[x] + 2*x*y[x] == 0, y[1] == 1/e}, y, {x, -3, 3}]`

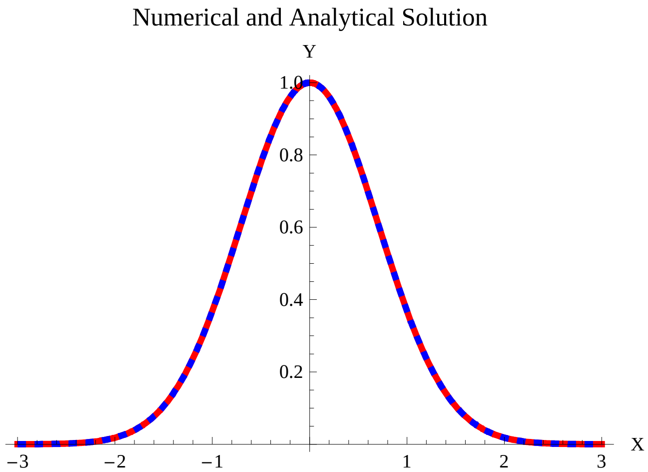
Out[12]= `{{y -> InterpolatingFunction[{{-3., 3.}}, <>]}}`

In[16]:=

```
nsol1plot = Plot[Evaluate[y[x] /. nsol1], {x, -3, 3},
  PlotRange -> All, PlotStyle -> {AbsoluteThickness[3], Blue, Dashed}];
```

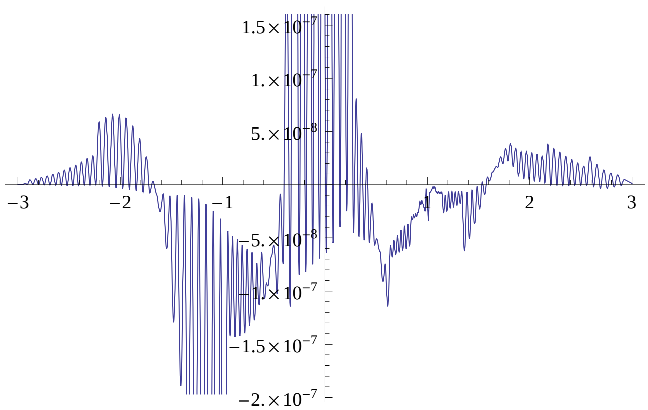
In[20]:=
```
Show[psol1plot, nsol1plot, AxesLabel → {"X", "Y"},
 PlotLabel → "Numerical and Analytical Solution"]
```

Out[20]=



In[67]:=
```
Plot[(y[x] /. psol1) - (y[x] /. nsol1), {x, -3, 3}]
```

Out[67]=




```
(*the accuracy of the numerical solution is very good. It
 differs from the analytic solution in the order of 10^-7 *)
```

(* 1.4.14 *)

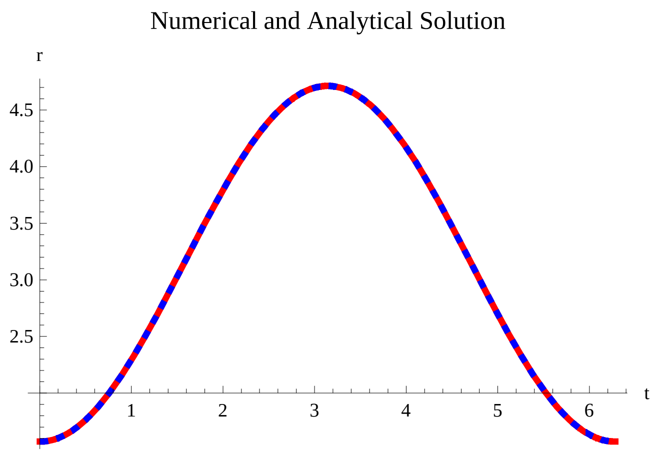
```
sol2 = DSolve[r'[t] == 0.5 * (r'[t] * Cos[t] + r[t] * Sin[t]), r[t], t]
psol2 = DSolve[{r'[t] == 0.5 * (r'[t] * Cos[t] + r[t] * Sin[t]), r[Pi / 2] == Pi}, r, t]
```

Out[56]= {{r[t] → C[1] (-2. + 1. Cos[t] + (0. + 0. ⅈ) Sin[t])}}

Out[57]= {{r → Function[{t}, 3.14159 - 1.5708 Cos[t] + (0. + 0. ⅈ) Sin[t]]}}

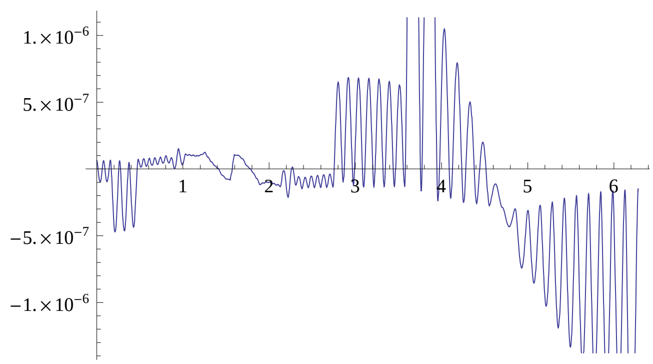
```
In[58]:= nsol2 = NDSolve[{r'[t] == 0.5 * (r'[t] * Cos[t] + r[t] * Sin[t]), r[Pi / 2] == Pi}, r, {t, 0, 2 * Pi}]

Out[58]= {{r → InterpolatingFunction[{{0., 6.28319}}, <>]}}

In[63]:= psol2plot =
           Plot[r[t] /. psol2, {t, 0, 2 * Pi}, AxesLabel → {t, r}, PlotStyle → {Red, AbsoluteThickness[3]}];

In[64]:= nsol2plot = Plot[Evaluate[r[t] /. nsol2], {t, 0, 2 * Pi},
            PlotRange → All, PlotStyle → {AbsoluteThickness[3], Blue, Dashed}];

In[65]:= Show[psol2plot, nsol2plot, AxesLabel → {"t", "r"},
          PlotLabel → "Numerical and Analytical Solution"]
```

Out[65]=




```
In[66]:= Plot[(r[t] /. psol2) - (r[t] /. nsol2), {t, 0, 2 * Pi}]
```

Out[66]=



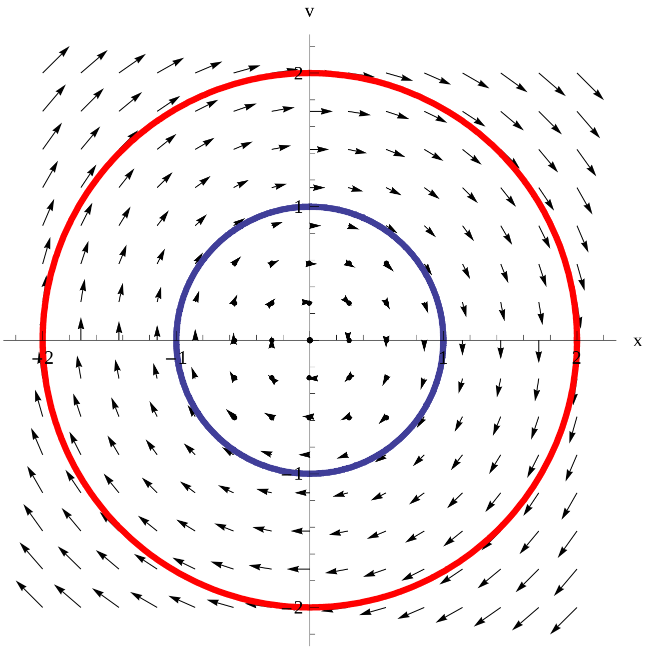
(* Problem 4 *)

(* Following the example discussed in class, plot the direction field for the equation for the following differential equation: $x'' - \epsilon(1-x^2)x' + x = 0$.

a) Start with the simpler (linear) case when $\epsilon=0$. Sketch a trajectory based on the direction field.What are the actual solutions of the linear equation and what do the trajectories look like in the phase plane? *)

```
(* general solution of x"+x=0 *)
DSolve[{x'[t] == v[t], v'[t] == -x[t]}, {x, v}, t]

{{x → Function[{t}, C[1] Cos[t] + C[2] Sin[t]], v → Function[{t}, C[2] Cos[t] - C[1] Sin[t]]}}

Needs["VectorFieldPlots`"]

dirPlot = VectorFieldPlot[{v, -x}, {x, -2, 2}, {v, -2, 2}];

(* one particular solution *)

partsol1 = DSolve[{x'[t] == v[t], v'[t] == -x[t], x[0] == 1, v[0] == 0}, {x, v}, t]

{{x → Function[{t}, Cos[t]], v → Function[{t}, -Sin[t]]}}

trajPlot1 = ParametricPlot[{x[t], v[t]} /. partsol1,
   {t, 0, 2*Pi}, AspectRatio → 1, PlotStyle → AbsoluteThickness[3]];

(* another particular solution *)

partsol2 = DSolve[{x'[t] == v[t], v'[t] == -x[t], x[0] == 2, v[0] == 0}, {x, v}, t]

{{x → Function[{t}, 2 Cos[t]], v → Function[{t}, -2 Sin[t]]}}

trajPlot2 = ParametricPlot[{x[t], v[t]} /. partsol2,
   {t, 0, 2*Pi}, AspectRatio → 1, PlotStyle → {Red, AbsoluteThickness[3]}];

Show[dirPlot, trajPlot1, trajPlot2, Axes → True, AxesLabel → {"x", "v"}]
```

```
(* in the phase plane the trajectories look like a set of circles *)
```



(* b) Now let ϵ=0.25.Again plot the direction field.Sketch some trajectories in the phase plane.Compare the behavior with the linear case.What qualitative change occurs?What does the solution look like?Does it depend on the initial conditions? *)

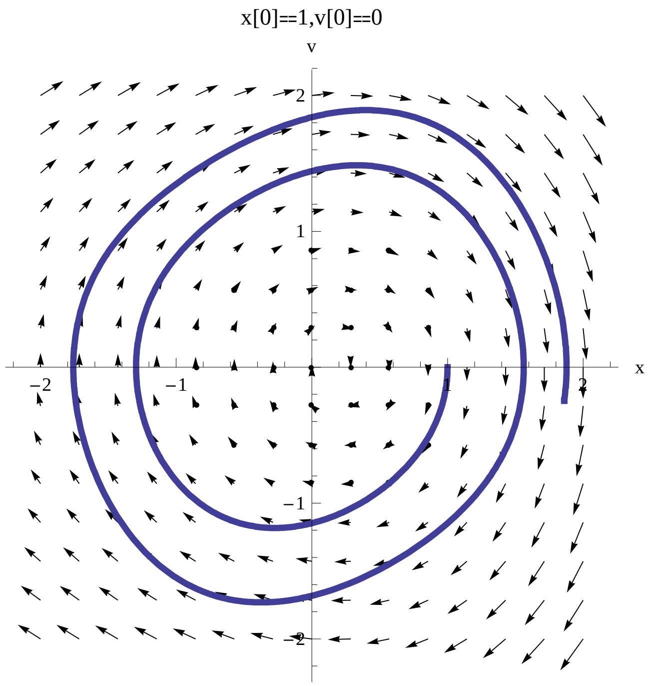
```
Clear[x, y]
numsol1 = NDSolve[
  {x'[t] == v[t], v'[t] == 0.25 * (1 - x[t]^2) * v[t] - x[t], x[0] == 1, v[0] == 0}, {x, v}, {t, 0, 4 * Pi}]
```

```
{{x → InterpolatingFunction[{{0., 12.5664}}, <>],
  v → InterpolatingFunction[{{0., 12.5664}}, <>]}}
```

```
numdirPlot1 = VectorFieldPlot[{v, 0.25 * (1 - x^2) - x}, {x, -2, 2}, {v, -2, 2}];
```

```
numtrajPlot1 = ParametricPlot[{x[t], v[t]} /. numsol1,
   {t, 0, 4 * Pi}, AspectRatio → 1, PlotStyle → AbsoluteThickness[3]];
```

```
Show[numdirPlot1, numtrajPlot1, Axes → True,
 AxesLabel → {"x", "v"}, PlotLabel → "x[0] == 1, v[0] == 0"]
```




```
Clear[x, v]
```

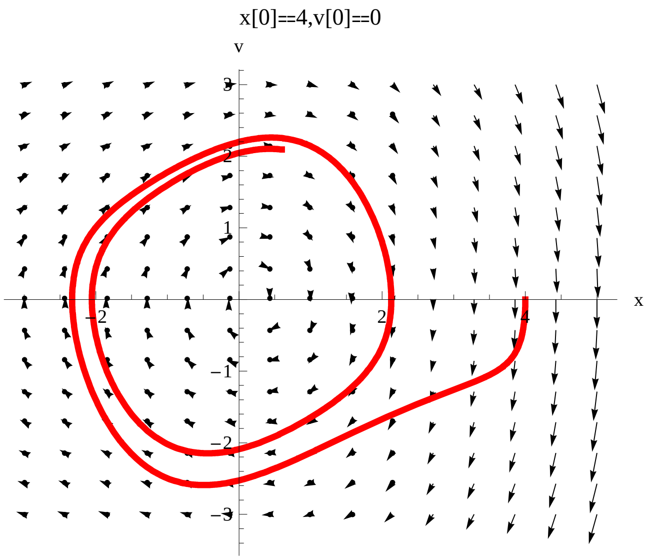
```
numsol2 = NDSolve[
  {x'[t] == v[t], v'[t] == 0.25 * (1 - x[t]^2) * v[t] - x[t], x[0] == 4, v[0] == 0}, {x, v}, {t, 0, 4 * Pi}]
```

```
{{x → InterpolatingFunction[{{0., 12.5664}}, <>],
  v → InterpolatingFunction[{{0., 12.5664}}, <>]}}
```

```
numdirPlot2 = VectorFieldPlot[{v, 0.25 * (1 - x^2) - x}, {x, -3, 5}, {v, -3, 3}];
```

```
numtrajPlot2 = ParametricPlot[{x[t], v[t]} /. numsol2,
   {t, 0, 4 * Pi}, AspectRatio → 1, PlotStyle → {Red, AbsoluteThickness[3]}];
```

```
Show[numdirPlot2, numtrajPlot2, Axes → True,
 AxesLabel → {"x", "v"}, PlotLabel → "x[0]==4,v[0]==0"]
```




```
Clear[x, v]
```

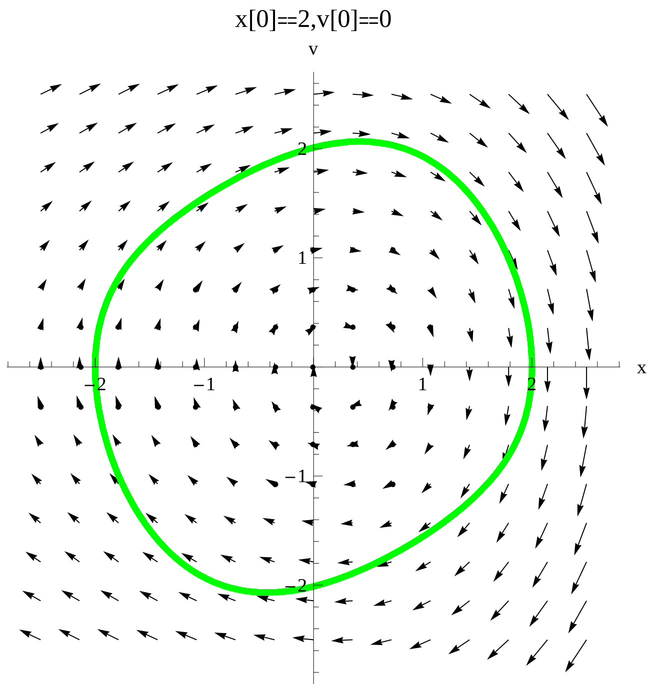
```
numsol3 = NDSolve[
  {x'[t] == v[t], v'[t] == 0.25 * (1 - x[t]^2) * v[t] - x[t], x[0] == 2, v[0] == 0}, {x, v}, {t, 0, 4 * Pi}]
```

```
{{x → InterpolatingFunction[{{0., 12.5664}}, <>],
  v → InterpolatingFunction[{{0., 12.5664}}, <>]}}
```

```
numdirPlot3 = VectorFieldPlot[{v, 0.25 * (1 - x^2) - x}, {x, -2.5, 2.5}, {v, -2.5, 2.5}];
```

```
numtrajPlot3 = ParametricPlot[{x[t], v[t]} /. numsol3,
   {t, 0, 4 * Pi}, AspectRatio → 1, PlotStyle → {Green, AbsoluteThickness[3]}];
```

```
Show[numdirPlot3, numtrajPlot3, Axes → True,
 AxesLabel → {"x", "v"}, PlotLabel → "x[0]==2,v[0]==0"]
```



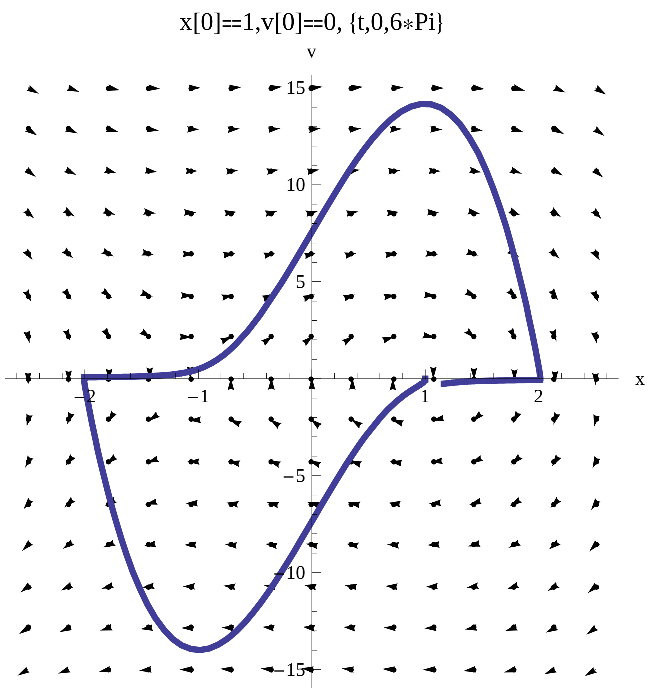

```
(* For weakly nonlinear case eps=0.25,
the solution trajectories in the phase plane start deviating from circle shape.
  For initial condition with amplitute less than 2, the trajectory grows
 toward a close curve that consists of 4 units in x and v (2 units from 0).
  For initial condition with amplitude more than 2, the trajectory circulates
 inward toward a close curve that consists of 4 units in x and v axes.
  For initial condition with amplitude equal to 2,
the trajectory is a closed curve with the size of 4 units in x and v axes as shown above.
  The results show that solutions for a long time
 do not depend on the initial conditions. For long time,
solutions differ from initial conditions seem to go toward the same trajectory. *)
```

(***

**c) Now let the nonlinearity be larger,
say ϵ=10. How does the direction field change?What does
the solution look like?Does it depend on initial conditions? *)**

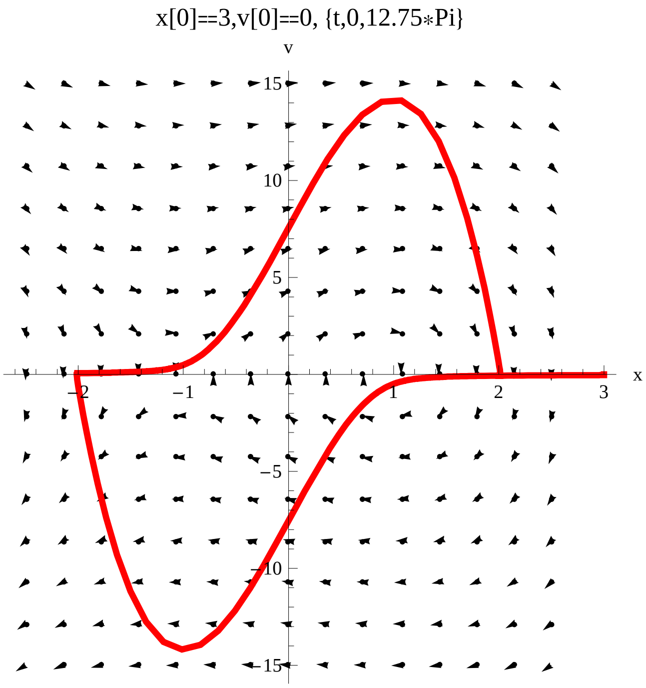
```
Clear[x, v]
```

```
numsol4c1 = NDSolve[
  {x'[t] == v[t], v'[t] == 10 * (1 - x[t]^2) * v[t] - x[t], x[0] == 1, v[0] == 0}, {x, v}, {t, 0, 6 * Pi}]

{{x → InterpolatingFunction[{{0., 18.8496}}, <>],
  v → InterpolatingFunction[{{0., 18.8496}}, <>]}}

numdirPlot4c1 = VectorFieldPlot[{v, 10 * (1 - x^2) - x}, {x, -2.5, 2.5}, {v, -15, 15}];

numtrajPlot4c1 =
  ParametricPlot[{x[t], v[t]} /. numsol4c1, {t, 0, 6 * Pi}, PlotStyle → AbsoluteThickness[3]];

Show[numdirPlot4c1, numtrajPlot4c1, Axes → True, AxesLabel → {"x", "v"},
 PlotLabel → "x[0]==1,v[0]==0, {t,0,6*Pi}", AspectRatio → 1]
```




```
Clear[x, v]

numsol4c2 = NDSolve[{x'[t] == v[t], v'[t] == 10 * (1 - x[t]^2) * v[t] - x[t], x[0] == 3, v[0] == 0},
  {x, v}, {t, 0, 12.75 * Pi}]

{{x → InterpolatingFunction[{{0., 40.0553}}, <>],
  v → InterpolatingFunction[{{0., 40.0553}}, <>]}}

numdirPlot4c2 = VectorFieldPlot[{v, 10 * (1 - x^2) - x}, {x, -2.5, 2.5}, {v, -15, 15}];

numtrajPlot4c2 = ParametricPlot[{x[t], v[t]} /. numsol4c2,
   {t, 0, 12.75 * Pi}, PlotStyle → {Red, AbsoluteThickness[3]}];
```

```
Show[numdirPlot4c2, numtrajPlot4c2, Axes → True, AxesLabel → {"x", "v"},
 PlotLabel → "x[0]==3,v[0]==0, {t,0,12.75*Pi}", AspectRatio → 1]
```




```
(* For the strong nonlinear equation,
the curve seems to have more "irregular" shape in the phase plane than from the weak
 nonlinear case. The trajectory does form a closed loop in the phase plane with
 a different period of time for different initial conditions, e.g. for x[0]=1,
it takes around 6*pi in t to close the loop while it takes more than 12*pi
  in t for the case of x[0]=3. However, for a long time, the trajectorie in
 phase plane of different initial conditions looks pretty much the same. *)
```

Is the given set a vector space? (Give a reason)

If yes, find the dimension and a basis.

7.9.1 All vectors in $\mathbb{R}^3$ satisfying $5v_1 - 3v_2 + 2v_3 = 0$

Sol$^{\underline{n}}$

$$5v_1 - 3v_2 + 2v_3 = 0 \qquad \text{—— (1)}$$

$$v_3 = \frac{3}{2}v_2 - \frac{5}{2}v_1$$

vector in $\mathbb{R}^3$ satisfying (1) $\Rightarrow \begin{bmatrix} v_1 \\ v_2 \\ \frac{3}{2}v_2 - \frac{5}{2}v_1 \end{bmatrix}$

Checking

1) Scalar multiplication ; let a = scalar

$$a\begin{bmatrix} v_1 \\ v_2 \\ \frac{3}{2}v_2 - \frac{5}{2}v_1 \end{bmatrix} = \begin{bmatrix} av_1 \\ av_2 \\ \frac{3}{2}av_2 - \frac{5}{2}av_1 \end{bmatrix}$$

$$= \begin{bmatrix} v_1' \\ v_2' \\ \frac{3}{2}v_2' - \frac{5}{2}v_1 \end{bmatrix} \Rightarrow \text{Still satisfy (1)}$$

$\therefore$ This set is closed in Scalar multiplication. *

2) Vector addition

$$\begin{bmatrix} v_1 \\ v_2 \\ \frac{3}{2}v_2 - \frac{5}{2}v_1 \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \\ \frac{3}{2}u_2 - \frac{5}{2}u_1 \end{bmatrix} = \begin{bmatrix} v_1 + u_1 \\ v_2 + u_2 \\ \frac{3}{2}(v_2 + u_2) - \frac{5}{2}(v_1 + u_1) \end{bmatrix}$$

$$= \begin{bmatrix} w_1 \\ w_2 \\ \frac{3}{2}w_2 - \frac{5}{2}w_1 \end{bmatrix} \; ; \; w_i = u_i + v_i \quad \Rightarrow \text{Still satisfy (1)}$$

$\therefore$ This set is closed in vector addition. **

$\therefore$ From (*) and (**), the given set is a vector space. Ans

This vector space is 2 dimension. Ans

A basis can be obtained by solving (1).

$$v_3 = \frac{3}{2}v_2 - \frac{5}{2}v_1$$

Arbitrary choose v_1 & v_2 such that $v_3 = 0$, we get

$$\underset{\sim}{b}_1 = \begin{bmatrix} 3 \\ 5 \\ 0 \end{bmatrix}$$ Ans

and $$v_2 = \frac{5}{3}v_1 + \frac{2}{3}v_3$$

Arbitrary choose v_1 & v_3 such that $v_2 = 0$, we get

$$\underset{\sim}{b}_2 = \begin{bmatrix} 2 \\ 0 \\ -5 \end{bmatrix}$$ Ans

---

7.9.5 All vectors in $\mathbb{R}^5$ with the first three components 0

Sol$^{\underline{n}}$ The given set is in the form $\begin{bmatrix} 0 \\ 0 \\ 0 \\ v_4 \\ v_5 \end{bmatrix}$

Checking

1) Scalar multiplication ; let a = scalar

$$a\begin{bmatrix} 0 \\ 0 \\ 0 \\ v_4 \\ v_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ av_4 \\ av_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ v_4' \\ v_5' \end{bmatrix}$$; $v_4' = av_4$, $v_5' = av_5$

$\Rightarrow$ Still satisfy the given condition.

2) Vector addition

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ v_4 \\ v_5 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ u_4 \\ u_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ v_4+u_4 \\ v_5+u_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ w_4 \\ w_5 \end{bmatrix}$$; $w_i = v_i + u_i$

$\Rightarrow$ Still satisfy the given condition.

∴ The given set is a vector space since it is closed in both scalar multiplication and vector addition. <u>Ans</u>

This is a 2D vector space <u>Ans</u>

Basis vectors are, e.g. <u>Ans</u>

$$\underset{\sim}{b}_1 = [0\ 0\ 0\ 1\ 0]^T$$
$$\underset{\sim}{b}_2 = [0\ 0\ 0\ 0\ 1]^T$$
<u>Ans</u>

<u>7.9.9</u> All polynomials with positive coefficients and degree 3 or less.

<u>Soln</u> Polynomial degree 3 or less : $a_3x^3 + a_2x^2 + a_1x + a_0$

a_is are positive.

Ck

1) Scalar multiplication ; let $c \equiv$ positive scalar value

$\Rightarrow -c \equiv$ negative

$$-c\,(a_3x^3 + a_2x^2 + a_1x + a_0) = -ca_3x^3 - ca_2x^2 - ca_1x - ca_0$$
$$= b_3x^3 + b_2x^2 + b_1x + b_0$$

$b_i = ca_i \equiv$ negative coefficient $\Rightarrow$ not belong to the given set.

∴ The given set is not closed in scalar multiplication, so it is not a vector space <u>Ans</u>

$\underset{\sim}{u} = \{1, 1, 1\}$; $\underset{\sim}{v} = \{1, 2, 3\}$; $\underset{\sim}{w} = \{3, 4, 12\}$

10 Compute the length of each vector

Sol^n

$$|\underset{\sim}{u}| = \sqrt{1^2 + 1^2 + 1^2}$$
$$= \sqrt{3} \quad \text{Ans}$$

$$|\underset{\sim}{v}| = \sqrt{1^2 + 2^2 + 3^2}$$
$$= \sqrt{14} \quad \text{Ans}$$

$$|\underset{\sim}{w}| = \sqrt{3^2 + 4^2 + 12^2}$$
$$= \sqrt{169} = 13 \quad \text{Ans}$$

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11 Are any of them parallel or perpendicular to one another?

Sol^n

If $\underset{\sim}{a} \cdot \underset{\sim}{b} = 0 \longrightarrow \underset{\sim}{a} \perp \underset{\sim}{b} \Rightarrow \theta = \frac{\pi}{2}$

If $|\underset{\sim}{a} \cdot \underset{\sim}{b}| = \pm|\underset{\sim}{a}||\underset{\sim}{b}| \longrightarrow \underset{\sim}{a} \,//\, \underset{\sim}{b} \Rightarrow \theta = 0, \pi$

$$\underset{\sim}{u} \cdot \underset{\sim}{v} = 1\cdot1 + 1\cdot2 + 1\cdot3$$
$$= 6$$

$\therefore$ $\underset{\sim}{u}$ and $\underset{\sim}{v}$ are neither parallel or perpendicular to one another. *

$$\underset{\sim}{u} \cdot \underset{\sim}{w} = 1\cdot3 + 1\cdot4 + 1\cdot12$$
$$= 19$$

$\therefore$ $\underset{\sim}{u}$ and $\underset{\sim}{w}$ are neither paraller or perpendicular to one another. *

$$\underset{\sim}{v} \cdot \underset{\sim}{w} = 1\cdot 3 + 2\cdot 4 + 3\cdot 12$$
$$= 47$$

$\therefore$ $\underset{\sim}{v}$ and $\underset{\sim}{w}$ are neither parallel or perpendicular to one another. *

$\Rightarrow$ None of them are parallel or perpendicular to one another. Ans

---

12 Create a unit vector parallel to each of the vectors.

Sol^{n} let $\hat{a}$ represents a unit vector parallel to $\underset{\sim}{a}$

$$\hat{u} = \frac{\underset{\sim}{u}}{|\underset{\sim}{u}|} = \frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix} \quad ; \ |\underset{\sim}{u}| = \sqrt{3} \text{ from } (10)$$

$$\hat{u} = \begin{bmatrix}1/\sqrt{3}\\1/\sqrt{3}\\1/\sqrt{3}\end{bmatrix}$$ Ans

$$\hat{v} = \frac{\underset{\sim}{v}}{|\underset{\sim}{v}|} = \frac{1}{\sqrt{14}}\begin{bmatrix}1\\2\\3\end{bmatrix} \quad ; \ |\underset{\sim}{v}| = \sqrt{14}$$

$$\hat{v} = \begin{bmatrix}1/\sqrt{14}\\2/\sqrt{14}\\3/\sqrt{14}\end{bmatrix}$$ Ans

$$\hat{w} = \frac{\underset{\sim}{w}}{|\underset{\sim}{w}|} = \frac{1}{13}\begin{bmatrix}3\\4\\12\end{bmatrix} \quad ; \ |\underset{\sim}{w}| = 13$$

$$\hat{w} = \begin{bmatrix}3/13\\4/13\\12/13\end{bmatrix}$$ Ans

---

13 Compute the projection of each vector onto the direction associated with the other vectors.

Sol$^{\underline{n}}$ Let $\underset{\sim}{p}_{mn}$ is a projection of $\underset{\sim}{m}$ onto $\underset{\sim}{n}$

$$\underset{\sim}{p}_{uv} = \frac{\underset{\sim}{u} \cdot \underset{\sim}{v}}{\underset{\sim}{v} \cdot \underset{\sim}{v}} \underset{\sim}{v}$$

$$= \frac{(1 \cdot 1 + 1 \cdot 2 + 1 \cdot 3)}{(1 \cdot 1 + 2 \cdot 2 + 3 \cdot 3)} \{1, 2, 3\} = \frac{6}{14} \{1, 2, 3\}$$

$$= \left\{ \frac{6}{14}, \frac{6}{7}, \frac{18}{14} \right\} \qquad \underline{\underline{\text{Ans}}}$$

$$\underset{\sim}{p}_{uw} = \frac{\underset{\sim}{u} \cdot \underset{\sim}{w}}{\underset{\sim}{w} \cdot \underset{\sim}{w}} \underset{\sim}{w}$$

$$= \frac{(1 \cdot 3 + 1 \cdot 4 + 1 \cdot 12)}{(3 \cdot 3 + 4 \cdot 4 + 12 \cdot 12)} \{3, 4, 12\} = \frac{19}{169} \{3, 4, 12\}$$

$$= \left\{ \frac{57}{169}, \frac{76}{169}, \frac{228}{169} \right\} \qquad \underline{\underline{\text{Ans}}}$$

$$\underset{\sim}{p}_{vu} = \frac{\underset{\sim}{v} \cdot \underset{\sim}{u}}{\underset{\sim}{u} \cdot \underset{\sim}{u}} \underset{\sim}{u}$$

$$= \frac{(1 \cdot 1 + 2 \cdot 1 + 3 \cdot 1)}{(1 \cdot 1 + 1 \cdot 1 + 1 \cdot 1)} \{1, 1, 1\} = 2 \{1, 1, 1\}$$

$$= \{2, 2, 2\} \qquad \underline{\underline{\text{Ans}}}$$

$$\underset{\sim}{p}_{vw} = \frac{\underset{\sim}{v} \cdot \underset{\sim}{w}}{\underset{\sim}{w} \cdot \underset{\sim}{w}} \underset{\sim}{w}$$

$$= \frac{(1 \cdot 3 + 2 \cdot 4 + 3 \cdot 12)}{(3 \cdot 3 + 4 \cdot 4 + 12 \cdot 12)} \{3, 4, 12\} = \frac{47}{169} \{3, 4, 12\}$$

$$= \left\{ \frac{141}{169}, \frac{188}{169}, \frac{564}{169} \right\} \qquad \underline{\underline{\text{Ans}}}$$

$$\underset{\sim}{p}_{wu} = \frac{\underset{\sim}{w}\cdot\underset{\sim}{u}}{\underset{\sim}{u}\cdot\underset{\sim}{u}}\,\underset{\sim}{u}$$

$$= \frac{(3\cdot 1 + 4\cdot 1 + 12\cdot 1)}{(1\cdot 1 + 1\cdot 1 + 1\cdot 1)}\{1,1,1\} = \frac{19}{3}\{1,1,1\}$$

$$= \left\{\frac{19}{3}, \frac{19}{3}, \frac{19}{3}\right\} \qquad \underline{\underline{\text{Ans}}}$$

$$\underset{\sim}{p}_{wv} = \frac{\underset{\sim}{w}\cdot\underset{\sim}{v}}{\underset{\sim}{v}\cdot\underset{\sim}{v}}\,\underset{\sim}{v}$$

$$= \frac{(3\cdot 1 + 4\cdot 2 + 12\cdot 3)}{(1\cdot 1 + 2\cdot 2 + 3\cdot 3)}\{1,2,3\} = \frac{47}{14}\{1,2,3\}$$

$$= \left\{\frac{47}{14}, \frac{47}{7}, \frac{141}{14}\right\} \qquad \underline{\underline{\text{Ans}}}$$

<u>14</u> Does the set of linear combinations of these vectors form a vector space? Why or why not? If so, what is the dimension of the space? Give a basis for the space.

<u>Sol^n</u> Let $\underset{\sim}{m}$ represents linear combinations of $\underset{\sim}{u}, \underset{\sim}{v}, \underset{\sim}{w}$

$$\underset{\sim}{m} = a\underset{\sim}{u} + b\underset{\sim}{v} + c\underset{\sim}{w} \quad ; a, b, c \text{ are scalar}$$

$$\underset{\sim}{m} = \begin{bmatrix} a + b + 3c \\ a + 2b + 4c \\ a + 3b + 12c \end{bmatrix}$$

1) Scalar multiplication ; d = scalar

$$d\underset{\sim}{m} = \begin{bmatrix} da + db + 3dc \\ da + 2db + 4dc \\ da + 3db + 12dc \end{bmatrix} = \begin{bmatrix} a' + b' + 3c' \\ a' + 2b' + 4c' \\ a' + 3b' + 12c' \end{bmatrix}$$

$\therefore$ Closure in scalar multiplication *

2) Vector addition

let $\underset{\sim}{n} = a'\underset{\sim}{u} + b'\underset{\sim}{v} + c'\underset{\sim}{w} \equiv$ linear combination of $\underset{\sim}{u}, \underset{\sim}{v}, \underset{\sim}{w}$

$$\underset{\sim}{n} = \begin{bmatrix} a' + b' + 3c' \\ a' + 2b' + 4c' \\ a' + 3b' + 12c' \end{bmatrix}$$

$$\underset{\sim}{m} + \underset{\sim}{n} = \begin{bmatrix} a+a' + b+b' + 3c+3c' \\ a+a' + 2b+2b' + 4c+4c' \\ a+a' + 3b+3b' + 12c+12c' \end{bmatrix}$$

$$= \begin{bmatrix} a_1 + b_1 + 3c_1 \\ a_1 + 2b_1 + 4c_1 \\ a_1 + 3b_1 + 12c_1 \end{bmatrix} \quad ; \quad \begin{aligned} a_1 &= a+a' \\ b_1 &= b+b' \\ c_1 &= c+c' \end{aligned}$$

$\therefore$ Closure in vector addition $**$

Thus, from (*) and (**), the set of linear combinations of $\underset{\sim}{u}$, $\underset{\sim}{v}$, and $\underset{\sim}{w}$ form a vector space. It satisfies vector space properties 1) and 2). Ans

This set is 3-dimension vector space Ans

Basis for the space, e.g. $\underset{\sim}{u}, \underset{\sim}{v}, \underset{\sim}{w}$ or

$\{0, 0, 1\}$, $\{0, 1, 0\}$, $\{1, 0, 0\}$

15 - Pick a random vector of length 3 and compute its projection onto $\underset{\sim}{u}, \underset{\sim}{v}, \underset{\sim}{w}$

- If you sum these projections, do you recover the original vector? Why or why not?
- How would you change your choice of vectors to avoid this difficulty.

Soln Let $\underset{\sim}{m} = \{3, 0, 0\}$

$$|\underset{\sim}{m}| = \sqrt{3^2 + 0^2 + 0^2} = 3$$

let $\underset{\sim}{p}_{mn}$ is a projection of $\underset{\sim}{m}$ onto $\underset{\sim}{n}$

$$\underset{\sim}{p}_{mu} = \frac{\underset{\sim}{m} \cdot \underset{\sim}{u}}{\underset{\sim}{u} \cdot \underset{\sim}{u}} \underset{\sim}{u}$$

$$= \frac{(3 \cdot 1 + 0 \cdot 1 + 0 \cdot 1)}{(1 \cdot 1 + 1 \cdot 1 + 1 \cdot 1)} \{1, 1, 1\} = \frac{3}{3}\{1, 1, 1\}$$

$$= \{1, 1, 1\} \quad \text{Ans}$$

$$\underset{\sim}{p}_{mv} = \frac{\underset{\sim}{m} \cdot \underset{\sim}{v}}{\underset{\sim}{v} \cdot \underset{\sim}{v}} \underset{\sim}{v}$$

$$= \frac{(3 \cdot 1 + 0 \cdot 2 + 0 \cdot 3)}{(1 \cdot 1 + 2 \cdot 2 + 3 \cdot 3)} \{1, 2, 3\} = \frac{3}{14}\{1, 2, 3\}$$

$$= \left\{\frac{3}{14}, \frac{3}{7}, \frac{9}{14}\right\} \quad \text{Ans}$$

$$\underset{\sim}{p}_{mw} = \frac{\underset{\sim}{m} \cdot \underset{\sim}{w}}{\underset{\sim}{w} \cdot \underset{\sim}{w}} \underset{\sim}{w}$$

$$= \frac{(3 \cdot 3 + 0 \cdot 4 + 0 \cdot 12)}{(3 \cdot 3 + 4 \cdot 4 + 12 \cdot 12)} \{3, 4, 12\}$$

$$= \frac{9}{169}\{3, 4, 12\} = \left\{\frac{27}{169}, \frac{36}{169}, \frac{108}{169}\right\} \quad \text{Ans}$$

$$\underset{\sim}{p}_{mu} + \underset{\sim}{p}_{mv} + \underset{\sim}{p}_{mw} = \left\{1 + \frac{3}{14} + \frac{27}{169},\ 1 + \frac{3}{7} + \frac{36}{169},\ 1 + \frac{9}{14} + \frac{108}{169}\right\}$$

$\underset{\sim}{p}_{mu} + \underset{\sim}{p}_{mv} + \underset{\sim}{p}_{mw} = \{1.374,\ 1.64\ ,\ 2.28\}$

$\therefore$ The original vector is not recovered since there are double counting due to nonorthogonal of $\underset{\sim}{u}$, $\underset{\sim}{v}$ and $\underset{\sim}{w}$ <u>Ans.</u>

To avoid this difficulty, we should choose $\underset{\sim}{u}'$, $\underset{\sim}{v}'$, $\underset{\sim}{w}'$ such that They are orthogonal. <u>Ans</u>