ME 564 - HW 1 - SOLUTION

1.1-13

* verify that y is a solution of the ODE
* determine from y the particular solution satisfying the initial condition
* sketch this solution

$$y' + 2xy = 0 \quad , \quad y = ce^{-x^2} \quad , \quad y(1) = \frac{1}{e}$$

SOLUTION:

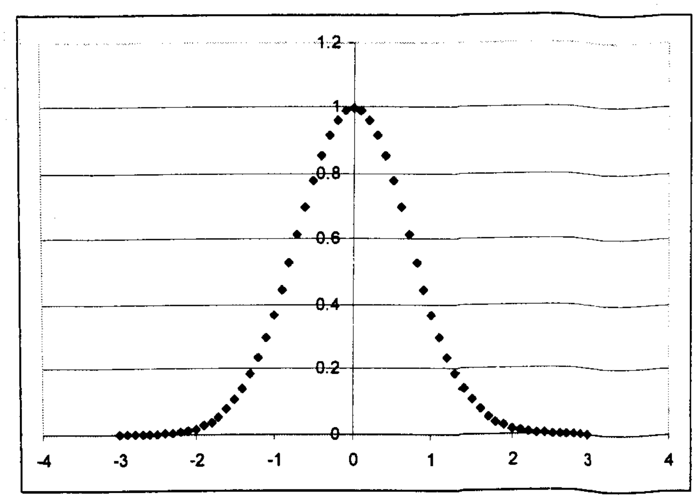
* $y = ce^{-x^2}$

$y' = -2xce^{-x^2}$

$-2xce^{-x^2} + 2x\left(ce^{-x^2}\right) = 0$ verified

* $y(1) = ce^{-1} = \frac{1}{e} \Rightarrow c = 1$

particular solution : $y = e^{-x^2}$



1.1-16

: Show that the ODE $y'^2 - xy' + y = 0$ has the general solution $y = cx - c^2$ and the singular solution $y = x^2/4$. Explain fig. 6

1) Substitute $y = cx - c^2$ and $y' = c$ in the ODE :

$c^2 - xc + (cx - c^2) = 0 \Rightarrow y = cx - c^2$ is the general solution of the above ODE

2) Substitute $y = x^2/4$ and $y' = \frac{x}{2}$ in the ODE ;

$\frac{x^2}{4} - \frac{x^2}{2} + \frac{x^2}{4} = 0 \Rightarrow y = \frac{x^2}{4}$ is the singular solution of the above ODE

3) Fig. 6 shows particular solutions of the ODE $y'^2 - xy' + y = 0$. All particular solutions are in forms of straight lines but the solution $y = x^2/4$ is in parabolic form. Thus $y = x^2/4$ is a solution of the given ODE but it doesn't belong to the family of the general solution and it's called **singular** solution.

$\boxed{1.3-18}$ L, R, b const

Find the particular solution (from the general sol.)

$$\frac{dr}{d\theta} = b\left[\frac{dr}{d\theta}\cos\theta + r\sin\theta\right], \quad r\left(\frac{\pi}{2}\right) = \pi, \quad 0 < b < 1$$

$$\frac{dr}{d\theta}\left[1 - b\cos\theta\right] = rb\sin\theta$$

$$\frac{dr}{r} = \frac{b\sin\theta\, d\theta}{1 - b\cos\theta} \Rightarrow \ln(r) = \ln(1 - b\cos\theta) + c^*$$

$$r = c(1 - b\cos\theta)$$

$$r\left(\frac{\pi}{2}\right) = c(1 - b\cdot 0) = \pi \Rightarrow c = \pi$$

$$\boxed{r = \pi(1 - b\cos\theta)}$$

$\boxed{1.3-32}$ From experiment, ΔS is proportional to S and small angle Δ with a proportionality constant of 0.15.

Find how many a rope must be wound around a bollard, so that a ma. can resist a force a thousand times he can exert.

$$\Delta S = 0.15\, S\, \Delta\phi$$

let $\Delta\phi \to 0$: $dS = 0.15\, S \cdot d\phi \to \int \frac{dS}{S} = \int 0.15\, d\phi$

$$\ln S = 0.15\,\phi + \tilde{C} \Rightarrow e^{\ln S} = e^{0.15\phi + \tilde{C}} \Rightarrow S = S_0\, e^{0.15\phi}, \quad S_0 = e^{\tilde{C}}$$

The angle should be so large that S equals 1000 times S_0.

Hence $\frac{S}{S_0} = 1000 = e^{0.15\phi} \Rightarrow \ln e^{0.15\phi} = \ln 1000 \to$

$$\phi = \frac{\ln 1000}{0.15} = 46 \text{ rad}$$

A bollard has a circular cross section $\to$ 2π rad/round

The rope must be wound $\frac{46}{2\pi} = 7.3 \approx$ <u>8 times</u>

$\boxed{1.4-1}$

$x^3 dx + y^3 dy = 0$

test for exactness : $M = x^3$, $N = y^3$

$$\frac{\partial M}{\partial y} = 0 \qquad \frac{\partial N}{\partial x} = 0 \quad \Rightarrow \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \rightarrow \text{ODE is exact} \quad \checkmark$$

since ODE is exact, its solution is in a form $u(x,y) = c$

$$\frac{\partial u}{\partial x} = M = x^3 \rightarrow u = \int x^3 dx + f(y) = \frac{x^4}{4} + f(y) \qquad (1)$$

$$\frac{\partial u}{\partial y} = N = y^3 \rightarrow u = \int y^3 dy + g(x) = \frac{y^4}{4} + g(x) \qquad (2)$$

from (1) and (2) we get $u(x,y) = \boxed{\frac{x^4}{4} + \frac{y^4}{4} = c}$ general solution

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$\boxed{1.4-11}$ (1.4-4 is on next page)

$-y\,dx + x\,dy = 0$

test for exactness : $\frac{\partial P}{\partial y} = \frac{\partial(-y)}{\partial y} = -1 \neq \frac{\partial Q}{\partial x} = \frac{\partial x}{\partial x} = 1$ ODE is not exact

let's find the integrating factor F, e.g. $F(x)$,

such that $\frac{\partial(FP)}{\partial y} = \frac{\partial(QF)}{\partial x}$

Since $F = F(x)$: $F\frac{\partial P}{\partial y} = \frac{\partial(QF)}{\partial x} \Rightarrow F\cdot(-1) = \frac{d(xF)}{dx} \Rightarrow$

$$-F = F + x\frac{dF}{dx} \Rightarrow x\frac{dF}{dx} = -2F \Rightarrow \frac{dF}{F} = -2\frac{dx}{x}$$

$\Rightarrow \ln F = -2\ln x + c^* \Rightarrow F = \frac{1}{x^2}$ (the integrating constant doesn't matter here since we multiply F on both sides of ODE)

then $-\frac{y}{x^2}dx + \frac{1}{x}dy = 0$ is an exact ODE

$$u = \int M dx + k(y) = \int -\frac{y}{x^2}dx + k(y) = \frac{y}{x} + k(y)$$

$$\frac{\partial u}{\partial y} = \frac{1}{x} + \frac{dk}{dy} = N = \frac{1}{x} \Rightarrow \frac{dk}{dy} = 0 \Rightarrow k = c^*$$

$u = \boxed{\frac{y}{x} = c} \Rightarrow$ general solution

$\boxed{1.4-4}$

$$\underbrace{(e^y - ye^x)}_{M}\,dx + \underbrace{(xe^y - e^x)}_{N}\,dy = 0$$

$$\frac{\partial M}{\partial y} = e^y - e^x = \frac{\partial N}{\partial x} = e^y - e^x \Rightarrow \text{ ODE is exact}$$

$$u = \int M\,dx + k(y) = \int (e^y - ye^x)\,dx + k(y) = xe^y - ye^x + k(y)$$

$$\frac{\partial u}{\partial y} = xe^y - e^x + \frac{dk}{dy} = N = xe^y - e^x \Rightarrow \frac{dk}{dy} = 0 \Rightarrow k = c^*$$

$$u(x,y) = \boxed{xe^y - ye^x = c} \quad \text{general solution}$$

$\boxed{1.4-14}$

$$\underbrace{(x^4 + y^2)}_{P}\,dx \underbrace{- xy}_{Q}\,dy = 0 \qquad y(2) = 1$$

$$\frac{\partial P}{\partial y} = 2y \neq \frac{\partial Q}{\partial x} = -y \Rightarrow \text{ not exact}$$

Find integrating factor :

$$R = \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = -\frac{1}{xy}(2y + y) = -\frac{3}{x} \Rightarrow \text{ R is a function of x onl}$$

$$\Rightarrow F(x) = \exp\int R\,dx = \exp(-3\ln x) = e^{\ln x^{-3}} = \frac{1}{x^3}$$

Multiply ODE by $F(x) = \frac{1}{x^3}$:

$$\underbrace{\frac{x^4 + y^2}{x^3}}_{M}\,dx - \underbrace{\frac{y}{x^2}}_{N}\,dy = 0$$

$$\frac{\partial M}{\partial y} = \frac{2y}{x^3}, \quad \frac{\partial N}{\partial x} = \frac{2y}{x^3} \Rightarrow \text{ ODE is exact}$$

$$u = \int N\,dy + k(x) = -\int \frac{y}{x^2}\,dy + k(x) = -\frac{y^2}{2x^2} + k(x)$$

$$\frac{\partial u}{\partial x} = \frac{y^2}{x^3} + \frac{dk}{dx} = M = \frac{x^4 + y^2}{x^3} \Rightarrow \frac{dk}{dx} = x \Rightarrow k(x) = \frac{x^2}{2} + c^*$$

$$u(x,y) = \boxed{-\frac{y^2}{2x^2} + \frac{x^2}{2} = c} \quad \text{general solution}$$

$$y(2) = 1 : \quad u(2,1) = -\frac{1}{8} + \frac{4}{2} = c = \frac{15}{8} \Rightarrow -\frac{y^2}{x^2} + x^2 = \frac{15}{4} \text{ or } \boxed{x^4 - y^2 - \frac{15}{4}x^2 = 0}$$

particular solution

$\boxed{1.4-18}$

$$\underbrace{\left(\cos xy + \frac{x}{y}\right)}_{P} dx + \left(1+ \underbrace{\frac{x}{y}\cos xy\right)}_{Q} dy = 0$$

$$\left.\begin{array}{l} \dfrac{\partial P}{\partial y} = -x \sin xy - \dfrac{x}{y^2} \\[2ex] \dfrac{\partial Q}{\partial x} = \dfrac{\cos xy}{y} \cancel{+y} \dfrac{x}{\cancel{y}} \sin xy \end{array}\right] \rightarrow \text{ODE is not exact}$$

let's use theorem 1 :

$$R = \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = \frac{\cancel{-x \sin xy} - \frac{x}{y^2} - \frac{\cos xy}{y} + \cancel{x \sin xy}}{1 + \frac{x}{y}\cos xy} = -\frac{1}{x} \, \frac{\frac{x}{y} + \cos xy}{\frac{y}{x} + \cos xy}$$

R fails because it's a function of both x and y

let's use theorem 2 :

$$R^* = \frac{1}{P}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) = \frac{\frac{\cos xy}{y} \cancel{- x \sin xy} \cancel{+ x \sin xy} + \frac{x}{y^2}}{\cos xy + \frac{x}{y}} = \frac{1}{y} \, \frac{\cos xy + \frac{x}{y}}{\cos xy + \frac{x}{y}} = \frac{1}{y}$$

$$F^*(y) = \exp \int R^* \, dy = \exp \int \frac{1}{y}\, dy = \exp(\ln y) = y$$

this gives the exact ODE : $F^* P \, dx + F^* Q \, dy = 0$

$$(\underbrace{y \cos xy + x}_{M}) \, dx + (\underbrace{y + x \cos xy}_{N}) \, dy = 0$$

$$\frac{\partial M}{\partial y} = \cos xy - xy \sin xy \quad , \quad \frac{\partial N}{\partial x} = \cos xy - xy \sin xy \quad \Rightarrow \text{exact ODE}$$

let's integrate M with respect to x :

$$u = \int M \, dx + k(y) = \int (y \cos xy + x) \, dx + k(y) = \sin xy + \frac{x^2}{2} + k(y)$$

$$\frac{\partial u}{\partial y} = x \cos xy + \frac{dk}{dy} = N = y + x \cos xy \Rightarrow \frac{dk}{dy} = y \Rightarrow k = \frac{1}{2} y^2 + c^*$$

$$u(x,y) = \boxed{\sin xy + \frac{x^2}{2} + \frac{y^2}{2} = c} \quad \text{general solution}$$

$\boxed{1.5-1}$ show that $e^{-\ln x} = \frac{1}{x}$ and $e^{-\ln(\sec x)} = \cos x$

1) $e^{-\ln x} = e^{\ln x^{-1}} = x^{-1} = \frac{1}{x}$

2) $e^{-\ln(\sec x)} = e^{\ln(\sec x)^{-1}} = \frac{1}{\sec x} = \cos x$