Solve by Cramer's rule and check by Gauss elimination and back substitution. (Show details.)

7.7.18

$$2x - 5y = 23$$
$$4x + 6y = -2$$

Solⁿ

$$\begin{bmatrix} 2 & -5 \\ 4 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 23 \\ -2 \end{bmatrix}$$

$$x = \frac{\begin{vmatrix} 23 & -5 \\ -2 & 6 \end{vmatrix}}{\begin{vmatrix} 2 & -5 \\ 4 & 6 \end{vmatrix}} = \frac{23 \times 6 - (-5)(-2)}{2 \times 6 - (-5)(4)} = \frac{128}{32} = 4$$

$$y = \frac{\begin{vmatrix} 2 & 23 \\ 4 & -2 \end{vmatrix}}{32} = \frac{2(-2) - 23 \times 4}{32} = \frac{-96}{32} = -3$$

Ans

Check by Gauss elimination

$$\left[\begin{array}{cc|c} 2 & -5 & 23 \\ 4 & 6 & -2 \end{array}\right] \begin{array}{l} R_1 \\ \frac{R_2}{2} - R_1 \end{array} \left[\begin{array}{cc|c} 2 & -5 & 23 \\ 0 & 8 & -24 \end{array}\right]$$

Back substitution

$$8y = -24 \longrightarrow y = -3$$
$$2x - 5y = 23 \longrightarrow x = \frac{23 + 5(-3)}{2} = 4$$

Ans

7.7.19

$$3y + 4z = 14.8$$
$$4x + 2y - z = -6.3$$
$$x - y + 5z = 13.5$$

Sol:

$$\begin{bmatrix} 0 & 3 & 4 \\ 4 & 2 & -1 \\ 1 & -1 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 14.8 \\ -6.3 \\ 13.5 \end{bmatrix}$$

$$x = \frac{\begin{vmatrix} 14.8 & 3 & 4 \\ -6.3 & 2 & -1 \\ 13.5 & -1 & 5 \end{vmatrix}}{\begin{vmatrix} 0 & 3 & 4 \\ 4 & 2 & -1 \\ 1 & -1 & 5 \end{vmatrix}} = \frac{14.8(10-1) - (-6.3)(15+4) + 13.5(-3-8)}{0(10-1) - 4(15+4) + 1(-3-8)}$$

$$= \frac{104.4}{-87} = -1.2$$

$$y = \frac{\begin{vmatrix} 0 & 14.8 & 4 \\ 4 & -6.3 & -1 \\ 1 & 13.5 & 5 \end{vmatrix}}{-87} = \frac{-14.8(20+1) + (-6.3)(0-4) - 13.5(0-16)}{-87}$$

$$= \frac{-69.6}{-87} = 0.8$$

$$z = \frac{\begin{vmatrix} 0 & 3 & 14.8 \\ 4 & 2 & -6.3 \\ 1 & -1 & 13.5 \end{vmatrix}}{-87} = \frac{14.8(-4-2) - (-6.3)(0-3) + 13.5(0-12)}{-87}$$

$$= \frac{-269.7}{-87} = 3.1$$

$\therefore\ x = -1.2,\ y = 0.8,\ z = 3.1$ Ans

Check by Gauss elimination

$$\left[\begin{array}{ccc|c} 0 & 3 & 4 & 14.8 \\ 4 & 2 & -1 & -6.3 \\ 1 & -1 & 5 & 13.5 \end{array}\right] \xrightarrow[R_1 \leftrightarrow R_2]{4R_3 - R_2 \text{ interchange}} \left[\begin{array}{ccc|c} 4 & 2 & -1 & -6.3 \\ 0 & 3 & 4 & 14.8 \\ 0 & -6 & 21 & 60.3 \end{array}\right] \begin{array}{c} R_1 \\ R_2 \\ \frac{R_3}{2} + R_2 \end{array} \longrightarrow \left[\begin{array}{ccc|c} 4 & 2 & -1 & -6.3 \\ 0 & 3 & 4 & 14.8 \\ 0 & 0 & 14.5 & 44.95 \end{array}\right]$$

Back substitution

$$14.5z = 44.95 \rightarrow z = 3.1$$
$$3y + 4z = 14.8 \rightarrow y = \frac{14.8 - 4(3.1)}{3} = 0.8$$
$$4x + 2y - z = -6.3 \rightarrow x = \frac{-6.3 + 3.1 - 2(0.8)}{4} = -1.2$$

Ans

7.2.24

7.7.24) Geometrical Applications: curves and surfaces through given points — The idea is to get an equation from the vanishing of the determinant of a homogeneous linear system as the condition for a nontrivial solution in Cramer's theorem. We explain the trick for obtaining such a system for the case of a line L through 2 given points $P_1: (x_1, y_1)$ and $P_2\ (x_2, y_2)$. The unknown line is $ax+by=-c$, say. We write it as $ax+by+c=0$. To get a nontrivial solution a, b, c, the determinant of the "coefficients" $x, y, 1$ must be zero. The system is:

$$\begin{aligned} ax + by + c\cdot 1 &= 0 \quad (\text{line } L) \\ ax_1 + by_1 + c\cdot 1 &= 0 \quad (P_1 \text{ on } L) \\ ax_2 + by_2 + c\cdot 1 &= 0 \quad (P_2 \text{ on } L) \end{aligned}$$

a) Line through 2 points. Derive from $D=0$ in (12) the familiar formula:

$$\frac{x-x_1}{x_1-x_2} = \frac{y-y_1}{y_1-y_2}$$

<u>matrix form</u>:

$$\begin{bmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Downarrow$

let this $= D$

$$\begin{bmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{bmatrix} \xrightarrow[\text{Row 2} - \text{Row 3}]{\text{Row 1} - \text{Row 2}} \begin{bmatrix} x-x_1 & y-y_1 & 0 \\ x_1-x_2 & y_1-y_2 & 0 \\ x_2 & y_2 & 1 \end{bmatrix} = D$$

↳ expand using this column

<u>Determinant of D</u>:

$$D = 0 = 0\begin{vmatrix} x_1-x_2 & y_1-y_2 \\ x_2 & y_2 \end{vmatrix} - 0\begin{vmatrix} x-x_1 & y-y_1 \\ x_2 & y_2 \end{vmatrix} + (1)\begin{vmatrix} x-x_1 & y-y_1 \\ x_1-x_2 & y_1-y_2 \end{vmatrix}$$

$$0 = (x-x_1)(y_1-y_2) - (x_1-x_2)(y-y_1)$$

$$(x_1-x_2)(y-y_1) = (x-x_1)(y_1-y_2)$$

$$\boxed{\left(\frac{y-y_1}{y_1-y_2}\right) = \left(\frac{x-x_1}{x_1-x_2}\right)}$$

✓

b) Plane. Find the analog of (12) for a plane through three given points. Apply it when the points are $(1,1,1)$, $(3,2,6)$, $(5,0,5)$:

- general equation for a plane: $ax + by + cz + d = 0$

$$\Downarrow$$

$$\begin{aligned} ax_1 + by_1 + cz_1 + d &= 0 \\ ax_2 + by_2 + cz_2 + d &= 0 \\ ax_3 + by_3 + cz_3 + d &= 0 \end{aligned}$$

unknown plane $ax + by + cz + d$

$$\Downarrow$$

$$\begin{bmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{bmatrix} = D = \begin{bmatrix} x & y & z & 1 \\ 1 & 1 & 1 & 1 \\ 3 & 2 & 6 & 1 \\ 5 & 0 & 5 & 1 \end{bmatrix}$$

- Find det(D):

$$\underbrace{x\begin{vmatrix} 1 & 1 & 1 \\ 2 & 6 & 1 \\ 0 & 5 & 1 \end{vmatrix}}_{(i)} - \underbrace{y\begin{vmatrix} 1 & 1 & 1 \\ 3 & 6 & 1 \\ 5 & 5 & 1 \end{vmatrix}}_{(ii)} + \underbrace{z\begin{vmatrix} 1 & 1 & 1 \\ 3 & 2 & 1 \\ 5 & 0 & 1 \end{vmatrix}}_{(iii)} - \underbrace{1\begin{vmatrix} 1 & 1 & 1 \\ 3 & 2 & 6 \\ 5 & 0 & 5 \end{vmatrix}}_{(iv)}$$

(i) $x\left(0\begin{vmatrix} 1 & 1 \\ 6 & 1 \end{vmatrix} - 5\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} + 1\begin{vmatrix} 1 & 1 \\ 2 & 6 \end{vmatrix}\right) = \left(-5(1-2) + (6-2)\right)x = \underline{\underline{9x}}$

(ii) $-y\left(1\begin{vmatrix} 6 & 1 \\ 5 & 1 \end{vmatrix} - 1\begin{vmatrix} 3 & 1 \\ 5 & 1 \end{vmatrix} + 1\begin{vmatrix} 3 & 6 \\ 5 & 5 \end{vmatrix}\right) = -y\left((6-5) - (3-5) + (15-30)\right) = \underline{\underline{12y}}$

(iii) $z\left(5\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} - 0\begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} + 1\begin{vmatrix} 1 & 1 \\ 3 & 2 \end{vmatrix}\right) = z\left(5(1-2) + 0 + (2-3)\right) = \underline{\underline{-6z}}$

(iv) $1\left(5\begin{vmatrix} 1 & 1 \\ 2 & 6 \end{vmatrix} - 0\begin{vmatrix} 1 & 1 \\ 3 & 6 \end{vmatrix} + 5\begin{vmatrix} 1 & 1 \\ 3 & 2 \end{vmatrix}\right) = \left(5(6-2) + 0 + 5(2-3)\right) = \underline{\underline{-15}}$

- Equation of plane:

$$\left(9x + 12y - 6z - 15 = 0\right)\tfrac{1}{3}$$

$$3x + 4y - 2z - 5 = 0$$

$$\boxed{3x + 4y - 2z = 5}$$

Find the eigenvalues and eigenvectors of the following matri
(Use the given λ or factors.)

8.1.5 $\begin{bmatrix} 5 & -2 \\ 9 & -6 \end{bmatrix}$

Sol^n $\begin{vmatrix} 5-\lambda & -2 \\ 9 & -6-\lambda \end{vmatrix} = (5-\lambda)(-6-\lambda) - (-2)(9)$

$= -30 + \lambda + \lambda^2 + 18$

Characteristic equation : $\lambda^2 + \lambda - 12 = 0$

$(\lambda + 4)(\lambda - 3) = 0$

The eigenvalues are $\lambda_1 = -4$, $\lambda_2 = 3$. Ans

For $\lambda_1 = -4$, the characteristic matrix is

$\begin{bmatrix} 5+4 & -2 \\ 9 & -6+4 \end{bmatrix} = \begin{bmatrix} 9 & -2 \\ 9 & -2 \end{bmatrix}$ row reduces to $\xrightarrow{R_2 - R_1}$ $\begin{bmatrix} 9 & -2 \\ 0 & 0 \end{bmatrix}$

$9v_1 - 2v_2 = 0$

$v_2 = \frac{9v_1}{2}$

Choose $v_1 = 2$, eigenvector is $\underset{\sim}{v}^{(1)} = \begin{bmatrix} 2 \\ 9 \end{bmatrix}$ Ans

For $\lambda_2 = 3$,

$\begin{bmatrix} 5-3 & -2 \\ 9 & -6-3 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ 9 & -9 \end{bmatrix} \xrightarrow[R_2 - \frac{9}{2}R_1]{R_1} \begin{bmatrix} 2 & -2 \\ 0 & 0 \end{bmatrix}$

$2v_1 - 2v_2 = 0$

$v_2 = v_1$

Choose $v_1 = 1$, eigenvector is $\underset{\sim}{v}^{(2)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ Ans

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8.1.9 $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

Sol^n $\begin{vmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{vmatrix} = (1-\lambda)(4-\lambda) - 2\times 2$

$= 4 - 5\lambda + \lambda^2 - 4$

Characteristic equation : $\lambda^2 - 5\lambda = 0$

$\lambda(\lambda - 5) = 0$

The eigenvalues are $\lambda_1 = 0$, $\lambda_2 = 5$. Ans

For $\lambda_1 = 0$

$$\begin{bmatrix} 1-0 & 2 \\ 2 & 4-0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \xrightarrow[R_2 - 2R_1]{R_1} \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

$V_1 + 2V_2 = 0$

$V_2 = \frac{-V_1}{2}$

Choose $V_1 = -2$, the eigenvector is $\underset{\sim}{v}^{(1)} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$. Ans

For $\lambda_2 = 5$

$$\begin{bmatrix} 1-5 & 2 \\ 2 & 4-5 \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \xrightarrow[2R_2 + R_1]{R_1} \begin{bmatrix} -4 & 2 \\ 0 & 0 \end{bmatrix}$$

$-4V_1 + 2V_2 = 0$

$V_2 = 2V_1$

Choose $v_1 = 1$, the eigenvector is $\underset{\sim}{v}^{(2)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Ans

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8.1.13 $\begin{bmatrix} 6 & 2 & -2 \\ 2 & 5 & 0 \\ -2 & 0 & 7 \end{bmatrix}$, $\lambda = 3$

Sol^n $\begin{vmatrix} 6-\lambda & 2 & -2 \\ 2 & 5-\lambda & 0 \\ -2 & 0 & 7-\lambda \end{vmatrix} = (5-\lambda)\left[(6-\lambda)(7-\lambda) - (-2)(-2)\right] - 2(7-\lambda)2$

$= -\lambda^3 + 18\lambda^2 - 99\lambda + 162$

Characteristic equation : $\lambda^3 - 18\lambda^2 + 99\lambda - 162 = 0$

$(\lambda - 3)(\lambda^2 - 15\lambda + 54) = 0$

$$(\lambda-3)(\lambda-6)(\lambda-9) = 0$$

The eigenvalues are $\lambda_1 = 3$, $\lambda_2 = 6$, $\lambda_3 = 9$. <u>Ans</u>

<u>For $\lambda_1 = 3$</u>

$$\begin{bmatrix} 6-3 & 2 & -2 \\ 2 & 5-3 & 0 \\ -2 & 0 & 7-3 \end{bmatrix} = \begin{bmatrix} 3 & 2 & -2 \\ 2 & 2 & 0 \\ -2 & 0 & 4 \end{bmatrix} \xrightarrow{\substack{R_1 \\ R_2+R_3 \\ R_3+\frac{2}{3}R_1}} \begin{bmatrix} 3 & 2 & -2 \\ 0 & 2 & 4 \\ 0 & 4/3 & 8/3 \end{bmatrix}$$

$$\xrightarrow{\substack{R_1 \\ R_2 \\ \frac{3}{2}R_3-R_2}} \begin{bmatrix} 3 & 2 & -2 \\ 0 & 2 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

$$2V_2 + 4V_3 = 0 \Rightarrow V_2 = -2V_3$$

$$3V_1 + 2V_2 - 2V_3 = 0 \Rightarrow V_1 = \frac{2V_3 - 2V_2}{3} = \frac{2V_3 - 2(-2V_3)}{3}$$

$$= 2V_3$$

Choose $V_3 = 1$, the eigenvector is $\underset{\sim}{V}^{(1)} = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$. <u>Ans</u>

<u>For $\lambda_2 = 6$</u>

$$\begin{bmatrix} 6-6 & 2 & -2 \\ 2 & 5-6 & 0 \\ -2 & 0 & 7-6 \end{bmatrix} = \begin{bmatrix} 0 & 2 & -2 \\ 2 & -1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \xrightarrow{\substack{R_1' = R_2 \\ R_2' = R_1 \\ R_3' = R_3}} \begin{bmatrix} 2 & -1 & 0 \\ 0 & 2 & -2 \\ -2 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{\substack{R_1 \\ R_2 \\ R_3+R_1}} \begin{bmatrix} 2 & -1 & 0 \\ 0 & 2 & -2 \\ 0 & -1 & 1 \end{bmatrix} \xrightarrow{\substack{R_1 \\ R_2 \\ 2R_3+R_2}} \begin{bmatrix} 2 & -1 & 0 \\ 0 & 2 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$2V_2 - 2V_3 = 0 \Rightarrow V_2 = V_3$$

$$2V_1 - V_2 = 0 \Rightarrow V_1 = \frac{V_2}{2} = \frac{V_3}{2}$$

Choose $V_3 = 2$, the eigenvector is $\underset{\sim}{V}^{(2)} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$. <u>Ans</u>

<u>For $\lambda_3 = 9$</u>

$$\begin{bmatrix} 6-9 & 2 & -2 \\ 2 & 5-9 & 0 \\ -2 & 0 & 7-9 \end{bmatrix} = \begin{bmatrix} -3 & 2 & -2 \\ 2 & -4 & 0 \\ -2 & 0 & -2 \end{bmatrix} \xrightarrow{\substack{R_1 \\ R_2+R_3 \\ R_3-\frac{2}{3}R_1}} \begin{bmatrix} -3 & 2 & -2 \\ 0 & -4 & -2 \\ 0 & -4/3 & -2/3 \end{bmatrix}$$

$$\begin{array}{c} R_1 \\ R_2 \\ 3R_3 - R_2 \\ \longrightarrow \end{array} \begin{bmatrix} -3 & 2 & -2 \\ 0 & -4 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$-4V_2 - 2V_3 = 0 \Rightarrow V_2 = -\frac{V_3}{2}$$

$$-3V_1 + 2V_2 - 2V_3 = 0 \Rightarrow V_1 = \frac{2V_2 - 2V_3}{3} = \frac{-V_3 - 2V_3}{3} = -V_3$$

Choose $V_3 = -2$, the eigenvector is $\underset{\sim}{v}^{(3)} = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$. **Ans**

8.1.14 $\begin{bmatrix} 2 & 0 & -2 \\ 0 & 0 & -2 \\ -2 & -2 & 1 \end{bmatrix}$, $\lambda = 1$

Sol^n

$$\begin{vmatrix} 2-\lambda & 0 & -2 \\ 0 & -\lambda & -2 \\ -2 & -2 & 1-\lambda \end{vmatrix} = -\lambda\left[(2-\lambda)(1-\lambda)-4\right] + 2(-2(2-\lambda))$$
$$= -\lambda^3 + 3\lambda^2 + 6\lambda - 8$$

Characteristic equation: $\lambda^3 - 3\lambda^2 - 6\lambda + 8 = 0$

$$(\lambda - 1)(\lambda^2 - 2\lambda - 8) = 0$$

$$(\lambda - 1)(\lambda - 4)(\lambda + 2) = 0$$

The eigenvalues are $\lambda_1 = -2$, $\lambda_2 = 1$, $\lambda_3 = 4$. **Ans**

For $\lambda_1 = -2$

$$\begin{bmatrix} 2+2 & 0 & -2 \\ 0 & 2 & -2 \\ -2 & -2 & 1+2 \end{bmatrix} = \begin{bmatrix} 4 & 0 & -2 \\ 0 & 2 & -2 \\ -2 & -2 & 3 \end{bmatrix} \begin{array}{c} R_1 \\ R_2 \\ 2R_3 + R_1 \\ \longrightarrow \end{array} \begin{bmatrix} 4 & 0 & -2 \\ 0 & 2 & -2 \\ 0 & -4 & 4 \end{bmatrix} \begin{array}{c} R_1 \\ R_2 \\ R_3 + 2R_2 \\ \longrightarrow \end{array} \begin{bmatrix} 4 & 0 & -2 \\ 0 & 2 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$2V_2 - 2V_3 = 0 \Rightarrow V_2 = V_3$$

$$4V_1 - 2V_3 = 0 \Rightarrow V_1 = \frac{V_3}{2}$$

Choose $V_3 = 2$, the eigenvector is $\underset{\sim}{v}^{(1)} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$. **Ans**

<u>For $\lambda = 1$</u>

$$\begin{bmatrix} 2-1 & 0 & -2 \\ 0 & -1 & -2 \\ -2 & -2 & 1-1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -2 \\ 0 & -1 & -2 \\ -2 & -2 & 0 \end{bmatrix} \begin{matrix} R_1 \\ R_2 \\ \xrightarrow{R_2+2R_1} \end{matrix} \begin{bmatrix} 1 & 0 & -2 \\ 0 & -1 & -2 \\ 0 & -2 & -4 \end{bmatrix} \begin{matrix} R_1 \\ R_2 \\ \xrightarrow{R_2-2R_2} \end{matrix} \begin{bmatrix} 1 & 0 & -2 \\ 0 & -1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$-V_2 - 2V_3 = 0 \Rightarrow V_2 = -2V_3$$

$$V_1 - 2V_3 = 0 \Rightarrow V_1 = 2V_3$$

Choose $v_3 = 1$, the eigenvector is $\underset{\sim}{v}^{(2)} = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$. <u>Ans</u>

<u>For $\lambda = 4$</u>

$$\begin{bmatrix} 2-4 & 0 & -2 \\ 0 & -4 & -2 \\ -2 & -2 & 1-4 \end{bmatrix} = \begin{bmatrix} -2 & 0 & -2 \\ 0 & -4 & -2 \\ -2 & -2 & -3 \end{bmatrix} \begin{matrix} R_1 \\ R_2 \\ \xrightarrow{R_3-R_1} \end{matrix} \begin{bmatrix} -2 & 0 & -2 \\ 0 & -4 & -2 \\ 0 & -2 & -1 \end{bmatrix} \begin{matrix} R_1 \\ R_2 \\ \xrightarrow{2R_3-R_2} \end{matrix} \begin{bmatrix} -2 & 0 & -2 \\ 0 & -4 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$-4V_2 - 2V_3 = 0 \Rightarrow V_2 = -\frac{V_3}{2}$$

$$-2V_1 - 2V_3 = 0 \Rightarrow V_1 = -V_3$$

Choose $v_3 = -2$, the eigenvector is $\underset{\sim}{v}^{(3)} = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$. <u>Ans</u>

<u>8.1.20</u> $\begin{bmatrix} 0 & 0 & -5 & 7 \\ 0 & 0 & 7 & -5 \\ 0 & 0 & 19 & -1 \\ 0 & 0 & -1 & 19 \end{bmatrix}$

<u>Sol^n</u> $\begin{vmatrix} 0-\lambda & 0 & -5 & 7 \\ 0 & 0-\lambda & 7 & -5 \\ 0 & 0 & 19-\lambda & -1 \\ 0 & 0 & -1 & 19-\lambda \end{vmatrix} = -\lambda \begin{vmatrix} -\lambda & 7 & -5 \\ 0 & 19-\lambda & -1 \\ 0 & -1 & 19-\lambda \end{vmatrix} = -\lambda\left[-\lambda\left\{(19-\lambda)(19-\lambda) - 1\right\}\right]$

$$= \lambda^2(\lambda^2 - 38\lambda + 360)$$

Characteristic equation; $\lambda^2(\lambda^2 - 38\lambda + 360) = 0$

$$\lambda^2(\lambda - 18)(\lambda - 20) = 0$$

The eigenvalues are $\lambda_1 = \lambda_2 = 0$, $\lambda_3 = 18$, $\lambda_4 = 20$. <u>Ans</u>

For $\lambda_1 = \lambda_2 = 0$

$$\begin{bmatrix} 0 & 0 & -5 & 7 \\ 0 & 0 & 7 & -5 \\ 0 & 0 & 19 & -1 \\ 0 & 0 & -1 & 19 \end{bmatrix} \xrightarrow{\begin{array}{c} R_1 - 5R_4 \\ R_2 + 7R_4 \\ R_3 + 19R_4 \\ R_4 \end{array}} \begin{bmatrix} 0 & 0 & 0 & -88 \\ 0 & 0 & 0 & 128 \\ 0 & 0 & 0 & 360 \\ 0 & 0 & -1 & 19 \end{bmatrix} \text{ imply that } v_4 = 0$$

$$0 \cdot v_1 + 0 \cdot v_2 - v_3 + 19 v_4 = 0 \quad \Rightarrow \quad v_3 = 19 v_4 = 0$$

$\Rightarrow$ v_1 and v_2 are arbitrary

Choose $v_1 = 1$, $v_2 = 0$, the eigenvector is $\underset{\sim}{v}^{(1)} = [1\ 0\ 0\ 0]^T$ Ans

Choose $v_1 = 0$, $v_2 = 1$, the eigenvector is $\underset{\sim}{v}^{(2)} = [0\ 1\ 0\ 0]^T$ Ans

For $\lambda_3 = 18$

$$\begin{bmatrix} -18 & 0 & -5 & 7 \\ 0 & -18 & 7 & -5 \\ 0 & 0 & 19-18 & -1 \\ 0 & 0 & -1 & 19-18 \end{bmatrix} = \begin{bmatrix} -18 & 0 & -5 & 7 \\ 0 & -18 & 7 & -5 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \xrightarrow{\begin{array}{c} R_1 \\ R_2 \\ R_3 \\ R_4 + R_3 \end{array}} \begin{bmatrix} -18 & 0 & -5 & 7 \\ 0 & -18 & 7 & -5 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$v_3 - v_4 = 0 \quad \Rightarrow \quad v_3 = v_4$$

$$-18 v_2 + 7 v_3 - 5 v_4 = 0 \quad \Rightarrow \quad v_2 = \frac{7v_3 - 5v_4}{18} = \frac{7v_4 - 5v_4}{18} = \frac{v_4}{9}$$

$$-18 v_1 - 5 v_3 + 7 v_4 = 0 \quad \Rightarrow \quad v_1 = \frac{7v_4 - 5v_3}{18} = \frac{7v_4 - 5v_4}{18} = \frac{v_4}{9}$$

Choose $v_4 = 9$; the eigenvector is $\underset{\sim}{v}^{(3)} = [1\ \ 1\ \ 9\ \ 9]^T$. Ans

For $\lambda_4 = 20$

$$\begin{bmatrix} -20 & 0 & -5 & 7 \\ 0 & -20 & 7 & -5 \\ 0 & 0 & 19-20 & -1 \\ 0 & 0 & -1 & 19-20 \end{bmatrix} = \begin{bmatrix} -20 & 0 & -5 & 7 \\ 0 & -20 & 7 & -5 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & -1 & -1 \end{bmatrix} \xrightarrow{\begin{array}{c} R_1 \\ R_2 \\ R_3 \\ R_4 - R_3 \end{array}} \begin{bmatrix} -20 & 0 & -5 & 7 \\ 0 & -20 & 7 & -5 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$-v_3 - v_4 = 0 \quad \Rightarrow \quad v_3 = -v_4$$

$$-20 v_2 + 7 v_3 - 5 v_4 = 0 \quad \Rightarrow \quad v_2 = \frac{7v_3 - 5v_4}{20} = \frac{7(-v_4) - 5v_4}{20} = \frac{-3}{5} v_4$$

$$-20 v_1 - 5 v_3 + 7 v_4 = 0 \quad \Rightarrow \quad v_1 = \frac{7v_4 - 5v_3}{20} = \frac{7v_4 - 5(-v_4)}{20} = \frac{3}{5} v_4$$

Choose $v_4 = -5$, the eigenvector is $\underset{\sim}{v}^{(4)} = [-3\ \ 3\ \ 5\ \ -5]^T$. Ans

8.1.30 Show that the inverse $\underline{\underline{A}}^{-1}$ exists if and only if none of the eigenvalues $\lambda_1, \dots, \lambda_n$ of $\underline{\underline{A}}$ is zero and then $\underline{\underline{A}}^{-1}$ has the eigenvalues $1/\lambda_1, \dots, 1/\lambda_n$

Sol$^{\underline{n}}$ From Theorem 1 (Section 7.8 p.316 in text)

$\Rightarrow \underline{\underline{A}}^{-1}$ exists if and only if $\det(\underline{\underline{A}}) \neq 0$.

Characteristic polynomial : $D(\lambda) = 0$

$$D(\lambda) = \det(\underline{\underline{A}} - \lambda \underline{\underline{I}}) = 0$$

In general form, for $n \times n$ matrix $\underline{\underline{A}}$,

$$\det(\underline{\underline{A}} - \lambda \underline{\underline{I}}) = (-1)^n (\lambda - \lambda_1)(\lambda - \lambda_2)\dots(\lambda - \lambda_n) = 0$$

Where $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_n$ are the eigenvalues of $\underline{\underline{A}}$.

$$\det(\underline{\underline{A}} - \lambda \underline{\underline{I}}) = \det(\underline{\underline{A}}) - \det(\lambda \underline{\underline{I}}) = (-1)^n (\lambda - \lambda_1)(\lambda - \lambda_2)\dots(\lambda - \lambda_n)$$

let $\lambda = 0$; $\det(\underline{\underline{A}}) = (-1)^n (-\lambda_1)(-\lambda_2)\dots(-\lambda_n)$

$\therefore$ If at least one of its eigenvalues is zero, $\det(\underline{\underline{A}}) = 0$ and from Theorem 1 above, A^{-1} does not exist.

$\Rightarrow$ The inverse $\underline{\underline{A}}^{-1}$ exists if and only if none of the eigenvalues of A is zero. Ans

Eigenvalues of $\underline{\underline{A}}$, λ, can be found from

$$\underline{\underline{A}}\,\underset{\sim}{x} = \lambda \underset{\sim}{x} . \qquad -(1)$$

Multiply $\underline{\underline{A}}^{-1}$ on both sides of (1),

$$\underline{\underline{A}}^{-1}\underline{\underline{A}}\,\underset{\sim}{x} = \lambda \underline{\underline{A}}^{-1}\underset{\sim}{x}$$

$\underline{\underline{A}}^{-1}\underline{\underline{A}} = \underline{\underline{I}}$ and $\underline{\underline{I}}\,\underset{\sim}{x} = \underset{\sim}{x}$,

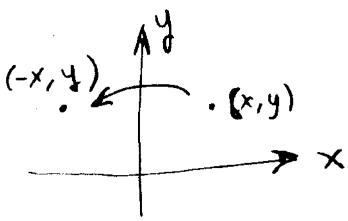
$$\underline{\underline{A}}^{-1}\underset{\sim}{x} = \frac{1}{\lambda}\underset{\sim}{x} \qquad \Rightarrow \text{Compare to (1)}$$

$\therefore$ $\underline{\underline{A}}^{-1}$ has the eigenvalues $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \dots \frac{1}{\lambda_n}$, where λ_i's are the eigenvalues of $\underline{\underline{A}}$. Ans

Find the matrix A in the indicated linear transformation $y = Ax$

Explain the geometric significance of the eigenvalues and eigenvectors of A

8.2.1 Reflection about the y-axis in $\mathbb{R}^2$



(x,y) is mapped onto $(-x,y)$

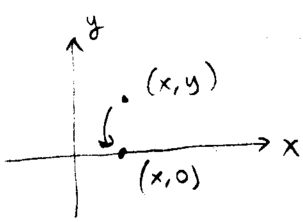
Hence : $\begin{pmatrix} -x \\ y \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

$A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, $|\lambda I - A| = \begin{vmatrix} \lambda+1 & 0 \\ 0 & \lambda-1 \end{vmatrix} = (\lambda+1)(\lambda-1)$

$\lambda_1 = 1 \rightarrow \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} v_1^{(1)} \\ v_2^{(1)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow v^{(1)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ any point on y-axis maps onto itself (positive eigenvalue)

$\lambda_2 = -1 \rightarrow \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} v_1^{(2)} \\ v_2^{(2)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow v^{(2)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ any point $(x,0)$ on x-ax[is] maps onto $(-x,0)$ (negative eigenvalue)

8.2.3 orthogonal projection ($\perp$ proj.) of $\mathbb{R}^2$ onto the x-axis



(x,y) maps onto $(x,0)$

Hence $\begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

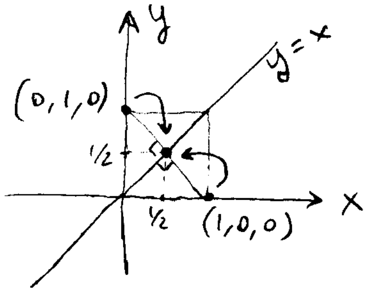
$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \rightarrow |\lambda I - A| = \begin{vmatrix} \lambda-1 & 0 \\ 0 & \lambda \end{vmatrix} = \lambda(\lambda-1)$

$\lambda_1 = 1 \rightarrow \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} v_1^{(1)} \\ v_2^{(1)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow v^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rightarrow$ any point on x-axis $(x,0)$ maps onto itself

$\lambda_2 = 0 \rightarrow \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} v_1^{(2)} \\ v_2^{(2)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow v^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \rightarrow$ any point on y-axis $(0,y)$ maps onto the origin

8.2.4 orthogonal projection ($\perp$ proj.) of $\mathbb{R}^3$ onto the plane $y = x$

it's convenient to sketch the line $y=x$ in the xy plane :



consider a point on the x-axis, e.g. $(1,0,0)$
its orthogonal proj. on the $y=x$ plane
is $(\frac{1}{2}, \frac{1}{2}, 0)$. Similarly for $(0,1,0)$

thus : $(1,0,0)$ maps onto $(\frac{1}{2}, \frac{1}{2}, 0)$
$(0,1,0)$ maps onto $(\frac{1}{2}, \frac{1}{2}, 0)$
$(0,0,1)$ maps onto $(0,0,1)$

Hence, the transformation matrix is :

$$A = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{so} \qquad \begin{pmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{x+y}{2} \\ \frac{x+y}{2} \\ z \end{pmatrix}$$

$$\begin{vmatrix} \lambda-\frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & \lambda-\frac{1}{2} & 0 \\ 0 & 0 & \lambda-1 \end{vmatrix} = (\lambda-1)\left[\left(\lambda-\tfrac{1}{2}\right)^2 - \tfrac{1}{4}\right] = (\lambda-1)\cdot\lambda\cdot(\lambda-1) = \lambda(\lambda-1)^2$$

$\lambda_{1,2} = 1$
$\lambda_3 = 0$

$$\lambda_{1,2} = 1 \longrightarrow \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} v_1^{(1)} \\ v_2^{(1)} \\ v_3^{(1)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \left(\begin{array}{ccc|c} \frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \rightarrow$$

$v_1^{(1)} = v_2^{(1)}$ & $v_3^{(1)}$ arbitrary

$$v^{(1)} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad v^{(2)} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$\Rightarrow$ which span the plane $y=x$
this indicates that every point of this plane is mapped onto itself

$$\lambda_3 = 0 \longrightarrow \left(\begin{array}{ccc|c} -\frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right) \rightarrow$$

$v_1^{(3)} = -v_2^{(3)}$
$v_2^{(3)}$ arbitrary
$v_3^{(3)} = 0$

$$v^{(3)} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

$\Rightarrow$ Any point on the line $y=-x$ $(z=0)$
(which is $\perp$ to the plane $y=x$)
is mapped onto the origin