HW 5, ME 564

1 For the data used in the class example $\{\{0,0\},\{1,8\},\{3,8\},\{4,20\}\}$ compute and plot the best fit quadratic and cubic curve. Briefly discuss the magnitude of the errors in each case.

i	t	b
1	0	0
2	1	8
3	3	8
4	4	20

a) Quadratic

Desired Form: $b = C + Dt + Et^2 \Rightarrow \underset{\underline{A}}{\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{pmatrix}} \underset{\underline{x}}{\begin{pmatrix} C \\ D \\ E \end{pmatrix}} = \underset{\underline{b}}{\begin{pmatrix} 0 \\ 8 \\ 8 \\ 20 \end{pmatrix}}$

- matrix $\underline{A}$ is incompatible with vector $\underline{x}$, multiply everything by A^T:

$$A^TAx = A^Tb$$

$$\Downarrow$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \\ 0 & 1 & 9 & 16 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{pmatrix}\begin{pmatrix} C \\ D \\ E \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \\ 0 & 1 & 9 & 16 \end{pmatrix}\begin{pmatrix} 0 \\ 8 \\ 8 \\ 20 \end{pmatrix}$$

$$\begin{pmatrix} 1+1+1+1 & 0+1+3+4 & 0+1+9+16 \\ 0+1+3+4 & 0+1+9+16 & 0+1+27+64 \\ 0+1+9+16 & 0+1+27+64 & 0+1+81+256 \end{pmatrix}\begin{pmatrix} C \\ D \\ E \end{pmatrix} = \begin{pmatrix} 0+8+8+20 \\ 0+8+24+80 \\ 0+8+72+320 \end{pmatrix}$$

$$\begin{pmatrix} 4 & 8 & 26 \\ 8 & 26 & 92 \\ 26 & 92 & 338 \end{pmatrix}\begin{pmatrix} C \\ D \\ E \end{pmatrix} = \begin{pmatrix} 36 \\ 112 \\ 400 \end{pmatrix}$$

- Gauss elimination:

$$\left(\begin{array}{ccc|c} 4 & 8 & 26 & 36 \\ 8 & 26 & 92 & 112 \\ 26 & 92 & 338 & 400 \end{array}\right) \text{Row}\cdot\tfrac{1}{4} \quad \left(\begin{array}{ccc|c} 1 & 2 & 13/2 & 9 \\ 8 & 26 & 92 & 112 \\ 26 & 92 & 338 & 400 \end{array}\right) \begin{array}{l} ②-8① \\ ③-26① \end{array} \quad \left(\begin{array}{ccc|c} 1 & 2 & 13/2 & 9 \\ 0 & 10 & 40 & 40 \\ 0 & 40 & 169 & 166 \end{array}\right) ② \times \tfrac{1}{10}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 13/2 & 9 \\ 0 & 1 & 4 & 4 \\ 0 & 40 & 169 & 166 \end{array}\right) ③-40② \quad \left(\begin{array}{ccc|c} 1 & 2 & 13/2 & 9 \\ 0 & 1 & 4 & 4 \\ 0 & 0 & 9 & 6 \end{array}\right)$$

$$\Downarrow$$

$9E = 6$, $E = \frac{6}{9} = \underline{\frac{2}{3}}$ $\qquad D + 4\left(\frac{2}{3}\right) = 4$, $\underline{D = 4/3}$ $\qquad C + 2\left(\frac{4}{3}\right) + \frac{13}{2}\left(\frac{2}{3}\right) = 9$, $\underline{C = 2}$

$$\boxed{\text{Quadratic fit: } b = 2 + \left(\tfrac{4}{3}\right)t + \left(\tfrac{2}{3}\right)t^2}$$

See attached for plot.

→

- error : quad

$$\underline{e} = \begin{pmatrix} 0 \\ 8 \\ 8 \\ 20 \end{pmatrix} - \left[2\begin{pmatrix} 0 \\ 1 \\ 3 \\ 4 \end{pmatrix} + \left(\tfrac{4}{3}\right)\begin{pmatrix} 0 \\ 1 \\ 3 \\ 4 \end{pmatrix} + \left(\tfrac{2}{3}\right)\begin{pmatrix} 0 \\ 1 \\ 9 \\ 16 \end{pmatrix} \right]$$

$$= \begin{pmatrix} 0 \\ 8 \\ 8 \\ 20 \end{pmatrix} - \left[\begin{pmatrix} 0 \\ 2 \\ 6 \\ 8 \end{pmatrix} + \begin{pmatrix} 0 \\ 4/3 \\ 4 \\ 16/3 \end{pmatrix} + \begin{pmatrix} 0 \\ 2/3 \\ 6 \\ 32/3 \end{pmatrix} \right] = \begin{pmatrix} 0 \\ 8 \\ 8 \\ 20 \end{pmatrix} - \begin{pmatrix} 0 \\ 4 \\ 16 \\ 24 \end{pmatrix}$$

$$\boxed{\underline{e} = \begin{pmatrix} 0 \\ 4 \\ -8 \\ -4 \end{pmatrix}}$$

5) $\boxed{\text{cubic}}$

Desired Form : $b = C + Dt + Et^2 + Ft^3 \Rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 3 & 9 & 27 \\ 1 & 9 & 16 & 64 \end{pmatrix}\begin{pmatrix} C \\ D \\ E \\ F \end{pmatrix} = \begin{pmatrix} 0 \\ 8 \\ 8 \\ 20 \end{pmatrix}$

$\underline{A} \quad \underline{x} \quad \underline{b}$

- matrix $\underline{A}$ is compatible with vector $\underline{x}$ ⇒ proceed to Gaussian Elimination

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 8 \\ 1 & 3 & 9 & 27 & 8 \\ 1 & 4 & 16 & 64 & 20 \end{array}\right) \begin{matrix} \\ ②-① \\ ③-① \\ ④-① \end{matrix} \rightarrow \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 8 \\ 0 & 3 & 9 & 27 & 8 \\ 0 & 4 & 16 & 64 & 20 \end{array}\right) \begin{matrix} \\ \\ ③-3② \\ ④-4② \end{matrix} \Rightarrow \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 8 \\ 0 & 0 & 6 & 24 & -16 \\ 0 & 0 & 12 & 60 & -12 \end{array}\right) ④-2③$$

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 8 \\ 0 & 0 & 6 & 24 & -16 \\ 0 & 0 & 0 & 12 & 20 \end{array}\right) \Rightarrow \quad 12F = 20 \quad 6E + 24\left(\tfrac{5}{3}\right) = -16 \quad D - (28/3) + (5/3) = 8$$

$$\underline{F = \tfrac{5}{3}} \qquad \underline{E = -28/3} \qquad \underline{D = 47/3} \qquad \underline{C = 0}$$

$$\boxed{\text{cubic fit : } b = 0 + \left(\tfrac{47}{3}\right)t + \left(-\tfrac{28}{3}\right)t^2 + \left(\tfrac{5}{3}\right)t^3}$$

- error : cubic

$$\underline{e} = \begin{pmatrix} 0 \\ 8 \\ 8 \\ 20 \end{pmatrix} - \left[\frac{47}{3}\begin{pmatrix} 0 \\ 1 \\ 3 \\ 4 \end{pmatrix} - \frac{28}{3}\begin{pmatrix} 0 \\ 1 \\ 9 \\ 16 \end{pmatrix} + \frac{5}{3}\begin{pmatrix} 0 \\ 1 \\ 27 \\ 64 \end{pmatrix} \right]$$

$$= \begin{pmatrix} 0 \\ 8 \\ 8 \\ 20 \end{pmatrix} - \left[\begin{pmatrix} 0 \\ 47/3 \\ 47 \\ 188/3 \end{pmatrix} + \begin{pmatrix} 0 \\ -28/3 \\ -252/3 \\ -448/3 \end{pmatrix} + \begin{pmatrix} 0 \\ 5/3 \\ 45 \\ 320/3 \end{pmatrix} \right] = \begin{pmatrix} 0 \\ 8 \\ 8 \\ 20 \end{pmatrix} - \begin{pmatrix} 0 \\ 8 \\ 8 \\ 20 \end{pmatrix}$$

$$\underline{e} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

*<u>Discussion of error</u> :

In the quadratic fit, we had an error vector of $[0 \ 4 \ -8 \ -4]^T$, while in the cubic fit we didn't have any discrepancy between the curve or the points. Although the cubic fit worked perfectly for these 4 points, it is not indicative of the nature of the behavior of the discrete set. Since 4 constants are required for the cubic fit, and we had a 4x4 [A] matrix, we should expect the cubic fit to run through all of the data points.

PROBLEM 1: QUADRATIC AND CUBIC FIT

Quadratic Fit

▪ Quadratic Fit Computations

```
t = {0, 1, 3, 4};
btemp = {0, 8, 8, 20};
Amat = Table[0, {i, 4}, {j, 3}];
For[i = 1, i ≤ Length[t],
  Amat[[i, 1]] = 1;
  Amat[[i, 2]] = t[[i]];
  Amat[[i, 3]] = t[[i]]^2;
  i++];
Print["A^Tb:"]
Bmat = Transpose[Amat].btemp
Print["A^TA:"]
Amat = Transpose[Amat].Amat
Print["Quadratic constants C, D, and E:"]
Const = Bmat.Inverse[Amat]
```

$A^T b$:

{36, 112, 400}

$A^T A$:

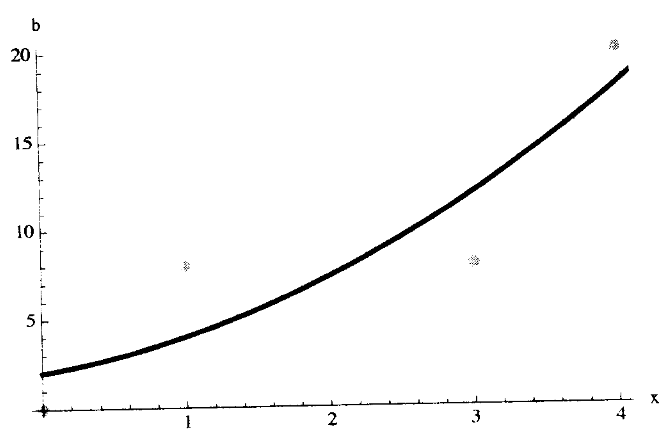
{{4, 8, 26}, {8, 26, 92}, {26, 92, 338}}

Quadratic constants C, D, and E:

$\left\{2, \frac{4}{3}, \frac{2}{3}\right\}$

▪ Quadratic Plot

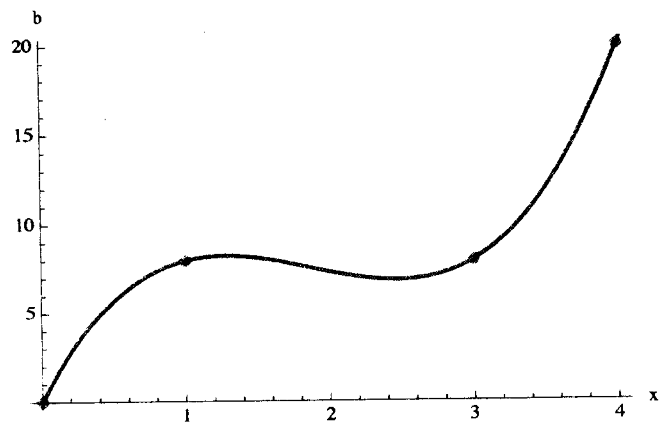
```
points = ListPlot[Table[{t[[i]], btemp[[i]]}, {i, 4}],
   PlotStyle → {Red, PointSize[0.02]}, AxesLabel → {"x", "b"}];
quadlinefit = Plot[Const[[1]] + Const[[2]] x + Const[[3]] x^2,
   {x, 0, 10}, PlotStyle → {Black, Thick}];
Show[points, quadlinefit]
```



Cubic Fit

▪ Cubic Plot

```
cubelinefit = Plot[(47 / 3) x - (28 / 3) x^2 + (5 / 3) x^3, {x, 0, 10}, PlotStyle → {Blue, Thick}];
Show[points, cubelinefit]
```



9.5.35

Plot the path, visualize the velocity $\underline{v}$ and acceleration vectors $\underline{a}$ at a sampling of points along the path

Determine $\underline{v}$ and $\underline{a}$ at the maximum and minimum y-values of the curve

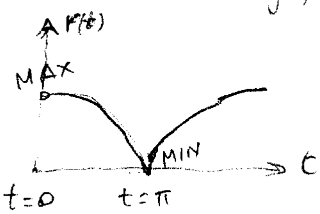
cycloid : $r(t) = (R\sin\omega t + \omega R t)\,\underline{i} + (R\cos\omega t + R)\,\underline{j}$

$$v(t) = \dot{r}(t) = \omega R(\cos\omega t + 1)\,\underline{i} - \omega R \sin\omega t\,\underline{j}$$

$$a(t) = \ddot{r}(t) = -\omega^2 R \sin\omega t\,\underline{i} - \omega^2 R \cos\omega t\,\underline{j} =$$

$$= -\omega^2 R(\sin\omega t\,\underline{i} + \cos\omega t\,\underline{j})$$

SEE PLOT next page



$\underline{v}$ and $\underline{a}$ at the MAXIMUM y-values of the curve :

$t = 0$: $\underline{v}(0) = 2\omega R\,\underline{i}$

$\underline{a}(0) = -\omega^2 R\,\underline{j}$

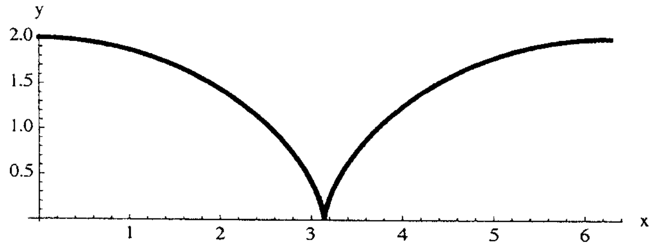
$\underline{v}$ and $\underline{a}$ at the MINIMUM y-values of the curve :

$t = \pi$ $\quad \underline{v}(\pi) = 0$

$\underline{a}(\pi) = \omega^2 R\,\underline{j}$

- **Let R = r = 1 and ω = 1 for simplicity**

```
r = 1; ω = 1;
disp = {r Sin[ω t] + r ω t, r Cos[ω t] + r};
cycloidplot =
 ParametricPlot[disp, {t, 0, 2 π}, PlotStyle → {Thick, Black}, AxesLabel → {"x", "y"}]
```



- **The velocity vectors 'v' and the acceleration vectors 'a' are the derivatives of the displacement and velocity vectors, respectively.**

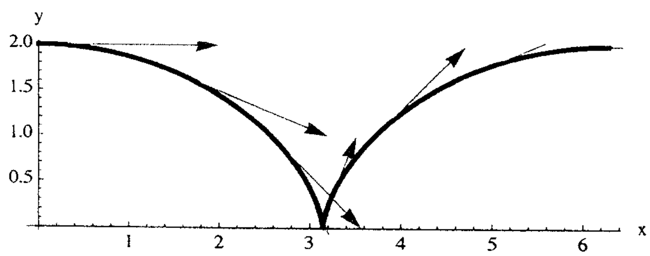
```
v = D[disp, t]
a = D[v, t]

{1 + Cos[t], -Sin[t]}

{-Sin[t], -Cos[t]}
```

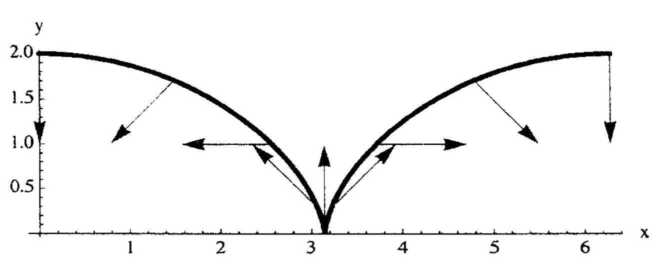
- **VISUALIZED VELOCITY VECTORS: 'vtable' is a list of values along the path of the wheel from 0 to 2π sampled every $\pi/4$. 'vvectplot' is a graphic of the velocity vectors at the points defined in 'ptable'. Maximum y-values occur at t = 0 and 2π and a minimum y-value occurs at t = π (for $0 \le t \le 2\pi$).**

```
vtable = Table[{disp, disp + v}, {t, 0, 2 π, π / 4}];
vvectplot = Show[Graphics[{Red, Arrow /@ ptable}]];
Show[cycloidplot, vvectplot]
```



- **VISUALIZED ACCELERATION VECTORS: 'atable' is a list of values along the path of the wheel from 0 to 2π sampled every $\pi/4$. 'avectplot' is a graphic of the acceleration vectors at the points defined in 'ptable'. Maximum y-values occur at t = 0 and 2π and a minimum y-value occurs at t = π (for $0 \le t \le 2\pi$).**

```
atable = Table[{disp, disp + a}, {t, 0, 2 π, π / 4}];
avectplot = Show[Graphics[{Red, Arrow /@ atable}]];
Show[cycloidplot, avectplot]
```



9.7-26 plot the 3D landscape, the const. height curve & the gradient vector field
describe relationship between these entities

if $z(x,y) = 2000 - 4x^2 - y^2$ [m] gives elevation of a mountain above sea level
what is the direction of steepest ascent at $P(3,-6)$?
what does the mountain look like?

the direction of the steepest ascent is ∇z

$$\nabla z = \frac{\partial z}{\partial x}\underline{i} + \frac{\partial z}{\partial y}\underline{j} = -8x\,\underline{i} - 2y\,\underline{j}$$

$$= -24\,\underline{i} + 12\,\underline{j} \quad \text{at } P(3,-6)$$

The direction of the steepest ascent is $[-2, 1]$

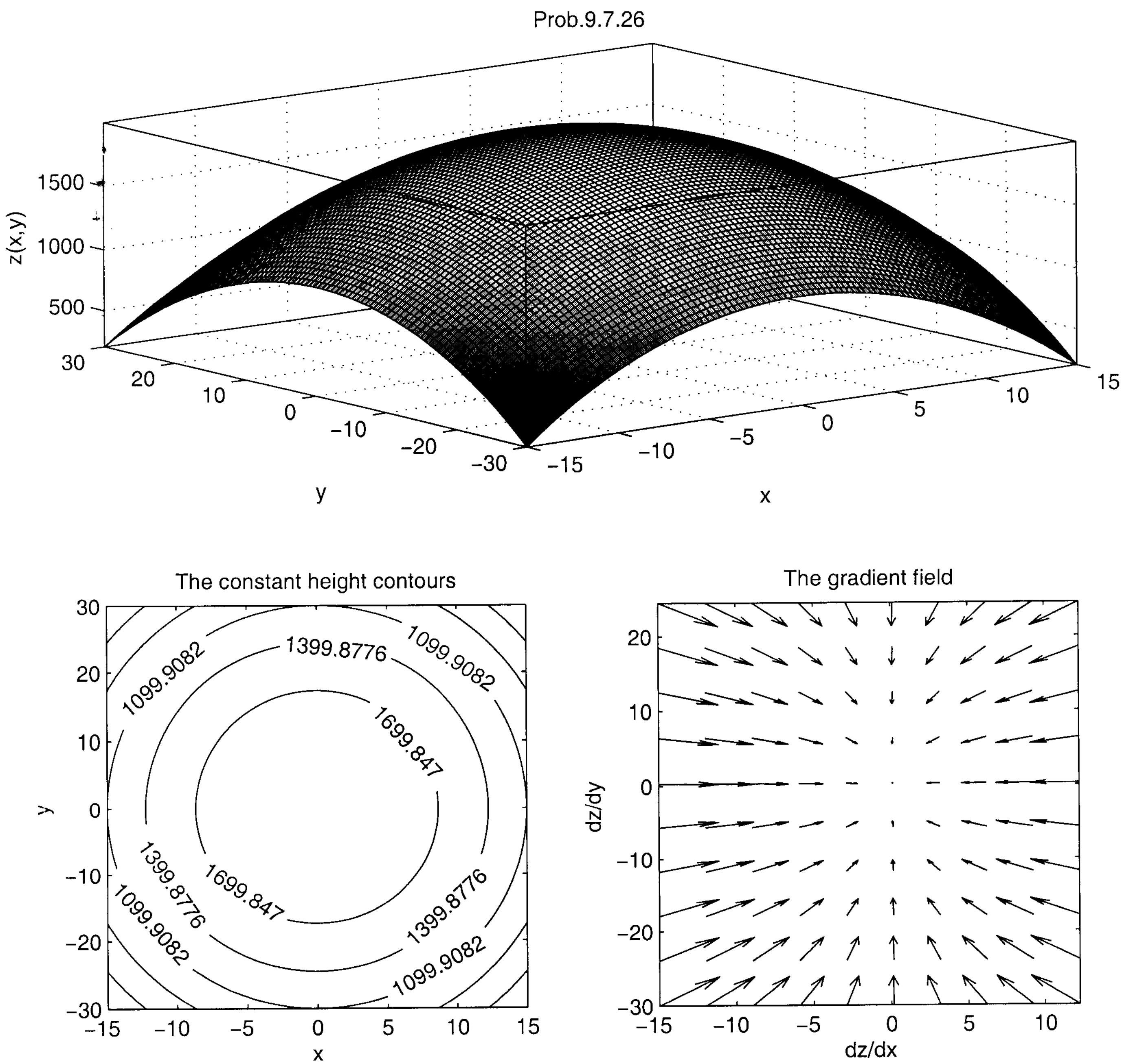
The mountain looks like a dome with the maximum height at $(x,y) = (0,0)$

The gradient is in the direction perpendicular to the constant height contours

```
% 9.7.26  Matlab
clear all
x=linspace(-15,15);
y=2.*x;
[X,Y]=meshgrid(x,y);
Z=2000-4.*X.^2-Y.^2;
figure(1); subplot(2,1,1);
surf(X,Y,Z);axis tight; box on;
xlabel('x');ylabel('y');zlabel('z');
title('Prob.9.7.26')

figure(1); subplot(2,2,3);
[c,h]=contour(X,Y,Z,5);clabel(c,h)
xlabel('x');ylabel('y');
title('The constant height contour');

[ZX,ZY]=gradient(Z);
figure(1); subplot(2,2,4);
quiver(x(1:10:100),y(1:10:100),ZX(1:10:100,1:10:100),ZY(1:1
0:100,1:10:100));
axis tight
xlabel('dz/dx');ylabel('dz/dy');
title('The gradient field');
```


Prob.9.7.26
z(x,y)
1500
1000
500
30
20
10
0
−10
−20
−30
y
−15
−10
−5
0
5
10
15
x
The constant height contours
1399.8776
1099.9082
1699.847
y
x
The gradient field
dz/dy
dz/dx

9.7.33

9.7.33) Find the directional derivative of f at P in the direction of a:

$f = x^2 + y^2 - z \qquad P = (1,1,-2) \qquad a = (1,1,2)$

$\nabla f = \frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath} + \frac{\partial f}{\partial z}\hat{k} = (2x)\hat{\imath} + (2y)\hat{\jmath} + (-1)\hat{k}$

$D_{\underline{a}} f|_p = \left(\frac{a}{|a|} f\right)\Big|_p = \frac{(1,1,2)}{\sqrt{1+1+2^2}}\left(2x\hat{\imath} + 2y\hat{\jmath} + (-1)\hat{k}\right)\Big|_{(1,1,-2)}$

$= \frac{1}{\sqrt{6}}(1,1,2)\cdot(2x\,i + 2y\hat{\jmath} + (-1)\hat{k})\Big|_{(1,1,-2)}$

$= \frac{1}{\sqrt{6}}(1,1,2)\,(2i + 2\hat{\jmath} + (-1)\hat{k}) = \frac{1}{\sqrt{6}}(2+2-2)$

$= \frac{2}{\sqrt{6}}$

$\boxed{D_a f(P) = \frac{2}{\sqrt{6}} = \sqrt{2/3}}$

9.7.35

9.7.35) $f = xyz \;,\quad P = (-1,1,3) \qquad a = (1,-2,2)$

$\nabla f = \frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath} + \frac{\partial f}{\partial z}\hat{k} = (yz)\hat{\imath} + (xz)\hat{\jmath} + (xy)\hat{k}$

$D_{\underline{a}} f|_P = \left(\frac{a}{|a|} f\right)\Big|_P = \frac{(1,-2,2)}{\sqrt{1^2+2^2+2^2}}\left(yz\hat{\imath} + xz\hat{\jmath} + xy\hat{k}\right)\Big|_{(-1,1,3)}$

$= \frac{1}{3}(1,-2,2)\cdot(yz\hat{\imath} + xz\hat{\jmath} + xy\hat{k})\Big|_{(-1,1,3)}$

$= \frac{1}{3}(1,-2,2)\cdot(3i - 3\hat{\jmath} - \hat{k}) = \frac{1}{3}(3+6-2)$

$\boxed{D_a f(P) = \frac{7}{3}}$

9.9.1

Find curl $\underline{v}$ for $\underline{v}$ given with respect of right-handed Cartesian coordin.

$\underline{v} = [y, 2x^2, 0]$

$$\text{curl } \underline{v} = \underline{\nabla} \times \underline{v} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & 2x^2 & 0 \end{vmatrix} = \left(\cancel{\frac{\partial 0}{\partial y}} - \cancel{\frac{\partial(2x^2)}{\partial z}}\right)\underline{i} + \left(\cancel{\frac{\partial y}{\partial z}} - \cancel{\frac{\partial 0}{\partial x}}\right)\underline{j} + \left(\frac{\partial(2x^2)}{\partial x} - \frac{\partial y}{\partial y}\right)\underline{k} =$$

$$= (4x - 1)\,\underline{k}$$

9.9.7

What direction does curl $\underline{v}$ have if $\underline{v}$ is a vector parallel to the xz-plane

$\underline{v} = [v_1, v_2, v_3]$ where $v_2 = c$ (constant)

and v_1 and v_3 can be function of x, z and c

$$\text{curl } \underline{v} = \left(\underbrace{\frac{\partial v_3}{\partial y}}_{0} - \overset{0}{\cancel{\frac{\partial c}{\partial z}}}\right)\underline{i} + \left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right)\underline{j} + \left(\overset{0}{\cancel{\frac{\partial c}{\partial x}}} - \underbrace{\frac{\partial v_1}{\partial y}}_{0}\right)\underline{k} = \left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right)\underline{j}$$

not function of y

The direction of curl $\underline{v}$ is perpendicular to the xz-plane

9.9.10 $\underline{v}$ = velocity vector of steady fluid flow

a) Is the flow irrotational?

b) incompressible?

c) Find the streamlines

$\underline{v} = [-y^2, 4, 0]$

$$\nabla \cdot \underline{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0 \Rightarrow \text{the flow is incompressible}$$

$$\nabla \times \underline{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^2 & 4 & 0 \end{vmatrix} = -\frac{\partial(-y^2)}{\partial y}\hat{k} = 2y\,\hat{k} \neq 0$$

$\Rightarrow$ the flow is ROTATIONAL

find the streamlines : Recall the streamlines are tangent to the velocity vector

$$\underline{v} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k} = -y^2\hat{i} + 4\hat{j}$$

$$\frac{dz}{dt} = 0 \Rightarrow z(t) = c_1$$

$$\frac{dy}{dt} = 4 \Rightarrow y(t) = 4t + c_2$$

$$\frac{dx}{dt} = -y^2 = -(4t+c_2)^2 \Rightarrow x(t) = -\frac{1}{12}(4t+c_2)^3 + c_3$$

9.9.13

$$\underline{v} = [x, -y, z]$$

$$\nabla \cdot \underline{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 1 - 1 + 1 = 1 \neq 0 \Rightarrow \text{flow is COMPRESSIBLE}$$

$$\nabla \times v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & -y & z \end{vmatrix} = 0 \Rightarrow \text{flow is IRROTATIONAL}$$

streamlines :

$$\underline{v} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}$$

$$\frac{dx}{dt} = x \Rightarrow \frac{dx}{x} = dt \Rightarrow \ln x = t + c^* \Rightarrow x(t) = c_1 e^{t}$$

$$\frac{dy}{dt} = -y \Rightarrow \frac{dy}{y} = -dt \Rightarrow \ln y = -t + c^* \Rightarrow y(t) = c_2 e^{-t}$$

$$\frac{dz}{dt} = z \Rightarrow \frac{dz}{z} = dt \Rightarrow \Rightarrow z(t) = c_3 e^{t}$$