Calculate $\int_C \underset{\sim}{F}(r) \cdot d\underset{\sim}{r}$

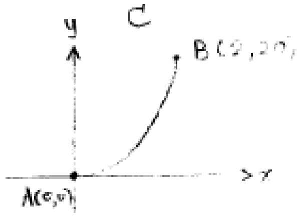
10.1.1 $F = [y^3, x^3]$, C the parabola $y = 5x^2$ from $A:(0,0)$ to $B:(2,20)$

Sol^n For parabola $y = 5x^2$, path c may be represented by

$$\underset{\sim}{r}(t) = [t, 5t^2] \Rightarrow \frac{d\underset{\sim}{r}}{dt} = [1, 10t]$$

and then $\underset{\sim}{F}(\underset{\sim}{r}(t)) = y^3\hat{i} + x^3\hat{j} = 125t^6\hat{i} + t^3\hat{j}$

$$\int_C \underset{\sim}{F}(\underset{\sim}{r}) \cdot d\underset{\sim}{r} = \int_{t=0}^{t=2} [125t^6, t^3] \cdot [1, 10t]\,dt$$

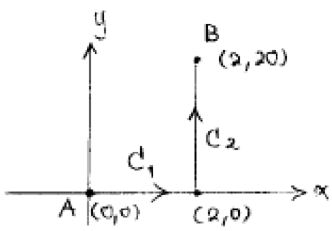


$$= \int_{t=0}^{t=2} (125t^6 + 10t^4)\,dt = \left[\frac{125t^7}{7} + \frac{\cancel{10}^{2}t^5}{\cancel{5}_{1}}\right]_0^2$$

$$= \frac{125(2)^7}{7} + 2(2)^5 = 2349.7 \quad \text{Ans}$$

10.1.3 F as in Prob 1, C from A straight to $(2,0)$ then vertically up to B

Sol^n



Let $C = C_1 + C_2$ (see figure)

C_1 : $\underset{\sim}{r}_1(t) = [t_1, 0]$; $t_1 \in [0, 2]$

C_2 : $\underset{\sim}{r}_2(t) = [2, t_2]$; $t_2 \in [0, 20]$

On C_1, $\underset{\sim}{F}(\underset{\sim}{r}_1(t)) = y^3\hat{i} + x^3\hat{j} = 0\hat{i} + t_1^3\hat{j}$

On C_2, $\underset{\sim}{F}(\underset{\sim}{r}_2(t)) = y^3\hat{i} + x^3\hat{j} = t_2^3\hat{i} + 8\hat{j}$

$$\int_C \underset{\sim}{F}(\underset{\sim}{r}) \cdot d\underset{\sim}{r} = \int_{C_1} \underset{\sim}{F}(\underset{\sim}{r}_1) \cdot d\underset{\sim}{r}_1 + \int_{C_2} \underset{\sim}{F}(\underset{\sim}{r}_2) \cdot d\underset{\sim}{r}_2$$

$$= \int_{C_1} \underset{\sim}{F}(\underset{\sim}{r}_1) \cdot \frac{d\underset{\sim}{r}_1}{dt_1}\,dt_1 + \int_{C_2} \underset{\sim}{F}(\underset{\sim}{r}_2) \cdot \frac{d\underset{\sim}{r}_2}{dt_2}\,dt_2$$

$$= \int_{t_1=0}^{t_1=2} [0, t_1^3] \cdot [1, 0]\,dt_1 + \int_{t_2=0}^{t_2=20} [t_2^3, 8] \cdot [0, 1]\,dt_2$$

$$= 0 + [8t_2]_{t_2=0}^{t_2=20} = 8(20) = 160 \quad \text{Ans}$$

- Show that the form under the integral sign is exact.
- Evaluate the integral.

(Show details of your work)

10.2.2 $\displaystyle\int_{(5,0)}^{(0,5)} (y^2 e^{2x} dx + y e^{2x} dy)$

Sol^n Let $\underset{\sim}{F}(x,y) = y^2 e^{2x} \hat{i} + y e^{2x} \hat{j}$

$F_1 = f_x = y^2 e^{2x}$; $F_2 = f_y = y e^{2x}$

$f = \int y^2 e^{2x} dx = \frac{1}{2} y^2 e^{2x} + g(y)$

$f_y = y e^{2x} + g'(y) = F_2 = y e^{2x} \Rightarrow g'(y) = 0$

$\Rightarrow g(y) = \text{Const}$

$\Rightarrow$ Choose $g(y) = 0$

$\therefore f(x,y) = \frac{1}{2} y^2 e^{2x}$ and $\underset{\sim}{F} = \nabla f \Rightarrow$ the integral is exact. **Ans**

$$\int_{(5,0)}^{(0,5)} (y^2 e^{2x} dx + y e^{2x} dy) = f(0,5) - f(5,0)$$
$$= \frac{1}{2} \cdot 5^2 \cdot e^{2(0)} - \frac{1}{2} \cdot (0)^2 e^{2(5)}$$
$$= \frac{25}{2} - 0 = 12.5 \quad \textbf{Ans}$$

10.2.7 $\displaystyle\int_{(1,0,0)}^{(7,8,0)} (2xy\, dx + x^2 dy + \sinh z\, dz)$

Sol^n Let $\underset{\sim}{F}(x,y,z) = 2xy\,\hat{i} + x^2 \hat{j} + \sinh z\, \hat{k}$

$F_1 = f_x = 2xy$, $F_2 = f_y = x^2$, $F_3 = f_z = \sinh z$

$f = \int 2xy\, dx = x^2 y + g(y,z)$

$f_y = x^2 + g_y(y,z) = x^2 \Rightarrow g_y(y,z) = 0 \Rightarrow g(y,z) = h(z)$

$\therefore f = x^2 y + h(z)$; $f_z = h'(z) = \sinh z \Rightarrow h(z) = \cosh z + C$

Choose $C = 0$, $f = x^2 y + \cosh z$

$\therefore$ The integral is exact since $\underset{\sim}{F} = \nabla f$ where $f = x^2 y + \cosh z$. **Ans**

$$\int_{(1,0,0)}^{(7,8,0)} (2xy\, dx + x^2 dy + \sinh z\, dz) = f(7,8,0) - f(1,0,0) = (7^2 \cdot 8 + \cosh 0) - (1^2 \cdot 0 + \cosh 0)$$
$$= 392 \quad \textbf{Ans}$$

Check for path independence and, if independent, integrate from $(0,0,0)$ to (a,b,c)

10.2.17

$$\underline{F} = \underset{F_1}{xyz^2}\,dx + \underset{F_2}{\tfrac{1}{2}x^2z^2}\,dy + \underset{F_3}{x^2yz}\,dz$$

if curl F = 0 then indep.

$$\frac{\partial F_3}{\partial y} = \frac{\partial F_2}{\partial z},\quad \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x},\quad \frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}$$

$$x^2z = \tfrac{2}{2}x^2z,\quad 2xyz = 2xyz,\quad xz^2 = xz^2 \quad ✓$$

$F = \text{grad}(f)$

$$F_1 = \frac{\partial f}{\partial x} = xyz^2 \Rightarrow f = \frac{x^2yz^2}{2} + g(y,z)$$

$$F_2 = \frac{\partial f}{\partial y} = \frac{1}{2}x^2z^2 \Rightarrow f = \frac{x^2yz^2}{2} + g(x,z) \quad \Rightarrow \quad f = \frac{x^2yz^2}{2}$$

$$F_3 = \frac{\partial f}{\partial z} = x^2yz \Rightarrow f = \frac{x^2yz^2}{2} + g(x,y)$$

$$\int_{(0,0,0)}^{(a,b,c)} xyz^2dx + \tfrac{1}{2}x^2z^2dy + x^2yz\,dz = f(a,b,c) - f(0,0,0) = \boxed{\frac{a^2bc^2}{2}}$$

10.2.18

$$\underline{F} = \underset{F_1}{yz\cosh x}\,dx + \underbrace{z\sinh x}_{F_2}\,dy + \underbrace{y\sinh x}_{F_3}\,dz$$

$$\frac{\partial F_3}{\partial y} = \sinh x = \frac{\partial F_2}{\partial x} = \sinh x$$

$$\frac{\partial F_1}{\partial z} = y\cosh x = \frac{\partial F_3}{\partial x} = y\cosh x \quad \text{path indipendent}$$

$$\frac{\partial F_2}{\partial x} = z\cosh x = \frac{\partial F_1}{\partial y} = z\cosh x$$

$F = \text{grad}\, f$

$$F_1 = \frac{\partial f}{\partial x} = yz\cosh x \Rightarrow f = yz\sinh x + g(y,z)$$

$$F_2 = \frac{\partial f}{\partial y} = z\sinh x \Rightarrow f = yz\sinh x + g(x,z) \quad \Rightarrow \quad f = yz\sinh x \Downarrow$$

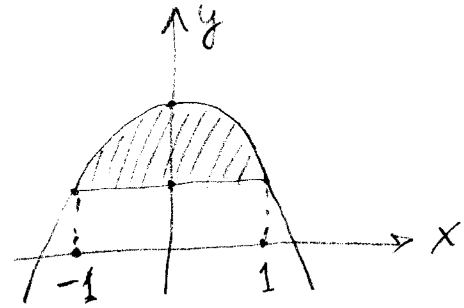
$$F_3 = \partial f/\partial z = y\sinh x \Rightarrow f = yz\sinh x + g(x,y)$$

So, $$\int_{0,0,0}^{a,b,c} yz\cosh x\,dx + z\sinh x\,dy + y\sinh x\,dz = f(a,b,c) - f(0,0,0) = \boxed{bc\sinh a}$$

Using Green's theorem, evaluate $\int_C F(r)\,dr$ counterclockwise around the boundary curve C of the region R. Sketch R

10.4.7

$F = [x^2+y^2,\ x^2-y^2]$, $R: 1 \le y \le 2-x^2$



C is a closed curve, so $\oint_C F(r)\cdot dr = \iint_R \left(\frac{\partial F_1}{\partial y} - \frac{\partial F_2}{\partial x}\right) =$

x varies from -1 to 1
y varies from 1 to $2-x^2$

$$\int_{-1}^{1}\left(\int_{1}^{2-x^2}(2x-2y)\,dy\right)dx = \int_{-1}^{1}\left[2xy-y^2\right]_1^{2-x^2}dx =$$

$$= \int_{-1}^{1}\left(2x(2-x^2)-(2-x^2)^2\right)-(2x-1)\,dx = \int_{-1}^{1}\left(4x-2x^3-4-x^4+4x^2-2x+1\right)dx$$

$$= \int_{-1}^{1}\left(3x-2x^3-3-x^4+4x^2\right)dx = \left[\frac{3}{2}x^2-\frac{x^4}{2}-3x-\frac{x^5}{5}+\frac{4}{3}x^3\right]_{-1}^{1} =$$

$$= 2\left(-3-\frac{1}{5}+\frac{4}{3}\right) = -\frac{56}{15}$$

10.4.8

$F = [e^x\cos y,\ -e^x\sin y]$, R the semidisk $x^2+y^2 \le a^2$, $x \ge 0$

C is a closed curve, so $\int F(r)\cdot dr = \oint F(r)\,dr = \oint_C (F_1dx + F_2dy) =$

by Green's theorem

$$= \iint_R \left(\frac{\partial F_2}{\partial y} - \frac{\partial F_1}{\partial x}\right)dxdy = \iint\left(-e^x\sin y - (-e^x\sin y)\right)dx\,dy = 0$$

10.4.9

$F = \mathrm{grad}\,(\underbrace{x^3\cos^2(xy)}_{G})$, R region in problem 7

$$F = [F_1, F_2] = \left[\frac{\partial G}{\partial x}, \frac{\partial G}{\partial y}\right]$$

$$\left.\begin{aligned}\frac{\partial F_1}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{\partial G}{\partial x}\right) = \frac{\partial^2 G}{\partial x\partial y}\\ \frac{\partial F_2}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{\partial G}{\partial y}\right) = \frac{\partial^2 G}{\partial x\partial y}\end{aligned}\right\} \Rightarrow \frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial y} \Rightarrow \iint_R \left(\frac{\partial F_2}{\partial y} - \frac{\partial F_1}{\partial x}\right)dxdy = 0$$

- Evaluate the integral by the divergence theorem (Show the details)

10.7.17 $\underset{\sim}{F} = [x, y, z]$, S the sphere $x^2 + y^2 + z^2 = 9$

Sol^n The divergence theorem: $\iint_S \underset{\sim}{F} \cdot \underset{\sim}{n}\, dA = \iiint_T \nabla \cdot \underset{\sim}{F}\, d\forall$

$$\underset{\sim}{F} = [x, y, z], \quad \nabla \cdot \underset{\sim}{F} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 1 + 1 + 1 = 3$$

$$\iint_S [x, y, z] \cdot \underset{\sim}{n}\, dA = \iiint_T 3\, d\forall = 3 \iiint_T d\forall$$

$$= 3\forall \quad , \text{ where } \forall \text{ is the volume of the sphere}$$

$$x^2 + y^2 + z^2 = 9 = 3^2$$

$$= \cancel{3}\left(\frac{4\pi\, 3^3}{\cancel{3}}\right) \quad ; \quad \forall_{sphere} = \frac{4}{3}\pi R^3$$

$$= 108\pi \quad \underline{\text{Ans}}$$

10.7.18 $\underset{\sim}{F} = [4x, 3z, 5y]$, S the surface of the cone $x^2 + y^2 \leq z^2$, $0 \leq z \leq 2$

Sol^n The divergence theorem: $\iint_S \underset{\sim}{F} \cdot \underset{\sim}{n}\, dA = \iiint_T \nabla \cdot \underset{\sim}{F}\, d\forall$

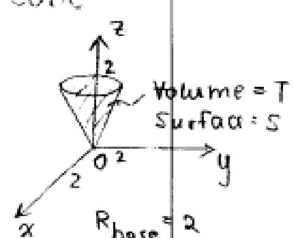
$$\underset{\sim}{F} = [4x, 3z, 5y], \quad \nabla \cdot \underset{\sim}{F} = \frac{\partial}{\partial x}(4x) + \frac{\partial}{\partial y}(3z) + \frac{\partial}{\partial z}(5y)$$
$$= 4 + 0 + 0 = 4$$

$$\iint_S [4x, 3z, 5y] \cdot \underset{\sim}{n}\, dA = \iiint_T 4\, d\forall = 4 \iiint_T d\forall$$

$$= 4\forall \quad ; \ \forall \text{ is the volume of the cone}$$

$$x^2 + y^2 \leq z^2 \ ; \ 0 \leq z \leq 2$$

$$= 4\left(\frac{1}{3} \times 2 \times \pi\, 2^2\right) \quad ; \ \forall_{Cone} = \frac{h}{3} \times \pi R_{base}^2$$



$$= \frac{32\pi}{3} \quad \underline{\text{Ans}}$$

10.8.9 Show that a region T with boundary surface S has the volume

$$\forall = \iint_S x\,dy\,dz = \iint_S y\,dz\,dx = \iint_S z\,dx\,dy = \frac{1}{3}\iint_S (x\,dy\,dz + y\,dz\,dx + z\,dx\,dy).$$

Solⁿ: Divergence theorem of Gauss

$$\iiint_T \nabla\cdot F\,d\forall = \iint_S F\cdot n\,dA$$

$$\iiint_T \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right)dx\,dy\,dz = \iint_S (F_1\,dy\,dz + F_2\,dz\,dx + F_3\,dx\,dy)$$

1) Let $F = [x, 0, 0]$

$$\iiint_T \left(\cancel{\frac{\partial x}{\partial x}}^{\,1} + 0 + 0\right)d\forall = \iint_S (x\,dy\,dz + 0 + 0)$$

$$\iiint_T d\forall = \forall = \iint_S x\,dy\,dz \qquad \underline{\underline{\text{Ans}}}$$

2) Let $F = [0, y, 0]$

$$\iiint_T \left(0 + \cancel{\frac{\partial y}{\partial y}}^{\,1} + 0\right)d\forall = \iint_S (0 + y\,dz\,dx + 0)$$

$$\iiint_T d\forall = \forall = \iint_S y\,dz\,dx \qquad \underline{\underline{\text{Ans}}}$$

3) Let $F = [0, 0, z]$

$$\iiint_T \left(0 + 0 + \cancel{\frac{\partial z}{\partial z}}^{\,1}\right)d\forall = \iint_S (0 + 0 + z\,dx\,dy)$$

$$\iiint_T d\forall = \forall = \iint_S z\,dx\,dy \qquad \underline{\underline{\text{Ans}}}$$

4) Let $F = [x, y, z]$

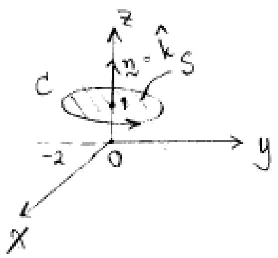
$$\underbrace{\iiint_T \left(\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z}\right)d\forall}_{} = \iint_S (x\,dy\,dz + y\,dz\,dx + z\,dx\,dy)$$

$$3\iiint_T d\forall = 3\forall \Rightarrow \forall = \frac{1}{3}\iint_S (x\,dy\,dz + y\,dz\,dx + z\,dx\,dy) \qquad \underline{\underline{\text{Ans}}}$$

- Calculate the line integral by Stokes's theorem, Clockwise as seen by a person standing at the origin. Assume the Cartesian coordinate to be right-handed. (Show the details.)

10.9.11 $\underset{\sim}{F} = [-3y, 3x, z]$, C the circle $x^2+y^2=4$, $z=1$

Sol$^{\underline{n}}$ Stokes's theorem : $\iint_S (\nabla \times \underset{\sim}{F}) \cdot \underset{\sim}{n}\, dA = \oint_C \underset{\sim}{F} \cdot \underset{\sim}{r}'(s)\, ds$



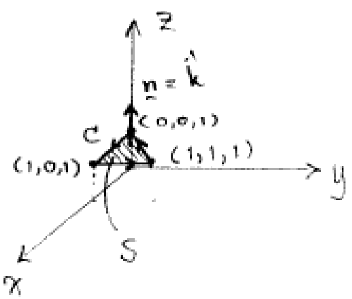
The surface S bounded by C is the plane circular disk $x^2+y^2 \leq 4$ in the plane $z=1$. With the clockwise direction, the $\underset{\sim}{n}$ in Stokes's theorem points in the positive z direction, $\underset{\sim}{n} = \hat{k}$. Hence

$$(\nabla \times \underset{\sim}{F}) \cdot \underset{\sim}{n} = (\nabla \times \underset{\sim}{F}) \cdot (\hat{k})$$
$$= \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \hat{k} \cdot (\hat{k}) = \frac{\partial (3x)}{\partial x} - \frac{\partial (-3y)}{\partial y}$$
$$= 3+3 = 6$$

$$\oint_C \underset{\sim}{F} \cdot \underset{\sim}{r}'(s)\, ds = \iint_S (\nabla \times \underset{\sim}{F}) \cdot \underset{\sim}{n}\, dA = \iint_S (6)\, dA$$
$$= 6A = 6(\pi 2^2) \quad ; \quad A = \text{area of a circular disk } x^2+y^2 \leq 4 = \pi R^2$$
$$= 24\pi \quad \underline{\underline{\text{Ans}}}$$

10.9.13 $\underset{\sim}{F} = [y^2, x^2, -x+z]$, around the triangle with vertices $(0,0,1)$, $(1,0,1)$, $(1,1,1)$

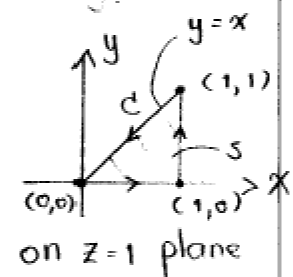
Sol$^{\underline{n}}$ Stokes's theorem : $\iint_S (\nabla \times \underset{\sim}{F}) \cdot \underset{\sim}{n}\, dA = \oint_C \underset{\sim}{F} \cdot \underset{\sim}{r}'(s)\, ds$



The surface S bounded by C is the plane triangle in the plane $z=1$. With the clockwise direction (wrt origin), the $\underset{\sim}{n}$ in the Stokes's theorem is $\underset{\sim}{n} = \hat{k}$.

$$(\nabla \times \underset{\sim}{F}) \cdot \underset{\sim}{n} = (\nabla \times \underset{\sim}{F}) \cdot (\hat{k})$$
$$= \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\hat{k} \cdot \hat{k} = \frac{\partial x^2}{\partial x} - \frac{\partial y^2}{\partial y}$$
$$= 2x - 2y$$

$$\oint_C [y^2, x^2, -x+z] \cdot \underset{\sim}{r}'(s)\, ds = \iint_S (\nabla \times \underset{\sim}{F}) \cdot \underset{\sim}{n}\, dA = \iint_S (2x - 2y)\, dA$$

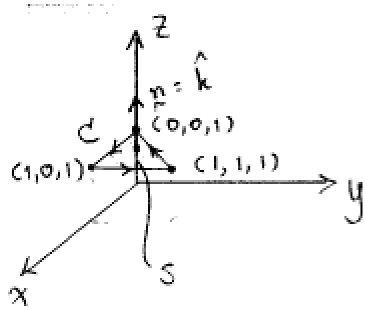


$$= \int_0^1 \int_0^x (2x - 2y)\, dy\, dx$$
$$= \int_0^1 \left[2xy - y^2\right]_{y=0}^{y=x} dx$$
$$= \int_0^1 (2x^2 - x^2)\, dx$$
$$= \int_0^1 x^2\, dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3} \qquad \underline{\underline{\text{Ans}}}$$

<u>10.9.18</u> $\underset{\sim}{F} = [z, x, y]$, C as in Prob. 13

<u>Sol</u>

Sol^n Stokes's theorem: $\iint_S (\nabla \times \underset{\sim}{F}) \cdot \underset{\sim}{n}\, dA = \oint_C \underset{\sim}{F} \cdot \underset{\sim}{r}'(s)\, ds$



As in Prob. 13, $\underset{\sim}{n} = \hat{k}$.

$$(\nabla \times \underset{\sim}{F}) \cdot \underset{\sim}{n} = (\nabla \times \underset{\sim}{F}) \cdot \hat{k}$$
$$= \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\hat{k} \cdot \hat{k} = \frac{\partial x}{\partial x} - \frac{\partial z}{\partial y}$$
$$= 1$$

$$\oint_C [z, x, y] \cdot \underset{\sim}{r}'(s)\, ds = \iint_S (\nabla \times \underset{\sim}{F}) \cdot \underset{\sim}{n}\, dA = \iint_S 1\, dA$$
$$= A = \frac{1}{2} \times 1 \times 1 \quad ; \text{ Area of triangle} = A$$
$$= \frac{1}{2} \qquad \underline{\underline{\text{Ans}}}$$