HW 7 ME 564

Index Notation Problems: Use index notation to verify the pairs of equations labeled as (14) and (16) in the Summary of Ch. 9 on p. 419.

14)

a) show $\text{div}(f\underline{v}) = f \text{ div } \underline{v} + \underline{v} \cdot \nabla f$

$\text{div}(f\underline{v}) = \partial_i f v_i = f \partial_i v_i = v_i \partial_i f =$

$= f \text{ div } \underline{v} + \mathbf{v} \cdot \nabla f$

b) Show $\text{div}(f \nabla g) = f \nabla^2 g + \nabla f \cdot \nabla g$

$\text{div}(f \nabla g) = \partial_i (f \partial_i g) = f \partial_i \partial_i g + \partial_i g \, \partial_i f =$

$= f \nabla^2 g + \nabla f \cdot \nabla g$

16)

a) Show $\text{curl}(f \underline{v}) = \nabla f \times \underline{v} + f \text{ curl } \underline{v}$

$\text{curl}(f\underline{v}) = \partial_i (f v_j) \varepsilon_{ijk} = \varepsilon_{ijk} v_j \partial_i f + f \varepsilon_{ijk} \partial_i v_j =$

$= \nabla f \times \underline{v} + f \text{ curl } \underline{v}$

b) Show $\text{div}(\underline{u} \times \underline{v}) = \underline{v} \cdot \text{curl } \underline{u} - \underline{u} \cdot \text{curl } \underline{v}$

$\text{div}(u \times v) = \partial_k (u_i v_j \epsilon_k) = \partial_k u_i \varepsilon_{ijk} \cdot \underline{v} + \partial_k v_j \varepsilon_{ijk} \cdot \underline{u} =$

$= \underline{v} \text{ curl } \underline{u} + \partial_k v_j (-\varepsilon_{kji}) \cdot \underline{u} =$

$= v \cdot \text{curl } \underline{u} - \underline{u} \cdot \text{curl } \underline{v}$

Reduce to first order and solve (Show each step in detail).

2.1.22 $(1-x^2)y'' - 2xy' + 2y = 0$, $y_1 = x$

Sol^n let $y = uy_1 = ux$ is the solution to the given problem

$$y' = u'x + u \ , \quad y'' = u''x + 2u'$$

$$(1-x^2)(u''x + 2u') - 2x(u'x + u) + 2ux = 0$$

$$(1-x^2)(u''x + 2u') - 2x^2u' - \cancel{2ux} + \cancel{2ux} = 0$$

$$(1-x^2)u''x + (2-4x^2)u' = 0$$

$$(x - x^3)u'' + (2-4x^2)u' = 0 \qquad (1)$$

let $v = u'$, Eq. (1) become 1^{st} order in v

$$(x - x^3)v' + (2-4x^2)v = 0 \qquad **$$

$$v' = -\frac{(2-4x^2)v}{(x-x^3)}$$

$$\frac{dv}{v} = -\frac{(2-4x^2)\,dx}{(x-x^3)}$$

$$= \left(\frac{-2}{x} + \frac{1}{1-x} - \frac{1}{1+x}\right)dx$$

$$\ln|v| = -2\ln|x| - \ln|1-x| - \ln|1+x|$$

$$= -[\ln x^2 + \ln|1-x| + \ln|1+x|]$$

$$= -\ln x^2|1-x^2|$$

$$v = (x^2(1-x^2))^{-1}$$

$$u = \int v\,dx = \int \frac{1}{x^2(1-x^2)}dx$$

$$= \int\left(\frac{1}{x^2} + \frac{1/2}{1-x} + \frac{1/2}{1+x}\right)dx$$

$$= -\frac{1}{x} - \frac{1}{2}\ln|1-x| + \frac{1}{2}\ln|1+x|$$

$$= -\frac{1}{x} + \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right|$$

The second solution is $y = ux = -1 + \frac{x}{2}\ln\left|\frac{1+x}{1-x}\right|$. **Ans**

$\boxed{2.2.29}$ $20y'' + 4y' + y = 0$, $y(0) = 3.2$, $y'(0) = 0$

Sol$^{\underline{n}}$ Characteristic eq. ; $20\lambda^2 + 4\lambda + 1 = 0$

$$\lambda = \frac{-4 \pm \sqrt{16 - 4(20)(1)}}{2(20)} = \frac{-4}{40} \pm \frac{\sqrt{-64}}{40}$$

$$= -0.1 \pm \frac{8i}{40} = -0.1 \pm 0.2i \quad ; \quad i = \sqrt{-1}$$

$$\therefore \quad y = e^{-0.1x}(A\cos(0.2x) + B\sin(0.2x)) \quad *$$

$$y(0) = 3.2 = e^{-0.1(0)}(A\cos(0.2\times 0) + B\underbrace{\cancel{\sin(0.2\times 0)}}_{0\ (\sin 0 = 0)})$$

$$A = 3.2 \quad *$$

$$y' = e^{-0.1x}(-0.2\times 3.2\sin(0.2x) + 0.2B\cos(0.2x))$$
$$+ (3.2\cos(0.2x) + B\sin(0.2x))(-0.1)e^{-0.1x}$$

$$y'(0) = 0 = e^{-0.1(0)}(-0.64\,\underbrace{\cancel{\sin(0.2\times 0)}}_{0} + 0.2B\cos(0.2\times 0))$$
$$-0.1e^{-0.1(0)}(3.2\cos(0.2\times 0) + B\underbrace{\cancel{\sin(0.2\times 0)}}_{0})$$

$$0 = 0.2B - 0.32$$

$$B = 1.6 \quad *$$

$$\therefore \quad y(x) = e^{-0.1x}[3.2\cos(0.2x) + 1.6\sin(0.2x)] \quad \underline{\underline{\text{Ans}}}$$

$$y'(x) = e^{-0.1x}[-0.64\sin(0.2x) + 0.32\cos(0.2x)]$$
$$-0.1e^{-0.1x}[3.2\cos(0.2x) + 1.6\sin(0.2x)]$$

Checking

$$y'(0) = \underbrace{e^0}_{1}[-0.64\underbrace{\cancel{\sin 0}}_{0} + 0.32\underbrace{\cancel{\cos 0}}_{1}]$$
$$-0.1\underbrace{e^0}_{1}[3.2\underbrace{\cancel{\cos 0}}_{1} + 1.6\underbrace{\cancel{\sin 0}}_{0}]$$

$$= 0.32 - 0.1\times 3.2 = 0 \quad \checkmark \text{ Ck I.C.} \quad \underline{\underline{\text{Ans}}}$$

$$y(0) = \underbrace{\cancel{e^0}}_{1}[3.2\underbrace{\cancel{\cos 0}}_{1} + 1.6\underbrace{\cancel{\sin 0}}_{0}]$$

$$= 3.2 \quad \checkmark \text{ Ck I.C.} \quad \underline{\underline{\text{Ans}}}$$

Rearrange; $y'(x) = e^{-0.1x}[-0.8\sin(0.2x)] = -0.8e^{-0.1x}\sin(0.2x)$

$$y''(x) = -0.16e^{-0.1x}\cos(0.2x) + 0.08e^{-0.1x}\sin(0.2x)$$

Plug y'', y' and y in the original ODE,

$$20[-0.16\cancel{e^{-0.1x}}\cos(0.2x) + 0.08\cancel{e^{-0.1x}}\sin(0.2x)] + 4[-0.8\cancel{e^{-0.1x}}\sin(0.2x)]$$
$$+ \cancel{e^{-0.1x}}(3.2\cos(0.2x) + 1.6\sin(0.2x)) = 0$$

$$-3.2\cancel{\cos(0.2x)} + 1.6\cancel{\sin(0.2x)} - 3.2\cancel{\sin(0.2x)} + 3.2\cancel{\cos(0.2x)}$$
$$+ 1.6\cancel{\sin(0.2x)} = 0 \quad \checkmark \quad \text{Ck the answer} \quad \underline{\underline{\text{Ans}}}$$

2.2.30 $y'' + 2ky' + (k^2+\omega^2)y = 0, \quad y(0) = 1, \; y'(0) = -k$

Sol^n Let k and ω are constant

Characteristic eqn: $\lambda^2 + 2k\lambda + (k^2+\omega^2) = 0$

$$\lambda = \frac{-2k \pm \sqrt{4k^2 - 4(1)(k^2+\omega^2)}}{2(1)} = -k \pm \frac{\sqrt{-4\omega^2}}{2}$$

$$= -k \pm |\omega| i$$

general sol^n; $y(x) = e^{-kx}\left[A\cos(|\omega|x) + B\sin(|\omega|x)\right]$ **Ans**

$$y'(x) = e^{-kx}\left[-|\omega|A\sin(|\omega|x) + |\omega|B\cos(|\omega|x)\right] - ke^{-kx}\left[A\cos(|\omega|x) + B\sin(|\omega|x)\right]$$

I.C. $y'(0) = -k = e^0\left[-|\omega|A\cancel{\sin 0}^{0} + |\omega|B\cos 0\right] - ke^0\left[A\cos 0 + B\cancel{\sin 0}^{0}\right]$

$$-k = |\omega|B - kA \quad \text{— (1)}$$

I.C. $y(0) = 1 = e^0\left[A\cos 0 + B\cancel{\sin 0}^{0}\right]$

$1 = A \; * \rightarrow$ plug in (1)

$-k = |\omega|B - k \longrightarrow B = 0 \; *$

particular sol^n; $y(x) = e^{-kx}\cos(|\omega|x)$ **Ans**

$$y'(x) = -|\omega|e^{-kx}\sin(|\omega|x) - ke^{-kx}\cos(|\omega|x)$$

$$y''(x) = -\omega^2 e^{-kx}\cos(|\omega|x) + |\omega|ke^{-kx}\sin(|\omega|x) + |\omega|ke^{-kx}\sin(|\omega|x) + k^2e^{-kx}\cos(|\omega|x)$$

Plug y'', y', y into the given ODE;

$$(k^2-\omega^2)\cancel{e^{-kx}}\cos(|\omega|x) + 2|\omega|k\cancel{e^{-kx}}\sin(|\omega|x) - 2k|\omega|\cancel{e^{-kx}}\sin(|\omega|x) - 2k^2\cancel{e^{-kx}}\cos(|\omega|x) + (k^2+\omega^2)\cancel{e^{-kx}}\cos(|\omega|x) = 0$$

$$\left[k^2 - \omega^2 - 2k^2 + \cancel{k^2} + \omega^2\right]\cos(|\omega|x) + \left[2|\cancel{\omega}|k - 2\cancel{k}|\omega|\right]\sin(|\omega|x) = 0$$

$0 = 0$ ✓ Ck for the answer. **Ans**

IC. $\left.\begin{aligned} y(0) &= e^0\cos 0 = 1 \\ y'(0) &= -|\omega|e^0\sin 0 - ke^0\cos 0 = -k \end{aligned}\right\}$ ck I.C. **Ans**

2.2.33 $-y'' - y = 0$, IC1 : $y(0) = 1$, $y'(0) = -1$

IC 2 : $y(0) = 1.001$, $y'(0) = -0.999$

- explain why small change of 0.001 at $x = 0$ causes a large change later

Solⁿ Characteristic eqn : $\lambda^2 - 1 = 0 = (\lambda-1)(\lambda+1)$

$\lambda = -1, +1$

general solⁿ ; $y(x) = Ae^{-x} + Be^{x}$

$y'(x) = -Ae^{-x} + Be^{x}$

IC 1 $y(0) = 1 = A + B$, $y'(0) = -1 = -A + B$ } $A = 1$, $B = 0$

general solⁿ for IC1 ; $y_1(x) = e^{-x}$ Ans

IC 2 $y(0) = 1.001 = A + B$, $y'(0) = -0.999 = -A + B$ } $A = 1$, $B = 0.001$

general solⁿ for IC 2 ; $y_2(x) = e^{-x} + 0.001e^{x}$ Ans

At $x = 10$; $y_1(10) = e^{-10}$

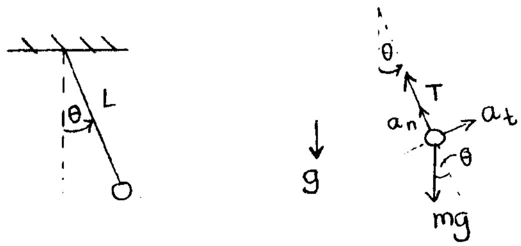
$y_2(10) = e^{-10} + 0.001e^{10}$

$y_2(10) - y_1(10) = 0.001e^{10} \approx 22$ *

A small change of the initial condition causes a large change later since in this example, when $y(0) = 1.001$ and $y'(0) = -0.999$, the solution include e^x into the total solution. e^x can cause an exponential change which makes large difference between the two solutions subjected to a small change of IC.

2.4.3 – Find the frequency of oscillation of a pendulum of length L, neglecting air resistance, weight of the rod, assume $\sin\theta \sim \theta$

Sol$^{\underline{n}}$



$\underset{\sim}{T}$ = tension force

m = mass of a pendulum ball

g = gravitational const

a_t = tangential acceleration

$$\Sigma F_t = m a_t \; ; \quad -\cancel{m} g \sin\theta = \cancel{m} a_t$$

$$\sin\theta \sim \theta \quad ; \quad -g\theta = a_t = L\alpha = L\theta''$$

Note that α = angular acceleration = θ''

$$L\theta'' + g\theta = 0$$

Characteristic eqn ; $L\lambda^2 + g = 0 \Rightarrow \lambda = \pm i\sqrt{\frac{g}{L}}$

$$\theta(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t)$$

Where $\omega_0 = \sqrt{\frac{g}{L}}$ $\frac{\text{rad}}{\text{s}}$ is the frequency of oscillation

or $\omega_0 = \frac{1}{2\pi}\sqrt{\frac{g}{L}}$ Hz Ans

Let IC for pendulum be $\theta(0) = \theta_0$, $\theta'(0) = 0$ (Velocity = 0 at t=0)

$$\theta'(t) = -\omega_0 A\sin(\omega_0 t) + \omega_0 B\cos(\omega_0 t)$$

$$\theta'(0) = 0 = \omega_0 B \Rightarrow B = 0$$

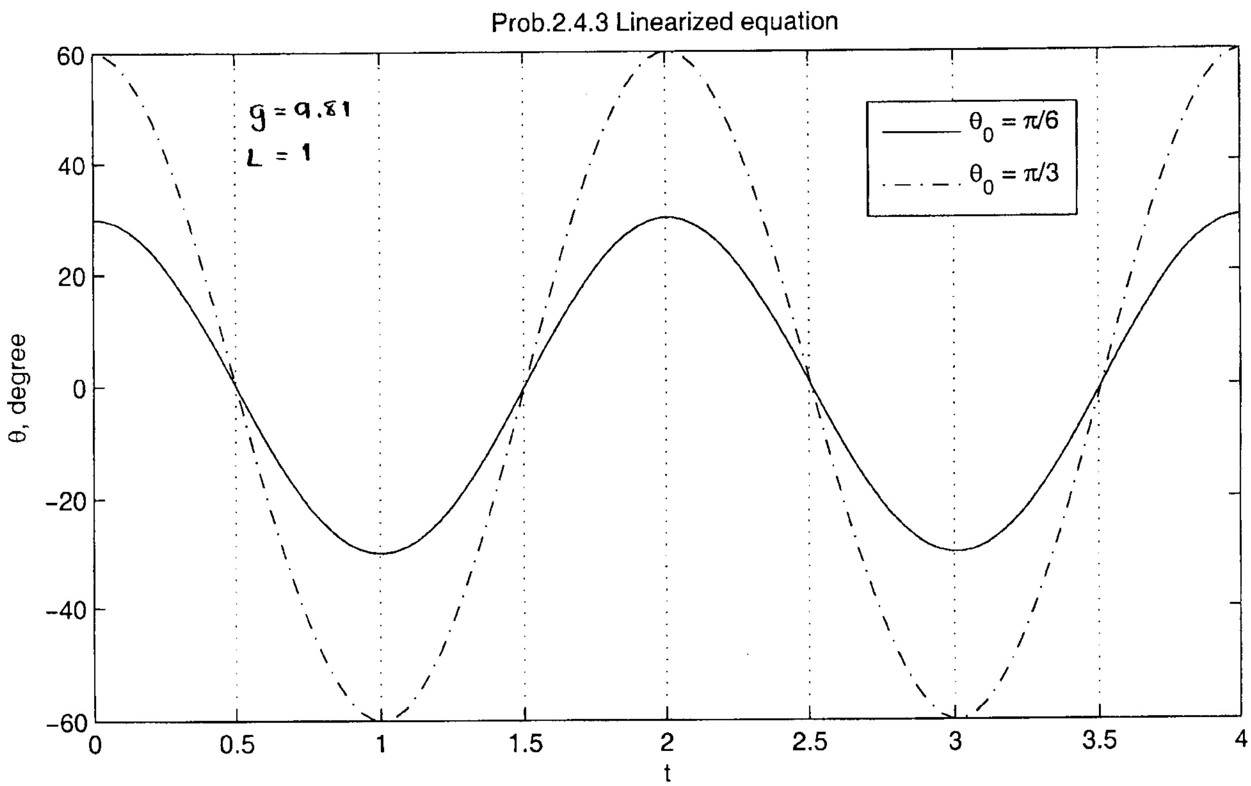
$$\theta(0) = \theta_0 = A\cancel{\cos 0}^{\,1} \Rightarrow A = \theta_0$$

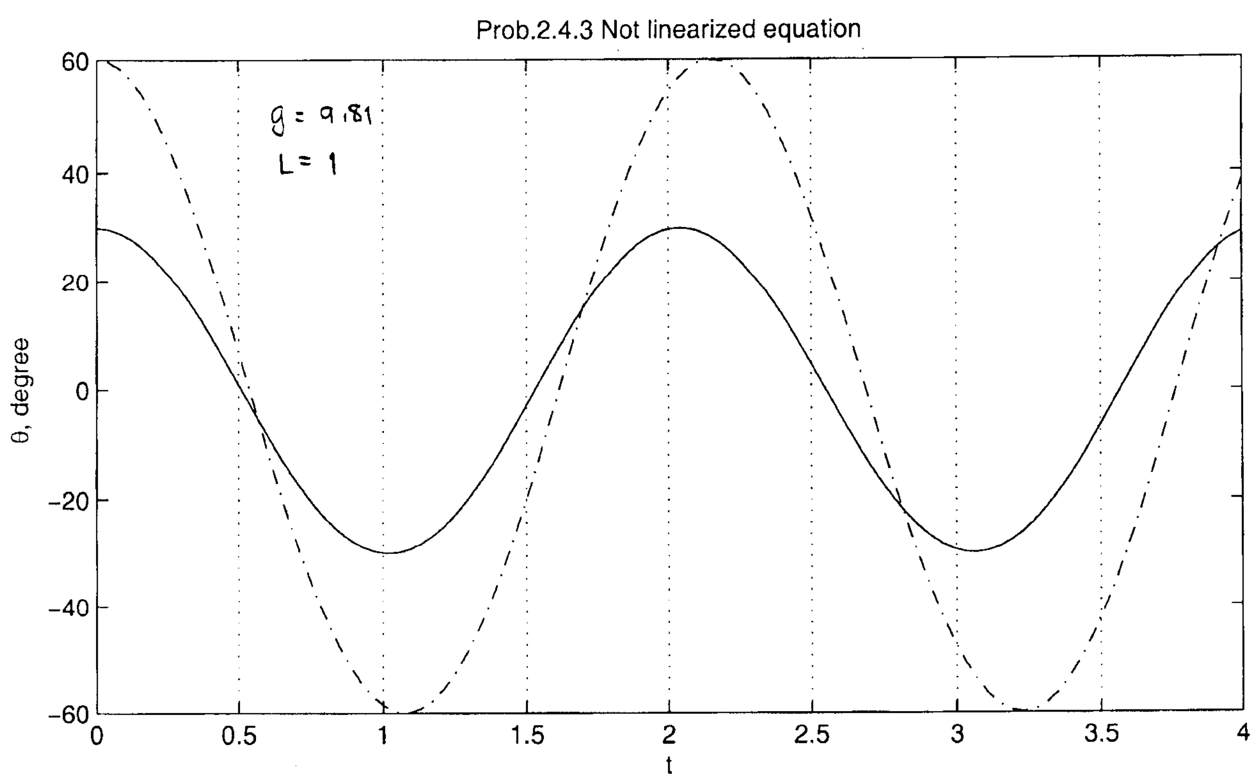
$$\therefore \quad \theta(t) = \theta_0 \cos(\omega_0 t)$$

If double the initial displacement, i.e. $\theta_{0,new} = 2\theta_0$, then

$$\theta_{new}(t) = 2\theta_0\cos(\omega_0 t)$$

$\therefore$ θ_{new} has the same period, $T = \frac{2\pi}{\omega_0}$, as θ since ω_0 remains unchange when the initial displacement is double. Only the amplitude changes. This is not the case when solving nonlinearized eq. In that case, both the amplitude and the period change (increase) as the initial displacement is double. (See graph)


Prob.2.4.3 Linearized equation
g = 9.81
L = 1
$\theta_0 = \pi/6$
$\theta_0 = \pi/3$
60
40
20
0
-20
-40
-60
θ, degree
0
0.5
1
1.5
2
2.5
3
3.5
4
t


Prob.2.4.3 Not linearized equation
g = 9.81
L = 1
60
40
20
0
-20
-40
-60
θ, degree
0
0.5
1
1.5
2
2.5
3
3.5
4
t

- Solve and graph the solution, show details of your work

2.5.11 $x^2y'' - 4xy' + 6y = 0$, $y(1) = 1$, $y'(1) = 0$

Sol^n Auxiliary eqn; $m^2 + (-4-1)m + 6 = 0$

$$m^2 - 5m + 6 = 0$$

$$(m-2)(m-3) = 0 \Rightarrow m = 2, 3$$

general sol^n; $y(x) = C_1x^2 + C_2x^3$ **Ans**

$$y'(x) = 2C_1x + 3C_2x^2$$

IC $y(1) = 1 = C_1 + C_2$, $y'(1) = 0 = 2C_1 + 3C_2$ $\Big\}$ $C_1 = 3$, $C_2 = -2$

particular sol^n; $y(x) = 3x^2 - 2x^3$ **Ans**

The graph is on the attached sheet

2.5.12 $x^2y'' + 3xy' + y = 0$, $y(1) = 4$, $y'(1) = -2$

Sol^n Auxiliary eqn; $m^2 + (3-1)m + 1 = 0$

$$m^2 + 2m + 1 = 0$$

$$(m+1)(m+1) = 0 \Rightarrow m = -1, -1$$

general sol^n; $y(x) = (C_1 + C_2\ln x)\, x^{-1}$ **Ans**

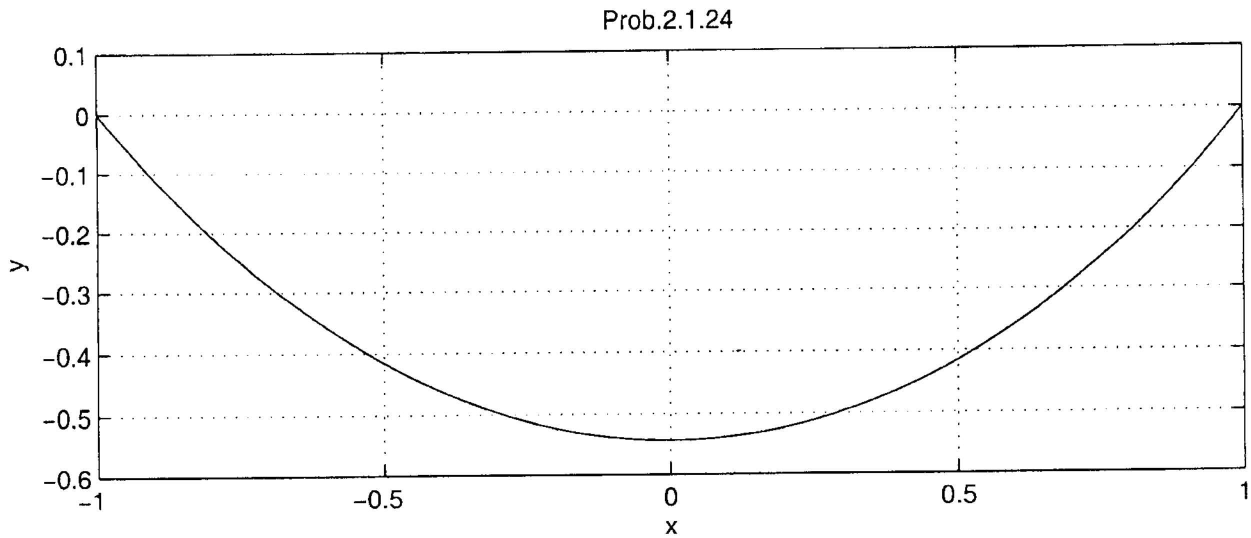
$$y'(x) = -(C_1 + C_2\ln x)\, x^{-2} + x^{-1}\frac{C_2}{x}$$

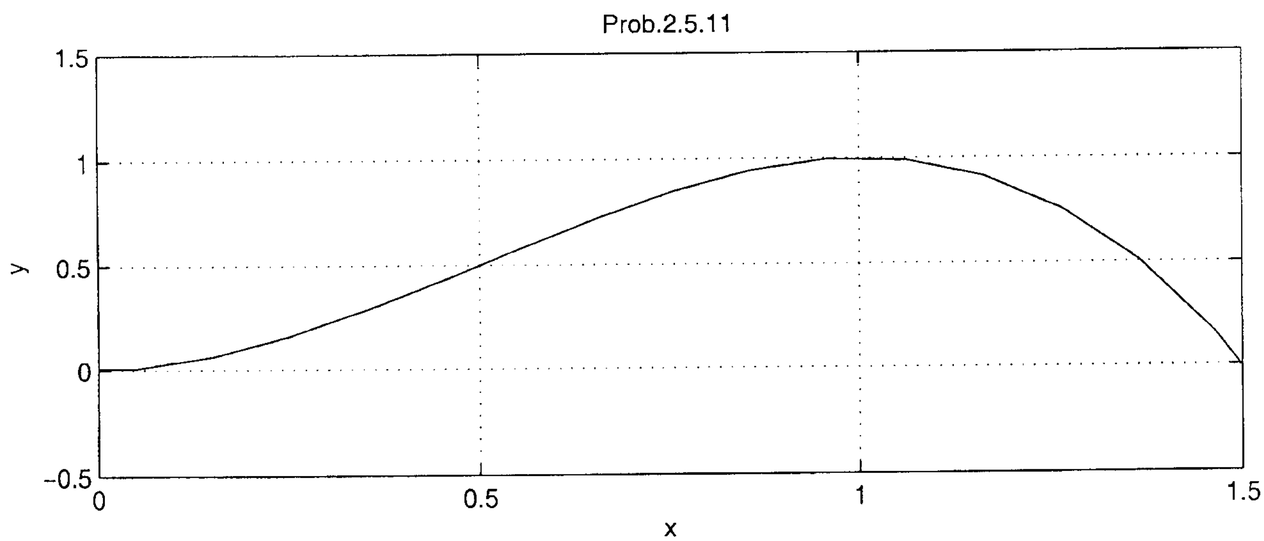
IC $y(1) = 4 = C_1 + C_2(0) \Rightarrow C_1 = 4$ *

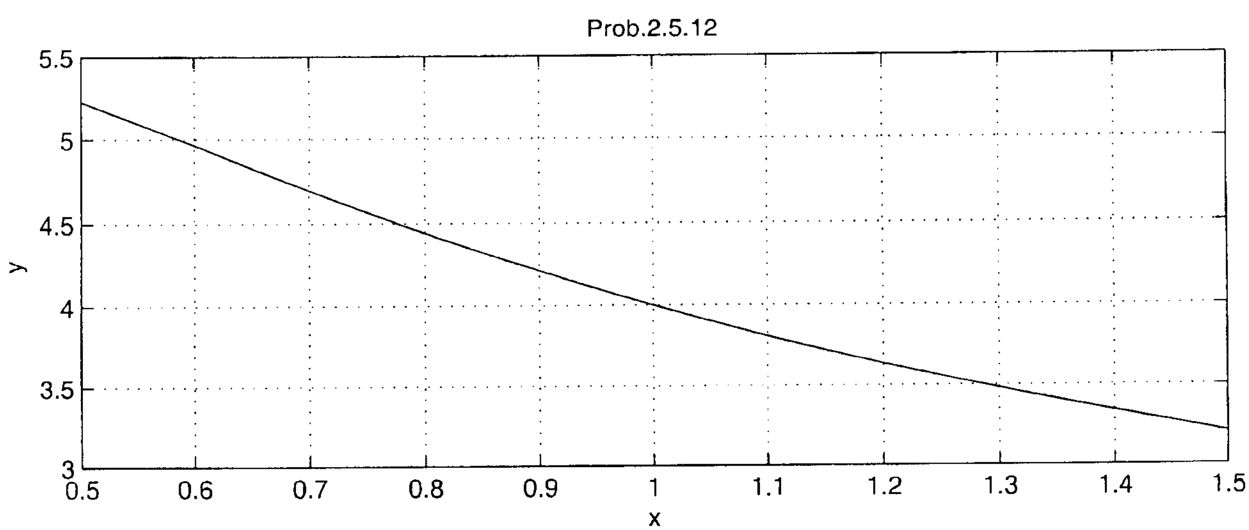
$y'(1) = -2 = -(4 + C_2(0)) \times 1 + 1 \times \frac{C_2}{1} \Rightarrow C_2 = 2$ *

particular sol^n; $y(x) = \dfrac{(4 + 2\ln x)}{x}$ **Ans**

The graph is on the attached sheet


Prob.2.1.24
y
x
0.1
0
−0.1
−0.2
−0.3
−0.4
−0.5
−0.6
−1
−0.5
0
0.5
1


Prob.2.5.11
y
x
1.5
1
0.5
0
−0.5
0
0.5
1
1.5


Prob.2.5.12
y
x
5.5
5
4.5
4
3.5
3
0.5
0.6
0.7
0.8
0.9
1
1.1
1.2
1.3
1.4
1.5

- Find a (real) general solution.
- Which rule are you using?
- Show each step of the calculation

2.7.5 $y'' + y' - 6y = 6x^3 - 3x^2 + 12x$ ——(1)

Sol^{n} 1) find homogeneous sol^{n} for the ODE, y_h

$$y'' + y' - 6y = 0 \qquad \text{——(2)}$$

Characteristic eqn of (2); $\lambda^2 + \lambda - 6 = 0$

$$(\lambda + 3)(\lambda - 2) = 0 \Rightarrow \lambda = -3, 2$$

Homogeneous sol^{n}; $y_h = c_1 e^{-3x} + c_2 e^{2x}$ *

2) find solution y_p of the nonhomogeneous ODE

Try $y_p = K_3 x^3 + K_2 x^2 + K_1 x + K_0$ → Using Basic Rule **Ans**

$$y_p' = 3K_3 x^2 + 2K_2 x + K_1$$

$$y_p'' = 6K_3 x + 2K_2$$

Substitute y_p'', y_p' and y_p in (1);

$$6K_3 x + 2K_2 + 3K_3 x^2 + 2K_2 x + K_1 - 6K_3 x^3 - 6K_2 x^2 - 6K_1 x - 6K_0$$
$$= 6x^3 - 3x^2 + 12x$$

Equating the coefficients,

$$-6K_3 = 6 \Rightarrow K_3 = -1$$

$$3K_3 - 6K_2 = -3 \Rightarrow K_2 = \frac{3K_3 + 3}{6} = \frac{3(-1)+3}{6} = 0$$

$$6K_3 + 2\cancel{K_2}^{\,0} - 6K_1 = 12 \Rightarrow K_1 = \frac{6K_3 - 12}{6} = \frac{6(-1)-12}{6} = -3$$

$$2\cancel{K_2}^{\,0} + K_1 - 6K_0 = 0 \Rightarrow K_0 = \frac{K_1}{6} = \frac{-3}{6} = -\frac{1}{2}$$

$$\therefore y_p = -x^3 - 3x - \frac{1}{2}$$

The general sol^{n} $y = y_h + y_p = c_1 e^{-3x} + c_2 e^{2x} - x^3 - 3x - \frac{1}{2}$; c_1 & c_2 = const. **Ans**

2.7.6 $y'' + 4y' + 4y = e^{-2x}\sin 2x$ — (1)

Sol^{n} 1) General solution of the homogeneous ODE, y_h

$$y'' + 4y' + 4y = 0 \quad — (2)$$

Characteristic eqn; $\lambda^2 + 4\lambda + 4 = 0$

$$(\lambda+2)(\lambda+2) = 0 \Rightarrow \lambda = -2, -2$$

general sol^{n}; $y_h = (C_1 + C_2 x)e^{-2x}$ *

2) Solution y_p of the nonhomogeneous ODE

Try $y_p = e^{-2x}(K\cos 2x + M\sin 2x)$ → Using Basic Rule Ans

$$y_p' = e^{-2x}(-2K\sin 2x + 2M\cos 2x) - 2e^{-2x}(K\cos 2x + M\sin 2x)$$

$$= -2e^{-2x}\left[(K-M)\cos 2x + (K+M)\sin 2x\right]$$

$$y_p'' = -2e^{-2x}\left[-2(K-M)\sin 2x + 2(K+M)\cos 2x\right]$$
$$+4e^{-2x}\left[(K-M)\cos 2x + (K+M)\sin 2x\right]$$

Substitute y_p'', y_p', y_p into (1);

$$4e^{-2x}\left[(K-M)\sin 2x - (K+M)\cos 2x\right] + 4e^{-2x}\left[(K-M)\cos 2x + (K+M)\sin 2x\right]$$
$$-8e^{-2x}\left[(K-M)\cos 2x + (K+M)\sin 2x\right] + 4e^{-2x}\left[K\cos 2x + M\sin 2x\right]$$
$$= e^{-2x}\sin 2x$$

Equating the coefficients,

$$-4(K+M) + 4(K-M) - 8(K-M) + 4K = 0 \Rightarrow -4(K+M+K-M-K) = 0$$
$$K = 0 \; *$$

$$4(K-M) + 4(K+M) - 8(K+M) + 4M = 1 \Rightarrow 4(\cancel{K}^{0} - M - \cancel{K}^{0} - M + M) = 1$$
$$M = -1/4$$

$\therefore$ $y_p = -\dfrac{e^{-2x}}{4}\sin 2x$

The general solution of the nonhomogeneous ODE is

$$y = y_h + y_p = (C_1 + C_2 x)e^{-2x} - \frac{1}{4}e^{-2x}\sin 2x \quad ; \; C_1 \,\&\, C_2 = \text{const}$$ Ans

- Solve the initial value problem
- State which rules you are using
- Show each step of the calculation in detail

2.7.18 $y'' - 2y' = 12e^{2x} - 8e^{-2x}$ —(1) , $y(0) = -2$, $y'(0) = 12$

Solⁿ 1) General solⁿ of the homogeneous ODE, y_h

$$y'' - 2y' = 0$$

Characteristic eqn ; $\lambda^2 - 2\lambda = 0 \Rightarrow \lambda(\lambda - 2) = 0$

$$\lambda = 0, 2$$

general solⁿ ; $y_h = C_1 + C_2 e^{2x}$

2) Solution y_p of the nonhomogeneous ODE

Try $y_p = Axe^{2x} + Be^{-2x}$ → Using Modification Rule and Sum Rule **Ans**

$$y_p' = 2Axe^{2x} + Ae^{2x} - 2Be^{-2x}$$

$$y_p'' = 4Axe^{2x} + 2Ae^{2x} + 2Ae^{2x} + 4Be^{-2x}$$

$$= 4Axe^{2x} + 4Ae^{2x} + 4Be^{-2x}$$

Substitute y_p'', y_p', y_p into (1)

$$4Axe^{2x} + 4Ae^{2x} + 4Be^{-2x} - 4Axe^{2x} - 2Ae^{2x} + 4Be^{-2x} = 12e^{2x} - 8e^{-2x}$$

Equating the coefficients

$$\cancel{4Ax} + 4A - \cancel{4Ax} - 2A = 12 \Rightarrow 2A = 12 \Rightarrow A = 6$$

$$4B + 4B = -8 \Rightarrow 8B = -8 \Rightarrow B = -1$$

$$\therefore y_p = 6xe^{2x} - e^{-2x}$$

The general solⁿ of the nonhomogeneous ODE is

$$y = y_h + y_p = C_1 + C_2e^{2x} + 6xe^{2x} - e^{-2x} = (C_2 + 6x)e^{2x} - e^{-2x} + C_1 \quad *$$

3) Solution of the initial value problem

IC $y(0) = -2 = (C_2 + 6(0))e^{2(0)} - e^{-2(0)} + C_1 \Rightarrow C_2 + C_1 = -1$ —(2)

$$y(x) = (C_2 + 6x)e^{2x} - e^{-2x} + C_1$$

$$y'(x) = 2(C_2 + 6x)e^{2x} + 6e^{2x} + 2e^{-2x}$$

IC. $y'(0) = 12 = 2(C_2 + 6(0))e^{2(0)} + 6e^{2(0)} + 2e^{-2(0)}$

$$12 = 2C_2 + 6 + 2 \Rightarrow C_2 = \frac{4}{2} = 2 \quad *$$

From (1) $\quad C_2 + C_1 = -1 \Rightarrow C_1 = -1 - C_2 = -1 - 2 = -3 \quad *$

$\therefore$ The solution to the given initial value problem is

$$y = (2 + 6x)e^{2x} - e^{-2x} - 3 \qquad \underline{\underline{\text{Ans}}}$$

- Solve the given nonhomogeneous ODE by variation of parameters or undetermined coefficients
- Give a general soln (Show the details of your work)

<u>2.10.3</u> $\quad x^2y'' - 2xy' + 2y = x^3\cos x \quad$ —(1)

<u>Soln</u> $\quad$ 1) General solution of the homogeneous ODE, y_h

$$x^2y'' - 2xy' + 2y = 0$$

Auxiliary eqn; $\quad m^2 + (-2-1)m + 2 = 0$

$$m^2 - 3m + 2 = 0$$

$$(m-2)(m-1) = 0 \Rightarrow m = 2, 1$$

The general soln, $\quad y_h = C_1x + C_2x^2 \quad *$

2) Solution y_p of the nonhomogeneous ODE

Using variation of parameters,

$$y_1 = x, \quad y_2 = x^2, \quad r = \frac{x^3}{x^2}\cos x = x\cos x$$

The Wronskien, $\quad W(y_1, y_2) = y_1y_2' - y_2y_1' = x \cdot 2x - x^2$

$$= x^2$$

$$y_p(x) = -x\int \frac{\cancel{x^2} \cdot x\cos x}{\cancel{x^2}}\,dx + x^2\int \frac{\cancel{x}\cos x}{\cancel{x^2}}\,dx$$

Consider $\int x\cos x\,dx = x\sin x - \int \sin x\,dx$; $\int u\,dv = uv - \int v\,du$

$$= x\sin x + \cos x + C$$

Choosing zero constants of integration,

$$y_p(x) = -x\,(x\sin x + \cos x) + x^2\sin x$$

$$= -x\cos x$$

The general solution of the given ODE is

$$y(x) = y_h(x) + y_p(x) = C_1 x + C_2 x^2 - x\cos x \,.$$ Ans

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2.10.8 $\quad y'' - 4y' + 4y = 12e^{2x}/x^4$

Sol$^{\underline{n}}$ 1) General solution of the homogeneous ODE, y_h

$$y'' - 4y' + 4y = 0$$

Characteristic eqn, $\lambda^2 - 4\lambda + 4 = 0 \Rightarrow (\lambda-2)(\lambda-2) = 0$

$$\lambda = 2, 2$$

general sol$^{\underline{n}}$; $\quad y_h(x) = (C_1 + C_2 x)e^{2x}$

2) Solution y_p of the nonhomogeneous ODE

Using variation of parameters,

$$y_1 = e^{2x}, \quad y_2 = xe^{2x}, \quad r = \frac{12e^{2x}}{x^4}$$

The Wronskien, $W(y_1, y_2) = y_1 y_2' - y_2 y_1'$

$$= e^{2x}(2xe^{2x} + e^{2x}) - xe^{2x}\cdot 2e^{2x}$$

$$= e^{4x}$$

$$y_p(x) = -e^{2x}\int \frac{x\cancel{e^{2x}}\cdot 12\cancel{e^{2x}}/x^4}{\cancel{e^{4x}}}\,dx + xe^{2x}\int \frac{\cancel{e^{2x}}\cdot 12e^{2x}/x^4}{\cancel{e^{4x}}}\,dx$$

$$= -e^{2x}\int 12x^{-3}\,dx + xe^{2x}\int 12x^{-4}\,dx = -12e^{2x}\frac{x^{-2}}{-2} + 12xe^{2x}\frac{x^{-3}}{-3}$$

$$= 6e^{2x}x^{-2} - 4e^{2x}x^{-2} = 2e^{2x}/x^2$$; choosing zero const. of integration

$\therefore$ The general sol$^{\underline{n}}$ $\quad y(x) = y_h + y_p = (c_1 + c_2 x)e^{2x} + \dfrac{2e^{2x}}{x^2}$ Ans

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