- Solve the following ODEs (Show the details)

3.3.1 $\quad y''' - 2y'' - 4y' + 8y = e^{-3x} + 8x^2$ —(1)

Sol^{n} 1) General solution of the homogeneous ODE., y_h

$$y''' - 2y'' - 4y' + 8y = 0$$

Characteristic eq^n; $\lambda^3 - 2\lambda^2 - 4\lambda + 8 = 0$

$$(\lambda - 2)(\lambda^2 - 4) = 0$$

$$(\lambda - 2)(\lambda - 2)(\lambda + 2) = 0 \Rightarrow \lambda = -2, 2, 2$$

$$y_h = c_1 e^{-2x} + (c_2 + c_3 x) e^{2x} \quad *$$

2) Find y_p

Try $y_p = Ae^{-3x} + Bx^2 + Cx + D \Rightarrow$ Use Sum Rule

$$y_p' = -3Ae^{-3x} + 2Bx + C$$

$$y_p'' = 9Ae^{-3x} + 2B$$

$$y_p''' = -27Ae^{-3x}$$

Substitute y_p''', y_p'', y_p', y_p into (1)

$$-27Ae^{-3x} - 18Ae^{-3x} - 4B + 12Ae^{-3x} - 8Bx - 4C + 8Ae^{-3x} + 8Bx^2$$

$$+ 8Cx + 8D = e^{-3x} + 8x^2$$

Equating coefficients;

$$-27A - 18A + 12A + 8A = 1 \Rightarrow -25A = 1, \quad A = -0.04$$

$$8B = 8 \Rightarrow B = 1$$

$$-8B + 8C = 0 \Rightarrow C = \frac{8B}{8} = 1$$

$$-4B - 4C + 8D = 0 \Rightarrow D = \frac{4B + 4C}{8} = 1$$

$$y_p = -0.04e^{-3x} + x^2 + x + 1 \quad *$$

$\therefore$ The general solution of the given ODE is

$$y = y_h + y_p = c_1 e^{-2x} + (c_2 + c_3 x)e^{2x} - 0.04e^{-3x} + x^2 + x + 1 \quad \underline{\text{Ans}}$$

3) Convert $y''' - 2y'' - 4y' + 8y = e^{-3x} + 8x^2$ to an equivalent system of 1^{st} order ODEs,

let $y_1 = y$, $y_2 = y'$, $y_3 = y''$, the given ODE can be rewritten as

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}' = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -8 & 4 & 2 \end{bmatrix}}_{\underline{\underline{A}}} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{-3x} + \begin{bmatrix} 0 \\ 0 \\ 8 \end{bmatrix} x^2}_{\underset{\sim}{f}}$$

Find eigenvalues: $|\underline{\underline{A}} - \lambda \underline{\underline{I}}| = 0$

$$\begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ -8 & 4 & 2-\lambda \end{vmatrix} = -\lambda[-\lambda(2-\lambda) - 4] - 1(+8)$$

$$= 2\lambda^2 - \lambda^3 + 4\lambda - 8 = 0$$

Characteristic eqn ; $\lambda^3 - 2\lambda^2 - 4\lambda + 8 = 0$

$$(\lambda - 2)(\lambda^2 - 4) = 0$$

$$(\lambda-2)(\lambda-2)(\lambda+2) = 0 \Rightarrow \lambda_1 = -2 , \lambda_2 = \lambda_3 = 2$$

Find associate eigenvectors

$\underline{\lambda_1 = -2}$; $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ -8 & 4 & 2+2 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ -8 & 4 & 4 \end{bmatrix} \begin{matrix} R_1 \\ R_2 \\ \xrightarrow{R_3 + 4R_1} \end{matrix} \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 8 & 4 \end{bmatrix} \begin{matrix} R_1 \\ R_2 \\ \xrightarrow{R_3 - 4R_2} \end{matrix} \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$$2V_1 + V_2 = 0 \Rightarrow V_1 = \frac{-V_2}{2}$$

$$2V_2 + V_3 = 0 \Rightarrow V_3 = -2V_2$$

Choose $V_2 = -2$; $\underset{\sim}{V}^{(1)} = [1 \ -2 \ 4]^T$, $\underset{\sim}{y}_{h_1} = e^{-2x}[1 \ -2 \ 4]^T$

$\underline{\lambda_2 = 2}$; $\begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ -8 & 4 & 2-2 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ -8 & 4 & 0 \end{bmatrix} \begin{matrix} R_1 \\ R_2 \\ \xrightarrow{R_3 - 4R_1} \end{matrix} \begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$$-2V_1 + V_2 = 0 \Rightarrow V_1 = \frac{V_2}{2}$$

$$-2V_2 + V_3 = 0 \Rightarrow V_3 = 2V_2$$

Choose $V_2 = 2$; $\underset{\sim}{V}^{(2)} = [1 \ 2 \ 4]^T$, $\underset{\sim}{y}_{h_2} = e^{2x}[1 \ 2 \ 4]^T$

$\underset{\sim}{y}_{h_3} = e^{2x}(\underset{\sim}{a}x + \underset{\sim}{b})$ where $\underset{\sim}{a} = \underset{\sim}{V}^{(2)} = [1 \ 2 \ 4]^T$

$$(\underline{\underline{A}} - \lambda_3 \underline{\underline{I}})\,\underset{\sim}{b} = \underset{\sim}{v}^{(2)}$$

$$\begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ -8 & 4 & 0 \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$$

$$R_3 - 4R_1 \begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} -2b_1 + b_2 &= 1 \\ -2b_2 + b_3 &= 2 \end{aligned}\right\} \text{Choose } b_3 = 0 \Rightarrow b_2 = -1,\ b_1 = -1$$

$$\underset{\sim}{b} = [-1 \ \ -1 \ \ 0]^T$$

$$\underset{\sim}{y}_{h_3} = e^{2x}\left[\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} x + \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix}\right]$$

$$\underset{\sim}{y}_h = c_1 e^{-2x}\begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix} + c_2 e^{2x}\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} + e^{2x}\left\{ c_3\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} x + c_4\begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}\right\}$$

If consider only $y_1 = y$; $\underset{\sim}{y} = [y \ \ y' \ \ y'']^T = [y_1 \ \ y_2 \ \ y_3]^T$

$$y = c_1 e^{-2x} + c_2 e^{2x} + c_3 x e^{2x} - c_4 e^{2x}$$

$$y = b_1 e^{-2x} + (b_2 + b_3 x) e^{2x} \qquad ; \ b_1 = c_1,\ b_2 = c_2 - c_4,\ b_3 = c_3$$

Which is the same as y_h obtained by solving 3^{rd} order ODE directly.

4) Find $\underset{\sim}{y}_P$ of the system of 1^{st} order ODE

Try
$$\underset{\sim}{y}_P = \begin{bmatrix} A \\ B \\ C \end{bmatrix} e^{-3x} + \begin{bmatrix} D \\ E \\ F \end{bmatrix} x^2 + \begin{bmatrix} G \\ H \\ I \end{bmatrix} x + \begin{bmatrix} J \\ K \\ L \end{bmatrix}$$

$$\underset{\sim}{y}_P' = -3e^{-3x}\begin{bmatrix} A \\ B \\ C \end{bmatrix} + 2x\begin{bmatrix} D \\ E \\ F \end{bmatrix} + \begin{bmatrix} G \\ H \\ I \end{bmatrix}$$

$\underset{\sim}{y}_P' = \underline{\underline{A}}\,\underset{\sim}{y}_P + \underset{\sim}{f}$;

$$-3e^{-3x}\begin{bmatrix} A \\ B \\ C \end{bmatrix} + 2x\begin{bmatrix} D \\ E \\ F \end{bmatrix} + \begin{bmatrix} G \\ H \\ I \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -8 & 4 & 2 \end{bmatrix}\begin{bmatrix} A \\ B \\ C \end{bmatrix} e^{-3x} + \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -8 & 4 & 2 \end{bmatrix}\begin{bmatrix} D \\ E \\ F \end{bmatrix} x^2$$
$$+ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -8 & 4 & 2 \end{bmatrix}\begin{bmatrix} G \\ H \\ I \end{bmatrix} x + \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -8 & 4 & 2 \end{bmatrix}\begin{bmatrix} J \\ K \\ L \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{-3x} + \begin{bmatrix} 0 \\ 0 \\ 8 \end{bmatrix} x^2$$

For each row of the above system, we got

$$-3Ae^{-3x} + 2Dx + G = Be^{-3x} + Ex^2 + Hx + K$$

$$-3Be^{-3x} + 2Ex + H = Ce^{-3x} + Fx^2 + Ix + L$$

$$-3Ce^{-3x} + 2Fx + I = (-8A + 4B + 2C)e^{-3x} + (-8D + 4E + 2F)x^2 + (-8G + 4H + 2I)x + (-8J + 4K + 2L) + e^{-3x} + 8x^2$$

Equating coefficients;

$-3A = B$; $2D = H$; $E = 0$ * ; $K = G$ *

$-3B = C$; $2E = I$; $F = 0$ * ; $L = H$ *

$-3C = -8A + 4B + 2C + 1 \Rightarrow 8A - 4B - 5C = 1$

$8A - 4(-3A) - 5(-3(-3A)) = 1$

$8A + 12A - 45A = 1 \Rightarrow A = \frac{-1}{25} = -0.04$ *

$B = -3A = 0.12$ *

$C = -3B = -0.36$ *

$\cancel{2F}^{0} = -8G + 4H + 2I \Rightarrow 8G - 4H - \cancel{2I}^{0} = 0 \Rightarrow 2G = H$ $(I = 2E$ & $E = 0)$

$-8D + 4E + \cancel{2F}^{0} + 8 = 0 \qquad 2D - \cancel{E}^{0} = 2 \Rightarrow D = 1$ * $\Rightarrow H = 2D = 2$ *

$G = \frac{H}{2} = 1$ *

$\cancel{I} = -8J + 4K + 2L \Rightarrow 8J = 4K + 2L = 4(1) + 2(2) = 8$

$0\ (I = 2E$ but $E = 0)$

$J = 1$ *

$L = H = 2$ *

$K = G = 1$ *

$$\therefore \quad \underset{\sim}{y}_p = \begin{bmatrix} -0.04 \\ 0.12 \\ -0.36 \end{bmatrix} e^{-3x} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} x^2 + \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

The general solution, $\underset{\sim}{y} = \underset{\sim}{y}_h + \underset{\sim}{y}_p$

$$\begin{bmatrix} y \\ y' \\ y'' \end{bmatrix} = c_1 e^{-2x}\begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix} + c_2 e^{2x}\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} + e^{2x}\left\{ c_3\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} x + c_4 \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}\right\} + \begin{bmatrix} -0.04 \\ 0.12 \\ -0.36 \end{bmatrix} e^{-3x} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} x^2 + \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

for y; $y = c_1 e^{-2x} + [(c_2 - c_4) + c_3 x]e^{2x} - 0.04e^{-3x} + x^2 + x + 1$

$= b_1 e^{-2x} + (b_2 + b_3 x)e^{2x} - 0.04e^{-3x} + x^2 + x + 1 \Rightarrow$ same as solving a 3rd order ODE

Ans

3.3.4 $y''' + 2y'' - 5y' - 6y = 100e^{-3x} + 18e^{-x}$ —(1)

Soln 1) General solution of the homogeneous ODE, y_h

$$y''' + 2y'' - 5y' - 6y = 0$$

Characteristic eqn; $\lambda^3 + 2\lambda^2 - 5\lambda - 6 = 0$

$$(\lambda+3)(\lambda+1)(\lambda-2) = 0 \Rightarrow \lambda = -3, -1, 2$$

$$y_h = c_1 e^{-3x} + c_2 e^{-x} + c_3 e^{2x} \quad *$$

2) Find y_p

Try

$$y_p = Axe^{-3x} + Bxe^{-x}$$

$$y_p' = -3Axe^{-3x} + Ae^{-3x} - Bxe^{-x} + Be^{-x}$$

$$y_p'' = 9Axe^{-3x} - 3Ae^{-3x} - 3Ae^{-3x} + Bxe^{-x} - Be^{-x} - Be^{-x}$$

$$= 9Axe^{-3x} - 6Ae^{-3x} + Bxe^{-x} - 2Be^{-x}$$

$$y_p''' = -27Axe^{-3x} + 9Ae^{-3x} + 18Ae^{-3x} - Bxe^{-x} + Be^{-x} + 2Be^{-x}$$

$$= -27Axe^{-3x} + 27Ae^{-3x} - Bxe^{-x} + 3Be^{-x}$$

Substitute y_p''', y_p'', y_p', y_p into (1);

$$-27\cancel{Ax}e^{-3x} + 27Ae^{-3x} - \cancel{Bx}e^{-x} + 3Be^{-x} + 18\cancel{Ax}e^{-3x} - 12Ae^{-3x} + 2\cancel{Bx}e^{-x} - 4Be^{-x}$$
$$+ 15\cancel{Ax}e^{-3x} - 5Ae^{-3x} + 5\cancel{Bx}e^{-x} - 5Be^{-x} - 6\cancel{Ax}e^{-3x} - 6\cancel{Bx}e^{-x} = 100e^{-3x} + 18e^{-x}$$

Equating coefficients;

$$27A - 12A - 5A = 100 \Rightarrow 10A = 100, \quad A = 10$$

$$3B - 4B - 5B = 18 \Rightarrow -6B = 18, \quad B = -3$$

$$y_p = 10xe^{-3x} - 3xe^{-x}$$

$\therefore$ The general solution of the given ODE is,

$$y = y_h + y_p = (c_1 + 10x)e^{-3x} + (c_2 - 3x)e^{-x} + c_3 e^{2x}. \quad \underline{\underline{\text{Ans}}}$$

3.3.10 $\quad y^{iv} - 16y = 128\cosh 2x$ — (1); $\quad y(0) = 1,\ y'(0) = 24,$

$y''(0) = 20,\quad y'''(0) = -160$

Sol^n 1) General solution of the homogeneous ODE, y_h

$$y^{iv} - 16y = 0$$

Characteristic eqn; $\quad \lambda^4 - 16 = 0 = (\lambda^2 - 4)(\lambda^2 + 4)$

$$(\lambda-2)(\lambda+2)(\lambda^2+4) = 0 \Rightarrow \lambda = -2, 2, -2i, 2i$$

$$y_h = C_1 e^{-2x} + C_2 e^{2x} + C_3\cos 2x + C_4 \sin 2x$$

2) Find y_p

Since $\quad \cosh 2x = \dfrac{e^{2x} + e^{-2x}}{2}$

Try
$$y_p = Axe^{-2x} + Bxe^{2x}$$
$$y_p' = -2Axe^{-2x} + Ae^{-2x} + 2Bxe^{2x} + Be^{2x}$$
$$y_p'' = 4Axe^{-2x} - 2Ae^{-2x} - 2Ae^{-2x} + 4Bxe^{2x} + 2Be^{2x} + 2Be^{2x}$$
$$= 4Axe^{-2x} - 4Ae^{-2x} + 4Bxe^{2x} + 4Be^{2x}$$
$$y_p''' = -8Axe^{-2x} + 4Ae^{-2x} + 8Ae^{-2x} + 8Bxe^{2x} + 4Be^{2x} + 8Be^{2x}$$
$$= -8Axe^{-2x} + 12Ae^{-2x} + 8Bxe^{2x} + 12Be^{2x}$$
$$y_p^{iv} = 16Axe^{-2x} - 8Ae^{-2x} - 24Ae^{-2x} + 16Bxe^{2x} + 8Be^{2x} + 24Be^{2x}$$
$$= 16Axe^{-2x} - 32Ae^{-2x} + 16Bxe^{2x} + 32Be^{2x}$$

Subsitute y_p^{iv}, y_p into (1);

$$16\cancel{Ax}e^{-2x} - 32Ae^{-2x} + 16\cancel{Bx}e^{2x} + 32Be^{2x} - 16\cancel{Ax}e^{-2x} - 16\cancel{Bx}e^{2x}$$
$$= 128\cosh 2x = \underset{64}{\cancel{128}}\left[\frac{e^{2x} + e^{-2x}}{\cancel{2}}\right]$$

Equating coefficients;

$$\left.\begin{aligned} -32A &= 64 \Rightarrow A = -2 \\ 32B &= 64 \Rightarrow B = 2 \end{aligned}\right\} y_p = -2xe^{-2x} + 2xe^{2x}$$

$\therefore$ The general solution of the given ODE is

$$y = y_h + y_p = (C_1 - 2x)e^{-2x} + (C_2 + 2x)e^{2x} + C_3\cos 2x + C_4 \sin 2x$$

$$\text{or} \quad y = c_1 e^{-2x} + c_2 e^{2x} + 4x\sinh 2x + c_3\cos 2x + c_4 \sin 2x$$

$$y(0) = 1 = c_1 + c_2 + c_3 \qquad \text{— (2)}$$

$$y' = -2c_1 e^{-2x} + 2c_2 e^{2x} + 8x\cosh 2x + 4\sinh 2x - 2c_3 \sin 2x + 2c_4\cos 2x$$

$$y'(0) = 24 = -2c_1 + 2c_2 + 2c_4 \Rightarrow -12 = c_1 - c_2 - c_4 \qquad \text{— (3)}$$

$$y'' = 4c_1 e^{-2x} + 4c_2 e^{2x} + 16x\sinh 2x + 8\cosh 2x + 8\cosh 2x$$

$$-4c_3\cos 2x - 4c_4 \sin 2x$$

$$y''(0) = 20 = 4c_1 + 4c_2 + 16 - 4c_3$$

$$1 = c_1 + c_2 - c_3 \qquad \text{— (4)}$$

$$y''' = -8c_1 e^{-2x} + 8c_2 e^{2x} + 32x\cosh 2x + 16\sinh 2x + 32\sinh 2x$$

$$+8c_3 \sin 2x - 8c_4\cos 2x$$

$$y'''(0) = -160 = -8c_1 + 8c_2 - 8c_4$$

$$20 = c_1 - c_2 + c_4 \qquad \text{— (5)}$$

$$\left.\begin{array}{ll} (2)+(4); & 2c_1 + 2c_2 = 2 \\ (3)+(5); & 2c_1 - 2c_2 = 8 \end{array}\right\} \quad c_1 = 2.5, \; c_2 = \frac{-6}{4} = -1.5$$

$$\text{from (2)}; \quad c_3 = 1 - c_1 - c_2 = 1 - 2.5 - (-1.5) = 0$$

$$\text{from (3)}; \quad c_4 = 12 + c_1 - c_2 = 12 + 2.5 - (-1.5) = 16$$

$$\therefore \; y = 2.5e^{-2x} - 1.5e^{2x} + 16\sin 2x + 4x\sinh 2x \qquad \underline{\underline{\text{Ans}}}$$

$$\text{or} \quad y = 4e^{-2x} - 1.5(e^{2x} + e^{-2x}) + 16\sin 2x + 4x\sinh 2x$$

$$= 4e^{-2x} - 3\cosh 2x + 4x\sinh 2x + 16\sin 2x \qquad \underline{\underline{\text{Ans}}}$$

Solve the following systems of linear 1st order ODEs

a) $x' = x - y + 4z$

$y' = 3x + 2y - z$

$z' = 2x + y - z + t$

Sol^n The above systems can be written as

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}' = \begin{bmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} t}_{\underset{\sim}{f}}$$

1) General solution of the homogeneous ODEs

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{bmatrix}}_{\underline{\underline{A}}} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Find eigenvalues : $|\underline{\underline{A}} - \lambda \underline{\underline{I}}| = 0$

$$\begin{vmatrix} 1-\lambda & -1 & 4 \\ 3 & 2-\lambda & -1 \\ 2 & 1 & -1-\lambda \end{vmatrix} = (1-\lambda)\left[(2-\lambda)(-1-\lambda)+1\right] + 3(-1-\lambda) + 2 + 4\left[3 - 2(2-\lambda)\right]$$

$$= -\lambda^3 + 2\lambda^2 + 5\lambda - 6 = 0$$

$$(\lambda - 3)(\lambda - 1)(\lambda + 2) = 0$$

$$\lambda_1 = -2, \ \lambda_2 = 1, \ \lambda_3 = 3$$

Find eigenvectors,

$\lambda = -2$; $\begin{bmatrix} 1+2 & -1 & 4 \\ 3 & 2+2 & -1 \\ 2 & 1 & -1+2 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 4 \\ 3 & 4 & -1 \\ 2 & 1 & 1 \end{bmatrix} \begin{matrix} R_1 \\ R_2 - R_1 \\ R_3 - \frac{2}{3}R_1 \end{matrix} \xrightarrow{} \begin{bmatrix} 3 & -1 & 4 \\ 0 & 5 & -5 \\ 0 & 5/3 & -5/3 \end{bmatrix} \begin{matrix} R_1 \\ R_2 \\ R_3 - \frac{R_2}{3} \end{matrix} \xrightarrow{}$

$$\begin{bmatrix} 3 & -1 & 4 \\ 0 & 5 & -5 \\ 0 & 0 & 0 \end{bmatrix}$$

$5v_2 - 5v_3 = 0 \Rightarrow v_2 = v_3$

$3v_1 - v_2 + 4v_3 = 0 \Rightarrow v_1 = \dfrac{v_2 - 4v_3}{3} = -v_3$

Choose $v_3 = 1$; $\underset{\sim}{v}^{(1)} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$ *

$\underline{\lambda = 1}$; $\begin{bmatrix} 1-1 & -1 & 4 \\ 3 & 2-1 & -1 \\ 2 & 1 & -1-1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 4 \\ 3 & 1 & -1 \\ 2 & 1 & -2 \end{bmatrix} \begin{matrix} R_1 \\ R_2 - \frac{3}{2}R_3 \\ R_3 \end{matrix} \begin{bmatrix} 0 & -1 & 4 \\ 0 & -1/2 & 4/2 \\ 2 & 1 & -2 \end{bmatrix}$

$\begin{matrix} R_1 \\ R_2 - \frac{R_1}{2} \\ R_3 \end{matrix} \longrightarrow \begin{bmatrix} 0 & -1 & 4 \\ 0 & 0 & 0 \\ 2 & 1 & -2 \end{bmatrix}$

$-v_2 + 4v_3 = 0 \Rightarrow v_2 = 4v_3$

$2v_1 + v_2 - 2v_3 = 0 \Rightarrow v_1 = \dfrac{2v_3 - v_2}{2} = -v_3$

Choose $v_3 = 1$; $\quad \underset{\sim}{v}^{(2)} = \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix}$ *

$\underline{\lambda = 3}$; $\begin{bmatrix} 1-3 & -1 & 4 \\ 3 & 2-3 & -1 \\ 2 & 1 & -1-3 \end{bmatrix} = \begin{bmatrix} -2 & -1 & 4 \\ 3 & -1 & -1 \\ 2 & 1 & -4 \end{bmatrix} \begin{matrix} R_1 \\ R_2 + \frac{3}{2}R_1 \\ R_3 + R_1 \end{matrix} \longrightarrow \begin{bmatrix} -2 & -1 & 4 \\ 0 & -5/2 & 5 \\ 0 & 0 & 0 \end{bmatrix}$

$-\frac{5}{2}v_2 + 5v_3 = 0 \Rightarrow v_2 = 2v_3$

$-2v_1 - v_2 + 4v_3 = 0 \Rightarrow v_1 = \dfrac{4v_3 - v_2}{2} = v_3$

Choose $v_3 = 1$; $\quad \underset{\sim}{v}^{(3)} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ *

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}_c = c_1 e^{-2t} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} + c_2 e^{t} \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix} + c_3 e^{3t} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \quad *$$

2) Find y_p

Try $\quad y_p = \begin{bmatrix} A \\ B \\ C \end{bmatrix} t + \begin{bmatrix} D \\ E \\ F \end{bmatrix}$

$y_p' = \begin{bmatrix} A \\ B \\ C \end{bmatrix}$

$\underset{\sim}{y_p'} = \underset{=}{A}\,\underset{\sim}{y_p} + \underset{\sim}{f}$; $\quad \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} t + \begin{bmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} D \\ E \\ F \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} t$

For each row of the above system;

$A = (A - B + 4C)t + (D - E + 4F)$

$B = (3A + 2B - C)t + (3D + 2E - F)$

$C = (2A + B - C)t + (2D + E - F) + t$

Equating coefficients,

$$A - B + 4C = 0 \quad -(1); \quad D - E + 4F = A \quad -(4)$$

$$3A + 2B - C = 0 \quad -(2); \quad 3D + 2E - F = B \quad -(5)$$

$$2A + B - C + 1 = 0 \quad -(3); \quad 2D + E - F = C \quad -(6)$$

$$\left.\begin{array}{ll} 2\times(1)+(2); & 5A + 7C = 0 \\ (1)+(3); & 3A + 3C = -1 \end{array}\right\} A = \frac{-7}{6}, \; C = \frac{5}{6}$$

Substitute in (1); $\frac{-7}{6} - B + 4\left(\frac{5}{6}\right) = 0 \Rightarrow B = \frac{13}{6}$

Thus (4) $\Rightarrow$ $D - E + 4F = \frac{-7}{6}$

(5) $\Rightarrow$ $3D + 2E - F = \frac{13}{6}$

(6) $\Rightarrow$ $2D + E - F = \frac{5}{6}$

$$\left.\begin{array}{ll} 2\times(4)+(5); & 5D + 7F = \frac{-1}{6} \\ (4)+(6); & 3D + 3F = -\frac{2}{6} \end{array}\right\} D = \frac{-11}{36}, \; F = \frac{7}{36}$$

Substitute in (6); $2\left(\frac{-11}{36}\right) + E - \frac{7}{36} = \frac{5}{6} \Rightarrow E = \frac{59}{36}$

$$\therefore \; y_p = \begin{bmatrix} -7 \\ 13 \\ 5 \end{bmatrix} \frac{t}{6} + \frac{1}{36}\begin{bmatrix} -11 \\ 59 \\ 7 \end{bmatrix}$$

The general solution of the given system is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = c_1 e^{-2t}\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} + c_2 e^{t}\begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix} + c_3 e^{3t}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + \frac{t}{6}\begin{bmatrix} -7 \\ 13 \\ 5 \end{bmatrix} + \frac{1}{36}\begin{bmatrix} -11 \\ 59 \\ 7 \end{bmatrix}. \; \underline{\underline{\text{Ans}}}$$

b) $x' = 3x - 3y + z$

$y' = 2x - y$

$z' = x - y + z$

$x(0) = 7 \; ; \; y(0) = 4 \; , \; z(0) = 3$

<u>Sol^{n}</u> The given system can be written as

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}' = \underbrace{\begin{bmatrix} 3 & -3 & 1 \\ 2 & -1 & 0 \\ 1 & -1 & 1 \end{bmatrix}}_{\underline{\underline{A}}} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

1) General solution of the homogeneous ODEs

Find eigenvalues : $|\underline{\underline{A}} - \lambda \underline{\underline{I}}| = 0$

$$\begin{vmatrix} 3-\lambda & -3 & 1 \\ 2 & -1-\lambda & 0 \\ 1 & -1 & 1-\lambda \end{vmatrix} = -2-(-1-\lambda)+(1-\lambda)\left[(3-\lambda)(-1-\lambda)+6\right]$$

$$= -\lambda^3 + 3\lambda^2 - 4\lambda + 2 = 0$$

$$(\lambda - 1)(\lambda^2 - 2\lambda + 2) = 0$$

$$\lambda_1 = 1 \; , \; \lambda_2 = \frac{2 + \sqrt{4 - 4(1)(2)}}{2(1)} = 1 + i \; , \; \lambda_3 = 1 - i$$

Find eigenvector

<u>$\lambda_1 = 1$</u> ;

$$\begin{bmatrix} 3-1 & -3 & 1 \\ 2 & -1-1 & 0 \\ 1 & -1 & 1-1 \end{bmatrix} = \begin{bmatrix} 2 & -3 & 1 \\ 2 & -2 & 0 \\ 1 & -1 & 0 \end{bmatrix} \begin{matrix} R_1 \\ R_2 \\ R_3 - \frac{R_2}{2} \end{matrix} \xrightarrow{\quad} \begin{bmatrix} 2 & -3 & 1 \\ 2 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$2v_1 - 2v_2 = 0 \Rightarrow v_2 = v_1$$

$$2v_1 - 3v_2 + v_3 = 0 \Rightarrow v_3 = 3v_2 - 2v_1 = v_1$$

Choose $v_1 = 1$; $\quad \underset{\sim}{v}^{(1)} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \; , \; \underset{\sim}{y}_1 = e^t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

<u>$\lambda_2 = 1 + i$</u> ;

$$\begin{bmatrix} 3-1-i & -3 & 1 \\ 2 & -1-1-i & 0 \\ 1 & -1 & 1-1-i \end{bmatrix} = \begin{bmatrix} 2-i & -3 & 1 \\ 2 & -2-i & 0 \\ 1 & -1 & -i \end{bmatrix} \begin{matrix} R_1 \\ R_2 - \frac{2}{(2-i)} R_1 \\ R_3 - \frac{R_2}{2} \end{matrix} \xrightarrow{\quad} \begin{bmatrix} 2-i & -3 & 1 \\ 0 & \frac{2}{5} + \frac{i}{5} & \frac{-4}{5} - \frac{2}{5} i \\ 0 & i/2 & -i \end{bmatrix}$$

$$\begin{array}{c} R_1 \\ 5R_2 \\ R_3 \end{array}\begin{bmatrix} 2-i & -3 & 1 \\ 0 & 2+i & -4-2i \\ 0 & i/2 & -i \end{bmatrix} \xrightarrow{\begin{array}{l} R_1 \\ R_2 \\ R_3 - \frac{iR_2/2}{(2+i)} \end{array}} \begin{bmatrix} 2-i & -3 & 1 \\ 0 & 2+i & -4-2i \\ 0 & 0 & 0 \end{bmatrix}$$

$$(2+i)v_2 - (4+2i)v_3 = 0 \quad \Rightarrow \quad v_2 = \frac{2\cancel{(2+i)}}{\cancel{(2+i)}} v_3 = 2v_3$$

$$(2-i)v_1 - 3v_2 + v_3 = 0 \quad \Rightarrow \quad v_1 = \frac{3v_2 - v_3}{(2-i)} = \frac{5v_3}{(2-i)}\frac{(2+i)}{(2+i)}$$

$$= (2+i)v_3$$

Choose $v_3 = 1$; $\quad \underset{\sim}{v}^{(2)} = \begin{bmatrix} 2+i \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} + i\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$$e^{\lambda_2 t}\underset{\sim}{v}^{(2)} = e^{(1+i)t}\left\{\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} + i\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right\}$$

$$= e^{t}(\cos t + i\sin t)\left\{\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} + i\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right\}$$

Real part ; $\quad \underset{\sim}{y}_2 = e^{t}\left\{\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}\cos t - \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\sin t\right\}$

Imaginary part; $\underset{\sim}{y}_3 = e^{t}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\cos t + \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}\sin t\right\}$

$$\underset{\sim}{y} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = c_1\underset{\sim}{y}_1 + c_2\underset{\sim}{y}_2 + c_3\underset{\sim}{y}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}e^{t}c_1 + \left\{\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}\cos t - \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\sin t\right\}e^{t}c_2$$

$$+\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\cos t + \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}\sin t\right\}e^{t}c_3$$

$$\begin{bmatrix} x(0) \\ y(0) \\ z(0) \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ 3 \end{bmatrix} = c_1\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} + c_3\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{array}{l} c_1 + 2c_2 + c_3 = 7 \\ c_1 + 2c_2 = 4 \\ c_1 + c_2 = 3 \end{array}\right\} \quad c_1 = 2, \; c_2 = 1, \; c_3 = 3$$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 2e^t\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + e^t\left\{\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}\cos t - \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\sin t\right\} + 3e^t\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\cos t + \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}\sin t\right\} \quad \text{Ans}$$

11.1.5

11.1.5) If $f(x)$ and $g(x)$ have period p, show that $h(x) = af(x) + bg(x)$ has the period p. Thus all functions of period p form a vector space.

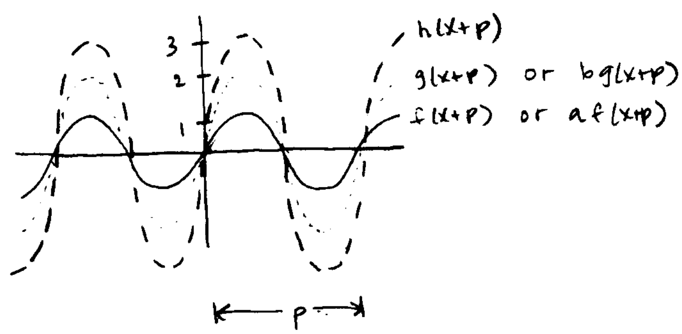
$f(x) = f(x+p)$
$g(x) = g(x+p)$

$\Downarrow$

$h(x) = af(x) + bg(x)$
$h(x+p) = af(x+p) + bg(x+p)$

$\Rightarrow$

VISUALLY :



- Find the Fourier series of the function $f(x)$, of period $p = 2L$.
- Sketch or graph the first three partial sum
- Show the details of your work

11.2.3 $\quad f(x) = x^2 \quad (-1 < x < 1)$, $p = 2$

Sol^n Fourier series of $f(x)$ is

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos \frac{n\pi}{L}x + b_n \sin \frac{n\pi}{L}x\right)$$

Here $L = 1$ $(p = 2L = 2)$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx = \frac{1}{2}\int_{-1}^{1} x^2\,dx = \frac{1}{2}\left[\frac{x^3}{3}\right]_{x=-1}^{x=1}$$

$$= \frac{1}{2}\left[\frac{1}{3} - \left(-\frac{1}{3}\right)\right] = \frac{1}{3} \quad *$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx = \frac{1}{1}\int_{-1}^{1} x^2 \cos n\pi x\,dx$$

Consider $\displaystyle\int x^2 \cos n\pi x\,dx = \frac{x^2 \sin n\pi x}{n\pi} - \int 2x \frac{\sin n\pi x}{n\pi}\,dx$

$\int u\,dv = uv - \int v\,du$

1) $u = x^2$, $du = 2x\,dx$
$dv = \cos n\pi x\,dx$
$v = \frac{\sin n\pi x}{n\pi}$

2) $u = x$, $du = dx$
$dv = \sin n\pi x\,dx$
$v = -\frac{\cos n\pi x}{n\pi}$

$$= \frac{x^2 \sin n\pi x}{n\pi} - \frac{2}{n\pi}\left[-\frac{x\cos n\pi x}{n\pi} + \int \frac{\cos n\pi x}{n\pi}\,dx\right]$$

$$= \frac{x^2 \sin n\pi x}{n\pi} + \frac{2x\cos n\pi x}{(n\pi)^2} - \frac{2\sin n\pi x}{(n\pi)^3}$$

$$a_n = \left[\frac{x^2 \sin n\pi x}{n\pi} + \frac{2x\cos n\pi x}{(n\pi)^2} - \frac{2\sin n\pi x}{(n\pi)^3}\right]_{x=-1}^{x=1}$$

$$= \cancelto{0}{\frac{\sin n\pi}{n\pi}} + \frac{2\cos n\pi}{(n\pi)^2} - \cancelto{0}{\frac{2\sin n\pi}{(n\pi)^3}} - \cancelto{0}{\frac{\sin(-n\pi)}{n\pi}} + \frac{2\cos(-n\pi)}{(n\pi)^2} + \cancelto{0}{\frac{2\sin(-n\pi)}{(n\pi)^3}}$$

$(\sin n\pi = 0 = \sin(-n\pi)\ ;\ n = 1, 2, 3 \ldots)$

$$a_n = \frac{4\cos n\pi}{(n\pi)^2} \quad ; \quad \cos(n\pi) = \cos(-n\pi)$$

$$= \frac{(-1)^n 4}{(n\pi)^2} \quad * \quad ; \quad \cos n\pi = (-1)^n$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx = \frac{1}{1}\int_{-1}^{1} x^2 \sin n\pi x\,dx = 0$$

odd function integrated over symmetrical interval

Consider $\int x^2 \sin n\pi x \, dx = -\frac{x^2 \cos n\pi x}{n\pi} + \int \frac{\cos n\pi x \cdot 2x dx}{n\pi}$

$\int u dv = uv - \int v du$

1) $u = x^2$, $du = 2x dx$
$dv = \sin n\pi x \, dx$
$v = -\frac{\cos n\pi x}{n\pi}$

2) $u = x$, $du = dx$
$dv = \cos n\pi x \, dx$
$v = \frac{\sin n\pi x}{n\pi}$

$$= -\frac{x^2 \cos n\pi x}{n\pi} + \frac{2}{n\pi}\int x \cos n\pi x \, dx$$

$$= -\frac{x^2 \cos n\pi x}{n\pi} + \frac{2}{n\pi}\left[\frac{x \sin n\pi x}{n\pi} - \int \frac{\sin n\pi x}{n\pi} dx\right]$$

$$= -\frac{x^2 \cos n\pi x}{n\pi} + \frac{2x \sin n\pi x}{(n\pi)^2} + \frac{2\cos n\pi x}{(n\pi)^3}$$

$$b_n = \left[-\frac{x^2 \cos n\pi x}{n\pi} + \frac{2x \sin n\pi x}{(n\pi)^2} + \frac{2\cos n\pi x}{(n\pi)^3}\right]_{x=-1}^{x=1}$$

$$= -\frac{\cos n\pi}{n\pi} + \frac{2\cancel{\sin n\pi}^{\,0}}{(n\pi)^2} + \frac{2\cos n\pi}{(n\pi)^3}$$

$$+ \frac{\cos(-n\pi)}{n\pi} - \frac{2\cancel{\sin(-n\pi)}^{\,0}}{(n\pi)^2} - \frac{2\cos(-n\pi)}{(n\pi)^3}$$

$$b_n = 0 \quad * \quad ; \cos(-n\pi) = \cos(n\pi)$$

$$\therefore \quad f(x) = x^2 = \frac{1}{3} + \sum_{n=1}^{\infty} \frac{(-1)^n \cdot 4}{(n\pi)^2} \cos n\pi x$$

$$= \frac{1}{3} - \frac{4}{\pi^2}\left[\cos \pi x - \frac{1}{4}\cos 2\pi x + \frac{1}{9}\cos 3\pi x + \dots\right]$$

Ans

See graph on the attached page

************ 11.2.10 on last pages ********************

Are the following functions even, odd, or neither even nor odd?

11.3.1 $|x|$, $x^2 \sin nx$, $x + x^2$, $e^{-|x|}$, $\ln x$, $x \cosh x$

Sol^n 1) $f(x) = |x| \Rightarrow f(-x) = |-x| = |-1||x| = |x| = f(x)$

$\therefore$ $|x|$ is an even function. Ans

2) $f(x) = x^2 \sin nx \Rightarrow f(-x) = (-x)^2 \sin(-nx) = -x^2 \sin nx$
$= -f(x)$

$\therefore$ $x^2 \sin nx$ is an odd function. Ans

3) $f(x) = x + x^2 \Rightarrow f(-x) = -x + (-x)^2 = -x + x^2 \neq f(x)$
$\neq -f(x)$

$\therefore$ $x + x^2$ is neither even nor odd function. Ans

4) $f(x) = e^{-|x|} \Rightarrow f(-x) = e^{-|-x|} = e^{-|-1||x|} = e^{-|x|} = f(x)$

$\therefore$ $e^{-|x|}$ is an even function. <u>Ans</u>

5) $f(x) = \ln x \Rightarrow f(-x) = \ln(-x) \Rightarrow$ not defined if $x > 0$
or if $x < 0$, $\ln x$ is not defined

$\therefore$ $\ln x$ is neither even nor odd function. <u>Ans</u>

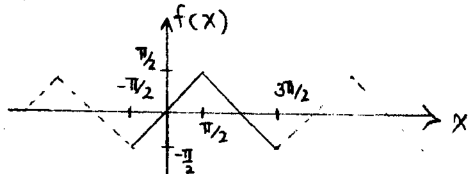
6) $f(x) = x\cosh x \Rightarrow f(-x) = -x\cosh(-x) = -x\cosh x = -f(x)$

$\therefore$ $x\cosh x$ is an odd function. <u>Ans</u>

- Is the given function even or odd?
- Find its Fourier series
- Sketch or graph the function and some partial sums (Show details of your work.)

<u>11.3.13</u) $\quad f(x) = \begin{cases} x & \text{if } -\pi/2 < x < \pi/2 \\ \pi - x & \text{if } \pi/2 < x < 3\pi/2 \end{cases}$

<u>Sol^n</u>



$f(x)$ is an odd function if $f(x)$ is periodic. (see sketch) <u>Ans</u>

Fourier series of $f(x)$;

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{L}x + b_n \sin\frac{n\pi}{L}x\right)$$

Here $L = \pi$ $(p = 2\pi = 2L)$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\int_{-\pi/2}^{\frac{3\pi}{2}} f(x)\,dx$$

$$= \frac{1}{2\pi}\left[\int_{-\pi/2}^{\pi/2} x\,dx + \int_{\pi/2}^{3\pi/2}(\pi - x)\,dx\right] = \frac{1}{2\pi}\left[\frac{x^2}{2}\Big|_{-\pi/2}^{\pi/2} + \left(\pi x - \frac{x^2}{2}\right)\Big|_{\frac{\pi}{2}}^{3\pi/2}\right]$$

$$= \frac{1}{2\pi}\left[\cancel{\frac{\pi^2}{8}} - \cancel{\frac{\pi^2}{8}} + \frac{3\pi^2}{2} - \frac{9\pi^2}{8} - \frac{\pi^2}{2} + \frac{\pi^2}{8}\right] = 0 \quad *$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx = \frac{1}{\pi}\int_{-\pi/2}^{3\pi/2} f(x)\cos nx\,dx$$

$$a_n = \frac{1}{\pi}\left[\int_{-\frac{\pi}{2}}^{\pi/2} x\cos nx\,dx + \int_{\pi/2}^{3\pi/2} (\pi - x)\cos nx\,dx\right]$$

Note

$\int u\,dv = uv - \int v\,du$

$u = x$, $du = dx$

$dv = \cos nx\,dx$

$v = \frac{\sin nx}{n}$

$$a_n = \frac{1}{\pi}\left[\frac{x\sin nx}{n}\Big|_{-\pi/2}^{\pi/2} - \int_{-\frac{\pi}{2}}^{\pi/2}\frac{\sin nx}{n}\,dx\right]$$

$$+\frac{1}{\pi}\left[\frac{\pi\sin nx}{n}\Big|_{\pi/2}^{3\pi/2} - \frac{x\sin nx}{n}\Big|_{\frac{\pi}{2}}^{3\pi/2} + \int_{\frac{\pi}{2}}^{3\pi/2}\frac{\sin nx}{n}\,dx\right]$$

$$= \frac{1}{\pi}\left[\frac{x\sin nx}{n}\Big|_{-\pi/2}^{\pi/2} + \frac{\cos nx}{n^2}\Big|_{-\frac{\pi}{2}}^{\pi/2} + \frac{\pi\sin nx}{n}\Big|_{\pi/2}^{3\pi/2}\right.$$

$$\left. - \frac{x\sin nx}{n}\Big|_{\pi/2}^{3\pi/2} - \frac{\cos nx}{n^2}\Big|_{\pi/2}^{3\pi/2}\right]$$

$$= \frac{1}{\pi}\left[\cancel{\frac{\pi}{2}\frac{\sin(n\pi/2)}{n}} + \cancel{\frac{\pi}{2}\frac{\sin(-n\pi/2)}{n}} + \cancel{\frac{\cos(n\pi/2)}{n^2}} - \cancel{\frac{\cos(-n\pi/2)}{n^2}}\right.$$

$$+ \pi\frac{\sin(3n\pi/2)}{n} - \pi\frac{\sin(n\pi/2)}{n} - \frac{3\pi}{2}\frac{\sin(3n\pi/2)}{n}$$

$$\left. + \frac{\pi}{2}\frac{\sin(n\pi/2)}{n} - \frac{\cos(3n\pi/2)}{n^2} + \frac{\cos(n\pi/2)}{n^2}\right]$$

$$= \frac{1}{\pi}\left[\frac{-\pi}{2n}\sin\left(\frac{3n\pi}{2}\right) \; \frac{-\pi}{2n}\sin\left(\frac{n\pi}{2}\right) - \frac{\cos}{n^2}(3n\pi/2) + \frac{\cos}{n^2}(n\pi/2)\right]$$

$$= \frac{1}{\pi}\left\{\frac{-\pi}{2n}\left[\sin\left(n\pi + \frac{n\pi}{2}\right) + \sin\left(n\pi - \frac{n\pi}{2}\right)\right]\right.$$

$$\left. + \frac{1}{n^2}\left[\cos\left(n\pi - \frac{n\pi}{2}\right) - \cos\left(n\pi + n\frac{\pi}{2}\right)\right]\right\}$$

$$= \frac{1}{\pi}\left\{\frac{-\pi}{\cancel{2}n}\cdot\cancel{2}\sin(n\pi)\cos\left(\frac{n\pi}{2}\right) + \frac{1}{n^2}\cdot 2\sin(n\pi)\sin\left(\frac{n\pi}{2}\right)\right\}$$

Note that: $\quad 2\sin x\cos y = \sin(x+y) + \sin(x-y)$

and $\quad 2\sin x\sin y = \cos(x-y) - \cos(x+y)$

$$a_n = -\frac{[\pi n\cos(n\pi/2) - 2\sin(n\pi/2)]\,\overset{=0}{\cancel{\sin(n\pi)}}}{n^2\pi}$$

$$= 0 \quad * \text{ since } \sin(n\pi) = 0 \text{ for } n = 1, 2, 3, \ldots$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx = \frac{1}{\pi}\int_{-\pi/2}^{3\pi/2} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi/2}^{\pi/2} x\sin nx\,dx + \int_{\pi/2}^{3\pi/2}(\pi-x)\sin nx\,dx\right.$$

Note

$\int u\,dv = uv - \int v\,du$

$u = x$, $du = dx$

$dv = \sin nx\,dx$

$v = \frac{-\cos nx}{n}$

$$b_n = \frac{1}{\pi}\left[-\frac{x\cos nx}{n}\Big|_{-\frac{\pi}{2}}^{\pi/2} + \int_{-\pi/2}^{\pi/2}\frac{\cos nx}{n}\,dx + \left[-\frac{\pi\cos nx}{n}\right]_{\frac{\pi}{2}}^{3\pi/2}\right.$$

$$\left.+ x\frac{\cos nx}{n}\Big|_{\frac{\pi}{2}}^{3\pi/2} - \int_{\pi/2}^{3\pi/2}\frac{\cos nx}{n}\,dx\right]$$

$$= \frac{1}{\pi}\left[\frac{-x\cos nx}{n}\Big|_{-\frac{\pi}{2}}^{\pi/2} + \frac{\sin nx}{n^2}\Big|_{-\frac{\pi}{2}}^{\pi/2} + \left[-\frac{\pi\cos nx}{n}\right]_{\pi/2}^{3\pi/2}\right.$$

$$\left.+ \frac{x\cos nx}{n}\Big|_{\pi/2}^{3\pi/2} - \frac{\sin nx}{n^2}\Big|_{\pi/2}^{3\pi/2}\right.$$

$$= \frac{1}{\pi}\left[\frac{-\pi}{2n}\cos(n\pi/2) - \frac{\pi}{2n}\cos(-n\pi/2) - \frac{\pi}{n}\cos(3n\pi/2)\right.$$

$$+ \frac{\pi}{n}\cos(n\pi/2) + \frac{3\pi}{2n}\cos(3n\pi/2) - \frac{\pi}{2n}\cos(n\pi/2)$$

$$\left.- \frac{\sin(3n\pi/2)}{n^2} + \frac{\sin(n\pi/2)}{n^2} + \frac{\sin(n\pi/2)}{n^2} - \frac{\sin(-\frac{n\pi}{2})}{n^2}\right]$$

$$= \frac{1}{\pi}\left\{\frac{-\pi}{2n}\cos(n\pi/2) + \frac{\pi}{2n}\cos(3n\pi/2) - \frac{\sin(3n\pi/2)}{n^2}\right.$$

$$\left.+ 3\frac{\sin(n\pi/2)}{n^2}\right\}$$

$$= \frac{1}{\pi}\left\{\frac{\pi}{2n}\left[\cos(n\pi - n\pi/2) + \cos\left(n\pi + \frac{n\pi}{2}\right)\right] - \frac{\pi}{2n}\cos\left(\frac{n\pi}{2}\right)\right.$$

$$\left.-\frac{\pi}{2n}\cos\left(\frac{n\pi}{2}\right) - \frac{1}{n^2}\left[\sin(n\pi + n\pi/2) - \sin\left(n\pi - \frac{n\pi}{2}\right)\right] + \frac{2}{n^2}\sin\left(\frac{n\pi}{2}\right)\right\}$$

$$= \frac{1}{\pi}\left\{\frac{\pi}{n}\cos(n\pi)\cos\left(\frac{n\pi}{2}\right) - \frac{\pi}{n}\cos\left(\frac{n\pi}{2}\right) - \frac{2}{n^2}\cos(n\pi)\sin\left(\frac{n\pi}{2}\right)\right.$$

$$\left.+ \frac{2}{n^2}\sin\left(\frac{n\pi}{2}\right)\right\}$$

Note that : $2\cos x\cos y = \cos(x-y) + \cos(x+y)$

and $2\cos x\sin y = \sin(x+y) - \sin(x-y)$

$$b_n = \frac{1}{\pi}\left[\frac{\pi}{n}\cos\left(\frac{n\pi}{2}\right)(\cos n\pi - 1) - \frac{2}{n^2}\sin\left(\frac{n\pi}{2}\right)(\cos n\pi - 1)\right]$$

$$= \frac{(-1+\cos n\pi)(\pi n\cos(n\pi/2) - 2\sin(n\pi/2))}{n^2\pi}$$

$$= \left[-1 + (-1)^n\right]\left[\pi n\cos(n\pi/2) - 2\sin(n\pi/2)\right]/(n^2\pi)$$

$$\therefore f(x) = \sum_{n=1}^{\infty} \frac{[-1+(-1)^n]}{n^2\pi}[\pi n\cos(n\pi/2) - 2\sin(n\pi/2)]\sin(nx)$$

$$= \frac{4}{\pi}\left[\sin x - \frac{1}{9}\sin 3x + \frac{1}{25}\sin 5x + \cdots\right]$$ Ans

See sketch on the attached page.

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c) Find a particular solution of $y'' + 4y = f(x)$ where

$$f(x) = \begin{cases} x & \text{if } -\pi/2 < x < \pi/2 \\ \pi - x & \text{if } \pi/2 < x < 3\pi/2 \end{cases}$$

Sol$^{\underline{n}}$ 1) General solution of the homogeneous ODE, y_c

$$y'' + 4y = 0$$

Characteristic eqn.; $\lambda^2 + 4 = 0 \Rightarrow \lambda = \pm\sqrt{-4}$

$$\lambda = -2i, 2i$$

$$y_c = A\cos 2x + B\sin 2x$$

2) Find y_p

From 11.3.13, we got

$$f(x) = \begin{cases} x & \text{if } -\pi/2 < x < \pi/2 \\ \pi - x & \text{if } \pi/2 < x < 3\pi/2 \end{cases}$$

$$= \frac{4}{\pi}\left[\sin x - \frac{1}{9}\sin 3x + \frac{1}{25}\sin 5x + \cdots\right]$$

Try $y_p = C_1\cos x + C_2\sin x + C_3\cos 3x + C_4\sin 3x + C_5\cos 5x + C_6\sin 5x$

$$y_p' = -C_1\sin x + C_2\cos x - 3C_3\sin 3x + 3C_4\cos 3x - 5C_5\sin 5x + 5C_6\cos 5x$$

$$y_p'' = -C_1\cos x - C_2\sin x - 9C_3\cos 3x - 9C_4\sin 3x - 25C_5\cos 5x - 25C_6\sin 5x$$

Substitute y_p'', y_p in $y_p'' + 4y_p = f(x)$;

$$-C_1\cos x - C_2\sin x - 9C_3\cos 3x - 9C_4\sin 3x - 25C_5\cos 5x - 25C_6\sin 5x$$

$$+ 4C_1\cos x + 4C_2\sin x + 4C_3\cos 3x + 4C_4\sin 3x + 4C_5\cos 5x + 4C_6\sin 5x$$

$$= \frac{4}{\pi}\left[\sin x - \frac{1}{9}\sin 3x + \frac{1}{25}\sin 5x + \cdots\right]$$

Equating coefficients;

$$-C_1 + 4C_1 = 0 \Rightarrow C_1 = 0 \quad *$$

$$-C_2 + 4C_2 = \frac{4}{\pi} \Rightarrow C_2 = \frac{4}{3\pi} \quad *$$

$$-9C_3 + 4C_3 = 0 \Rightarrow C_3 = 0 \quad *$$

$$-9C_4 + 4C_4 = \frac{-4}{9\pi} \Rightarrow C_4 = \frac{4}{45\pi} \quad *$$

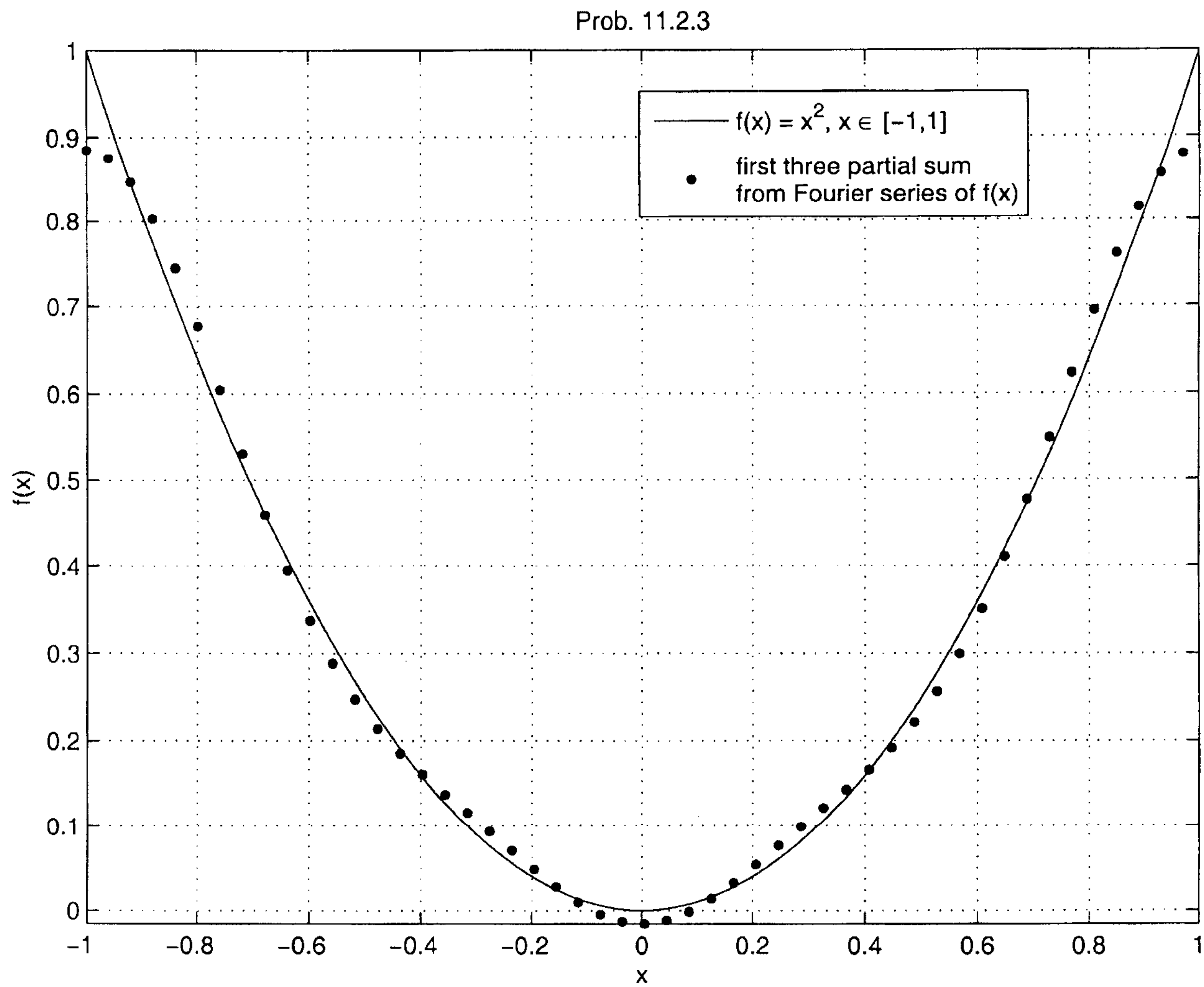
$$-25C_5 + 4C_5 = 0 \Rightarrow C_5 = 0 \quad *$$

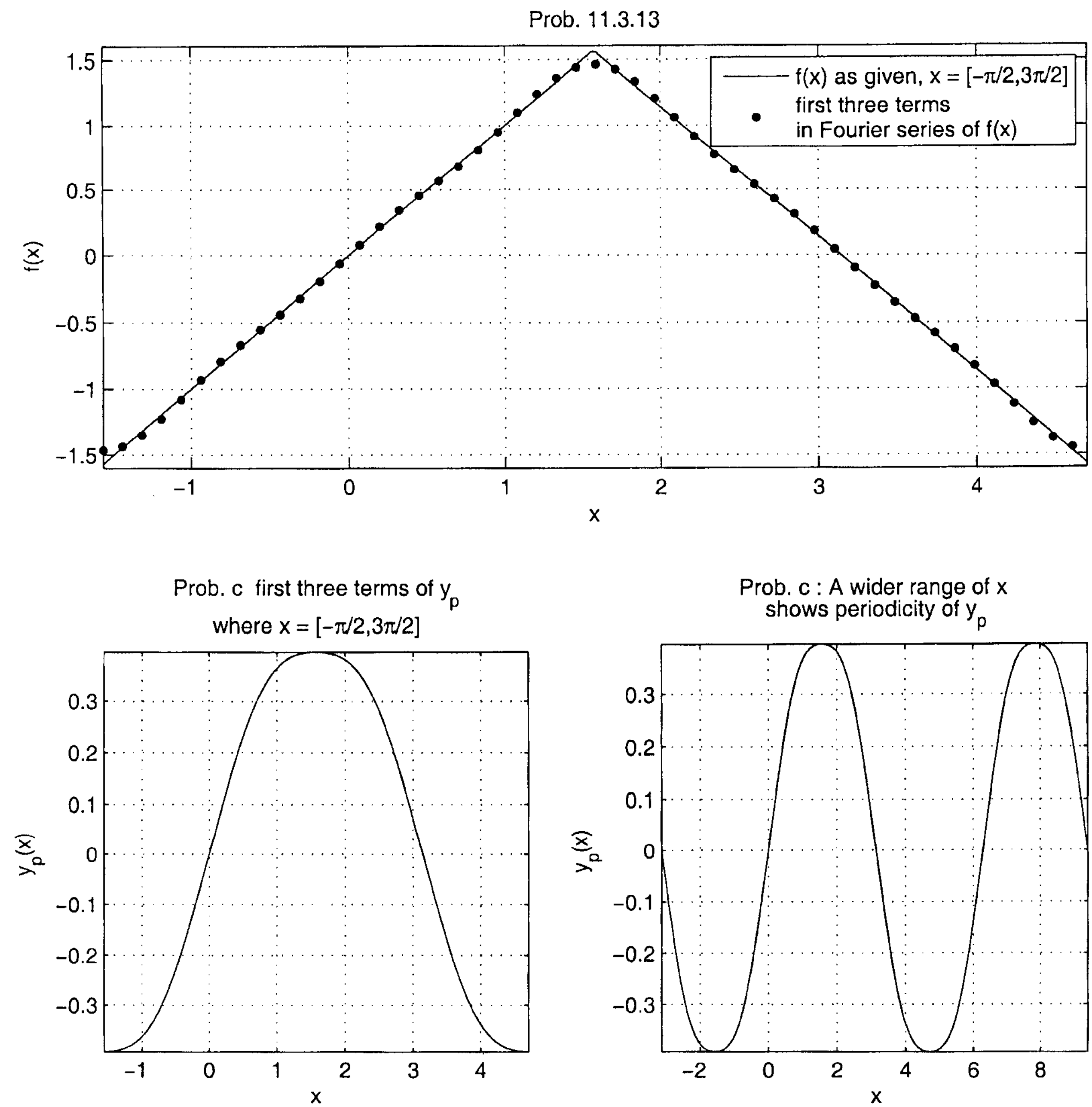
$$-25C_6 + 4C_6 = \frac{4}{25\pi} \Rightarrow C_6 = \frac{-4}{525\pi} \quad *$$

$$\left.\begin{aligned} \therefore\ y_P &= \frac{4}{3\pi}\sin x + \frac{4}{45\pi}\sin 3x - \frac{4}{525\pi}\sin 5x \quad \text{(first 3 terms)} \\ y = y_c + y_P &= A\cos 2x + B\sin 2x + \frac{4}{3\pi}\sin x + \frac{4}{45\pi}\sin 3x - \frac{4}{525\pi}\sin 5x + \ldots \end{aligned}\right\} \text{Ans}$$

Plot of y_P is on the attached page.

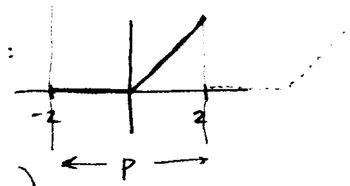
From the plot, we can see that y_P is periodic. This is because y_P is obtained from $f(x)$ in the Fourier form which is assumed periodicity in the range of 2π. y_P is also periodic in 2π. Ans


Prob. 11.2.3
f(x) = x², x ∈ [−1,1]
first three partial sum
from Fourier series of f(x)
f(x)
x
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
−1
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1


Prob. 11.3.13
f(x) as given, x = [−π/2,3π/2]
first three terms
in Fourier series of f(x)
f(x)
x
Prob. c first three terms of y_p
where x = [−π/2,3π/2]
y_p(x)
x
Prob. c : A wider range of x
shows periodicity of y_p
y_p(x)
x

11-2-10) $f(x) = 0 \quad (-2 < x < 0)$
$f(x) = x \quad (0 < x < 2)$ $\qquad p = 4 \rightarrow p = 2L, \; L = 2$

$f(x)$:



- Apply euler formulas

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$$
$$= \frac{1}{4}\left(\int_{-L}^{0} 0\cdot dx + \int_0^L x\,dx\right) = \frac{1}{4}\left(\cancel{\int_{-2}^{0} 0\cdot dx}^{\,0} + \int_0^2 x\,dx\right)$$
$$= \frac{1}{4}\int_0^2 x\,dx = \frac{1}{4}\left(\frac{1}{2}x^2\right)\Big|_0^2 = \frac{1}{4}(2) = \frac{1}{2} = a_0$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx = \frac{1}{2}\left(\cancel{\int_{-2}^{0} 0\cdot\cos\frac{n\pi x}{L}}^{\,0} + \int_0^2 x\cos\frac{n\pi x}{2}\,dx\right)$$
$$= \frac{1}{2}\int_0^2 x\cos\frac{n\pi x}{2}\,dx \longrightarrow \text{evaluated using Mathematica}$$
$$= \frac{-2 + 2\cos n\pi + 2n\pi\cancel{\sin n\pi}^{\,0}}{n^2\pi^2} = \frac{-2 + 2\cos n\pi}{n^2\pi^2} \quad \begin{cases} n\text{-odd} & -4/n^2\pi^2 \\ n\text{-even} & 0 \end{cases} = a_n$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx = \frac{1}{2}\left(\cancel{\int_{-2}^{0} 0\cdot\sin\frac{n\pi x}{L}}^{\,0} + \int_0^2 x\sin\frac{n\pi x}{2}\,dx\right)$$
$$= \frac{1}{2}\int_0^2 x\sin\frac{n\pi x}{2}\,dx \longrightarrow \text{evaluated using Mathematica}$$
$$= \frac{-2n\pi\cos n\pi + 2\cancel{\sin n\pi}^{\,0}}{n^2\pi^2} = \frac{-2n\pi\cos n\pi}{n^2\pi^2} \quad \begin{cases} n\text{-odd} & \frac{2n\pi}{n^2\pi^2} = \frac{2}{n\pi} \\ n\text{-even} & -\frac{2}{n\pi} \end{cases} = b_n$$

- Fourier series:

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L} + b_n\sin\frac{n\pi x}{L}\right)$$

$$\boxed{\begin{aligned} f(x) &= \frac{1}{2} - \frac{4}{\pi^2}\left(\cos\frac{\pi x}{2} + \frac{1}{9}\cos\frac{3\pi x}{2} + \frac{1}{25}\cos\frac{5\pi x}{2} + \cdots\right) \\ &\quad + \frac{2}{\pi}\left(\sin\frac{\pi x}{2} - \frac{1}{2}\sin\pi x + \frac{1}{3}\sin\frac{3\pi x}{2} - \frac{1}{4}\sin 2\pi x + \cdots\right)\end{aligned}}$$

see attached for plot.

Problem (11.2.10)

```
ClearAll[ao, a, b, partsum]
ao = (1/4) Integrate[x, {x, 0, 2}];
a[n_] := (1/2) * Integrate[x * Cos[n π x / 2], {x, 0, 2}]
b[n_] := (1/2) * Integrate[x * Sin[n π x / 2], {x, 0, 2}]
Print["Fourier coefficient a[n] evaluates to:"]
a[n]
Print["Fourier coefficient b[n] evaluates to:"]
b[n]
```

Fourier coefficient a[n] evaluates to:

$$\frac{2\,(-1 + \text{Cos}[n\,\pi] + n\,\pi\,\text{Sin}[n\,\pi])}{n^2\,\pi^2}$$

Fourier coefficient b[n] evaluates to:

$$\frac{-4\,n\,\pi\,\text{Cos}[n\,\pi] + 4\,\text{Sin}[n\,\pi]}{2\,n^2\,\pi^2}$$

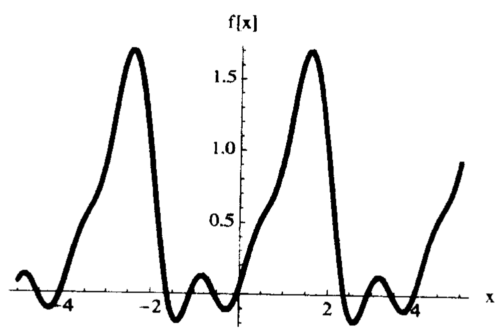
```
partsum[x_, k_] := ao + Sum[a[n] * Cos[n π x / 2] + b[n] * Sin[n π x / 2], {n, 1, k}]
Print["First few partial sums of Fourier series:"]
partsum[x, 3]
```

First few partial sums of Fourier series:

$$\frac{1}{2} - \frac{4\,\text{Cos}\left[\frac{\pi x}{2}\right]}{\pi^2} - \frac{4\,\text{Cos}\left[\frac{3\pi x}{2}\right]}{9\,\pi^2} + \frac{2\,\text{Sin}\left[\frac{\pi x}{2}\right]}{\pi} - \frac{\text{Sin}[\pi\,x]}{\pi} + \frac{2\,\text{Sin}\left[\frac{3\pi x}{2}\right]}{3\,\pi}$$

- **Plot with first 3 partial sums**

```
Plot[Evaluate[partsum[x, 3]], {x, -5, 5}, PlotStyle → {Black, Thick}, AxesLabel → {"x", "f[x]"}]
```



- **Plot with first 15 partial sums**

```
Plot[Evaluate[partsum[x, 15]], {x, -5, 5}, PlotStyle → {Black, Thick}, AxesLabel → {"x", "f[x]"}]
```