Introduction to Index Notation for Cartesian Vector Representations

We have now spent several weeks dealing with vector quantities using the most direct and explicit notation available. However, such explicit forms can be cumbersome for many calculations. (For example, compute $\vec{a}\times(\vec{b}\times\vec{c})$ and compare with the version below. This handout presents an introduction to manipulating vectors (represented in Cartesian coordinate systems) using index notation. Useful symbols include:

Kronecker Delta: $\delta_{ij}=\begin{Bmatrix}1, & i=j\\ 0, & i\neq j\end{Bmatrix}$

Alternator: $\varepsilon_{ijk}=\begin{Bmatrix}1, & (i,j,k)\text{ cyclic, i.e. }(1,2,3),(2,3,1),(3,1,2)\\ -1, & (i,j,k)\text{ anticyclic, i.e. }(3,2,1),(2,1,3),(1,3,2)\\ 0, & \text{if any two indices are equal}\end{Bmatrix}$

In <u>index notation</u> vector is represented by its typical component: $\vec{a}=a_i$; $\vec{b}=b_i$.

The scalar (dot, inner) product and cross (vector) product are written as:

$\vec{a}\cdot\vec{b}=a_ib_i \quad \textit{and} \quad \vec{a}\times\vec{b}=a_ib_j\varepsilon_{ijk}$

where repeated subscripts are automatically summer from 1 to 3 (Einstein Summation Convention); e.g. $a_ib_i=a_1b_1+a_2b_2+a_3b_3$.

Useful properties of the alternator: $\begin{matrix}\varepsilon_{ijk}\varepsilon_{lmk}=\delta_{il}\delta_{jm}-\delta_{im}\delta_{jl}\\ \varepsilon_{ijk}=\varepsilon_{jki}=\varepsilon_{kij}=-\varepsilon_{ikj}=-\varepsilon_{kji}=-\varepsilon_{jik}\end{matrix}$.

Repeated subscripts are dummies and may be replaced by another letter without affecting the value of the sum: $a_ib_i=a_jb_j$.

Note that if δ_{ij} appears in an expression summing both *i* and *j*, then *j* can be set equal to *i* and δ_{ij} removed: $a_ib_j\delta_{ij}=a_ib_i$.

Example:

$$\begin{aligned}\vec{a}\times(\vec{b}\times\vec{c})&=\vec{a}\times(b_jc_k\varepsilon_{jkl})\\ &=a_ib_jc_k\varepsilon_{jkl}\varepsilon_{ilm}=a_ib_jc_k\varepsilon_{jkl}\varepsilon_{mil}\\ &=a_ib_jc_k\delta_{jm}\delta_{ki}-a_ib_jc_k\delta_{ji}\delta_{km}\\ &=a_ib_mc_i-a_ib_ic_m=\vec{b}(\vec{a}\cdot\vec{c})-\vec{c}(\vec{a}\cdot\vec{b})\end{aligned}$$

Partial derivative with respect to x_i is abbreviated as: $\partial(\cdot)/\partial x_i=\partial_i(\cdot)$

Operators involving derivatives (grad, div, curl) become:

$\text{grad}\,\varphi=\nabla\varphi=\partial_i\varphi, \quad \text{div}\,\vec{u}=\nabla\cdot\vec{u}=\partial_iu_i, \quad \text{curl}\,\vec{u}=\partial_iu_j\varepsilon_{ijk}$

Example: Show $\nabla\times\nabla\varphi=\vec{0}$.

$$\begin{aligned}\nabla\times\nabla\varphi&=\partial_i\partial_j\varphi\varepsilon_{ijk}=\partial_j\partial_i\varphi\varepsilon_{ijk}\ \text{(change order of differentiation)}\\ &=\partial_i\partial_j\varphi\varepsilon_{jik}\ \text{(rename dummy indices}: i\to j,\ j\to i)\\ &=-\partial_i\partial_j\varphi\varepsilon_{ijk}\ \text{(exchanging adjacent alternator indices changes sign)}\\ &=-\nabla\times\nabla\varphi=0 \quad (\vec{u}=-\vec{u}\Rightarrow\vec{u}=\vec{0})\end{aligned}$$

Exercises - Derive the following relations using index notation:

1) $(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d})=(\vec{a}\cdot\vec{c})(\vec{b}\cdot\vec{d})-(\vec{a}\cdot\vec{d})(\vec{b}\cdot\vec{c})$

2) $\nabla\cdot(\nabla\times\vec{u})=0$

3) $\nabla\times(\nabla\times\vec{u})=\nabla(\nabla\cdot\vec{u})-\nabla^2\vec{u}$ (In the last term, the Laplacian acts on each component of the vector.)