HW1 - ME 565 - WI 09

11.4.2

Show that the complex Fourier Coeff. of an even function are real and those of an odd function are pure imaginary.

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-inx}\, dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\left[\cos(nx) - i\cdot\sin(nx)\right] dx$$

$$= \frac{1}{2}(a_n - i b_n)$$

→ for even functions: $b_n = 0 \Rightarrow c_n = \frac{a_n}{2}$ real

→ for odd functions: $a_n = 0 \Rightarrow c_n = i\frac{b_n}{2}$ pure imaginary

11.4.3

Show that a) $a_0 = c_0$

b) $a_n = c_n + c_{-n}$

c) $b_n = i(c_n - c_{-n})$

a)

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\, dx$$

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(x)\, e^{-\frac{in\pi x}{L}}\, dx \Rightarrow c_0 = \frac{1}{2L}\int_{-L}^{L} f(x) = a_0$$

b)

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(x)\, e^{-\frac{in\pi x}{L}}\, dx$$

$$a_n = \frac{1}{2}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}$$

$$K_n = c_{-n} = \frac{1}{2L}\int_{-L}^{L} f(x)\, e^{\frac{in\pi x}{L}}\, dx$$

$$c_n + c_{-n} = \frac{1}{2L}\int_{-L}^{L} f(x)\left(e^{-\frac{in\pi x}{L}} + e^{+\frac{in\pi x}{L}}\right) = \frac{1}{2L}\int_{-L}^{L} f(x)\cdot 2\cdot\cos\left(\frac{n\pi x}{L}\right) dx = a_n$$

c)

$$c_n - c_{-n} = \frac{1}{2L}\int_{-L}^{L} f(x)\left(e^{-\frac{in\pi x}{L}} - e^{+\frac{in\pi x}{L}}\right) = \frac{1}{2L}\int_{-L}^{L} f(x)(-2i)\cdot\sin\left(\frac{n\pi x}{L}\right) dx$$

$i^2 = -1$

$$i(c_n - c_{-n}) = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right) dx = b_n$$

Find the complex Fourier series (Show the details)

11.4.11 $f(x) = x^2 \quad (-\pi < x < \pi)$

Sol^n

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

$$C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-inx}\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2 e^{-inx}\,dx$$

Consider $\displaystyle\int_{-\pi}^{\pi} x^2 e^{-inx}\,dx = -\left.\frac{x^2 e^{-inx}}{in}\right|_{x=-\pi}^{x=\pi} - \int_{-\pi}^{\pi} \frac{-e^{-inx}}{in}\cdot 2x\,dx$

Note: $\int u\,dv = uv - \int v\,du$

$u = x^2$, $du = 2x\,dx$

$dv = e^{-inx}dx$, $v = -\dfrac{e^{-inx}}{in}$

$$= -\left.\frac{x^2 e^{-inx}}{in}\right|_{x=-\pi}^{x=\pi} + \frac{2}{in}\int_{-\pi}^{\pi} x e^{-inx}\,dx$$

$$= \left.\frac{x^2 i e^{-inx}}{n}\right|_{x=-\pi}^{x=\pi} + \frac{2}{in}\left[\frac{e^{-inx}}{(in)^2}(-inx-1)\right]_{x=-\pi}^{x=\pi}$$

$$= \frac{\pi^2 i e^{-in\pi}}{n} - \frac{\pi^2 i e^{in\pi}}{n} + \frac{2}{in}\left[\frac{e^{-in\pi}}{-n^2}(-in\pi-1) + \frac{e^{in\pi}}{n^2}(in\pi-1)\right]$$

Since $e^{in\pi} = e^{-in\pi} = \cos(n\pi)$,

$$\int_{-\pi}^{\pi} x^2 e^{-inx}\,dx = \frac{\pi^2 i e^{-in\pi}}{n} - \frac{\pi^2 i e^{in\pi}}{n} + \frac{2}{in}\left[+\frac{e^{-in\pi}}{n^2}\cdot in\pi + \frac{e^{-in\pi}}{n^2} + \frac{e^{in\pi}}{n^2}in\pi - \frac{e^{in\pi}}{n^2}\right]$$

$$= \frac{2\pi e^{-in\pi}}{n^2} + \frac{2\pi e^{in\pi}}{n^2} = \frac{4\pi e^{in\pi}}{n^2} \quad ; \quad e^{in\pi} = e^{-in\pi}$$

for $n \neq 0$; $\displaystyle C_n = \frac{1}{2\pi}\left[\frac{4\pi e^{in\pi}}{n^2}\right] = \frac{2e^{in\pi}}{n^2} = \frac{2\cos(n\pi)}{n^2} = \frac{(-1)^n \cdot 2}{n^2}$ *

for $n = 0$; $\displaystyle C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-inx}\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2\,dx$

$$= \frac{1}{2\pi}\left[\frac{x^3}{3}\right]_{x=-\pi}^{x=\pi} = \frac{1}{2\pi}\left[\frac{\pi^3}{3} + \frac{\pi^3}{3}\right] = \frac{\pi^2}{3} \quad *$$

∴ The complex Fourier series is

$$x^2 = \frac{\pi^2}{3} + 2\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{(-1)^n}{n^2} e^{inx}$$

Ans

11.4.12 | Convert the series in Problem 11 to real form

Let $k = |n|$

for $n = 1, 2, 3, \cdots$

$$e^{inx} = e^{ikx} = \cos(kx) + i\sin(kx) \qquad \text{— (1)}$$

for $n = -1, -2, -3, \cdots$

$$e^{inx} = e^{-ikx} = \cos(kx) - i\sin(kx) \qquad \text{— (2)}$$

$$(1) + (2) = \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} e^{inx} = \sum_{k=1}^{\infty} 2\cos(kx)$$

∴ The real Fourier series is

$$x^2 = \frac{\pi^2}{3} + 4\sum_{k=1}^{\infty} \frac{(-1)^n}{n^2} \cos(kx)$$

$$= \frac{\pi^2}{3} - 4\left(\cos x - \frac{1}{4}\cos 2x + \frac{1}{9}\cos 3x + \cdots\right)$$

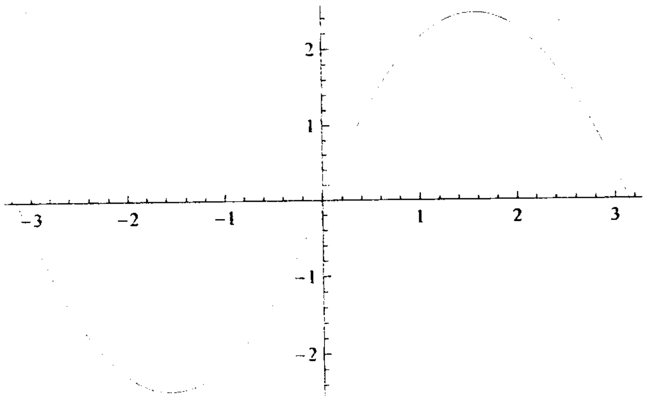
Which is the same as the Fourier cosine series in 11.3.24 **Ans**

```
"Section 11.6 # 9"
ClearAll[x]
f[x_] = Piecewise[{{x * (x + π), -π < x < 0}, {x * (-x + π), 0 < x < π}}]
Plot[f[x], {x, -π, π}]
```

11.6.9

Section 11.6 # 9

$$\begin{cases} x\,(\pi + x) & -\pi < x < 0 \\ (\pi - x)\,x & 0 < x < \pi \end{cases}$$




```
b[n_] := Integrate[(1 / π) * f[x] * Sin[n * x], {x, -π, π}]
Assuming[n ∈ Integers, b[n]]
b[3] // N
```

$$-\frac{4\left(-1+(-1)^{n}\right)}{n^{3}\,\pi}$$

0.094314

```
oddPsum[x_, i_] := Sum[b[n] * Sin[n * x], {n, 1, i}]

estar1 = Integrate[(f[x] - oddPsum[x, 1])^2, {x, -π, π}] // N
estar2 = Integrate[(f[x] - oddPsum[x, 2])^2, {x, -π, π}] // N
estar3 = Integrate[(f[x] - oddPsum[x, 3])^2, {x, -π, π}] // N
estar4 = Integrate[(f[x] - oddPsum[x, 4])^2, {x, -π, π}] // N
estar5 = Integrate[(f[x] - oddPsum[x, 5])^2, {x, -π, π}] // N
```

0.0294796

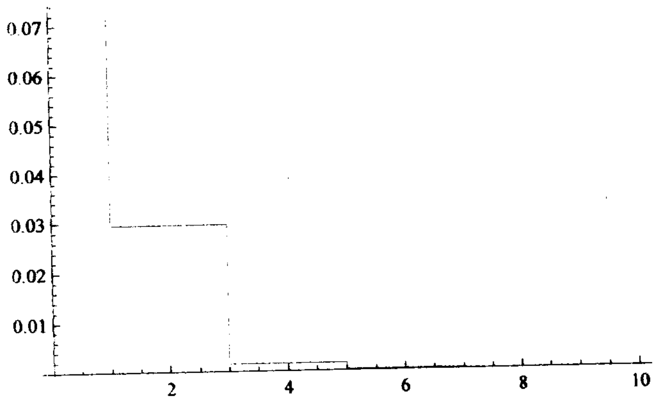
0.0294796

0.0015347

0.0015347

0.000230905

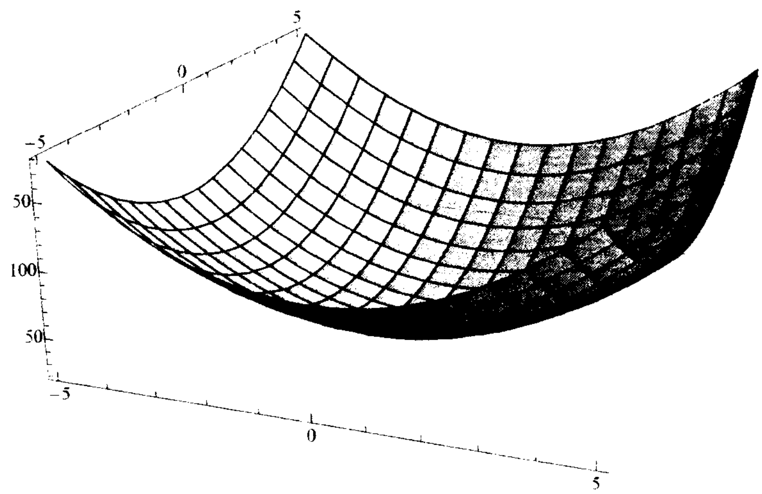
```
"Plot of estar vs N";
estar[v_] := Integrate[(f[x] - oddPsum[x, v])^2, {x, -π, π}] // N
Plot[estar[v], {v, 0, 10}]
```



```
"plotting the E* as both an and bn change from -5 to 5";

asubn[w_] := w
bsubn[z_] := z
TOTALsum[x_] := Sum[asubn[w] * Cos[n * x] + (b[n] + bsubn[z]) * Sin[n * x], {n, 0, 3, 1}]
TOTALsum[x]
Plot3D[Integrate[(f[x] - TOTALsum[x])^2, {x, -π, π}], {w, -5, 5}, {z, -5, 5}]
```

$$w + w\,\mathrm{Cos}[x] + w\,\mathrm{Cos}[2\,x] + w\,\mathrm{Cos}[3\,x] + \left(\frac{8}{\pi} + z\right)\mathrm{Sin}[x] + z\,\mathrm{Sin}[2\,x] + \left(\frac{8}{27\,\pi} + z\right)\mathrm{Sin}[3\,x]$$



```
FindMinimum[Integrate[(f[x] - TOTALsum[x])^2, {x, -π, π}], {w, -5, 5}, {z, -5, 5}]
```

$\{0.0015347,\ \{w \to 6.54928 \times 10^{-10},\ z \to 6.65692 \times 10^{-18}\}\}$

```
"The results from finding the minimum the plot prove that for (an
   + w) and (bn + z) the minimum value of the error is as w and z approach
   zero and are of magnitude 10^-10 or smaller.  The error at this minimum
   of an+w = 0 and bn +z = bn for N=3 is the same as found above.  This
   proves that the fourier constants do provide optimal approximation.";
```