ME565, HW 2

Problem 1

Part a)

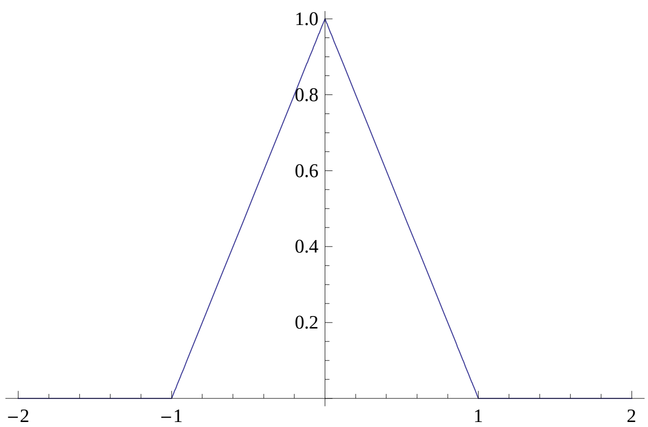
Let' s pick a function for the spectrum such that g(omega)=0 for $|\omega|>\omega0$
In this case $\omega0=1$ (cut-off frequency).

For sake of simplicity, let's select an even function so that the inverse FT will give a real function (if the spectrum is odd then the inverse FT will give a pure imaginary function)

In[36]:=
```
gomega = Piecewise[{{1 - ω, 0 <= ω < 1}, {1 + ω, -1 < ω < 0}}]
```

Out[36]=
$$\begin{cases} 1-\omega & 0 \le \omega < 1 \\ 1+\omega & -1 < \omega < 0 \end{cases}$$

In[37]:=
```
Plot[gomega, {ω, -2, 2}]
```

Out[37]=


Part b)

```
ω0 = 1
```

1

Let' s apply the Inverse Fourier Transform to the spectrum g (omega) to find f(t)
Note: we could also apply a cosine FT for 0<ω<ω0)

```
foft[t_] = Integrate[gomega * Exp[I * t * ω], {ω, -ω0, ω0}]
```

$$-\frac{e^{-i t}\left(-1+e^{i t}\right)^2}{t^2}$$

```
Expand[%]
```

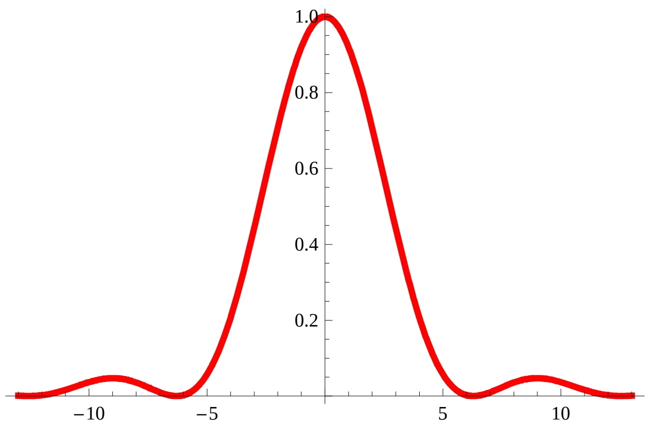
$$\frac{2}{t^2}-\frac{e^{-i t}}{t^2}-\frac{e^{i t}}{t^2}$$

This is equivalent to :

```
ft[t_] = (2 - 2 * Cos[t]) / t^2
```

$$\frac{2-2\,\mathrm{Cos}[t]}{t^2}$$

```
plot1 = Plot[ft[t], {t, -13, 13}, PlotStyle → {Red, AbsoluteThickness[3]}]
```



Part c)

To apply Shannon' s theorem select a sampling frequency fs = ω0/Pi

```
fs = ω0 / Pi
```

$$\frac{1}{\pi}$$

```
dt = 1 / fs (* this is T, the period *)
```

π

```
samples = Table[foft[n*dt], {n, -5, 5}]
```

Power::infy : Infinite expression $\frac{1}{0^2}$ encountered. ≫

∞::indet : Indeterminate expression 0 ComplexInfinity encountered. ≫

$\left\{\frac{4}{25\pi^2}, 0, \frac{4}{9\pi^2}, 0, \frac{4}{\pi^2}, \text{Indeterminate}, \frac{4}{\pi^2}, 0, \frac{4}{9\pi^2}, 0, \frac{4}{25\pi^2}\right\}$

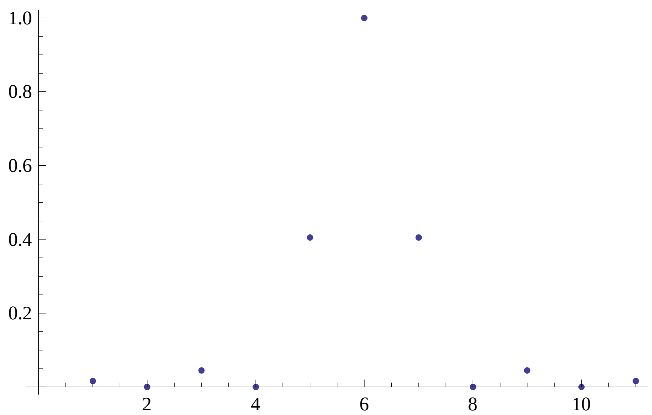
Since the result for the function at zero is Indeterminate, let' s calculate it by hand and substitute it back in the samples :

- **Note :** $\frac{2-2\,\text{Cos}[t]}{t^2}$ **at zero** $=$ $\frac{\text{Sin}[t]}{t}$ **at zero** $= \text{Cos}[0] = 1$

samples $= \left\{\frac{4}{25\pi^2}, 0, \frac{4}{9\pi^2}, 0, \frac{4}{\pi^2}, 1, \frac{4}{\pi^2}, 0, \frac{4}{9\pi^2}, 0, \frac{4}{25\pi^2}\right\}$

$\left\{\frac{4}{25\pi^2}, 0, \frac{4}{9\pi^2}, 0, \frac{4}{\pi^2}, 1, \frac{4}{\pi^2}, 0, \frac{4}{9\pi^2}, 0, \frac{4}{25\pi^2}\right\}$

```
ListPlot[samples]
```

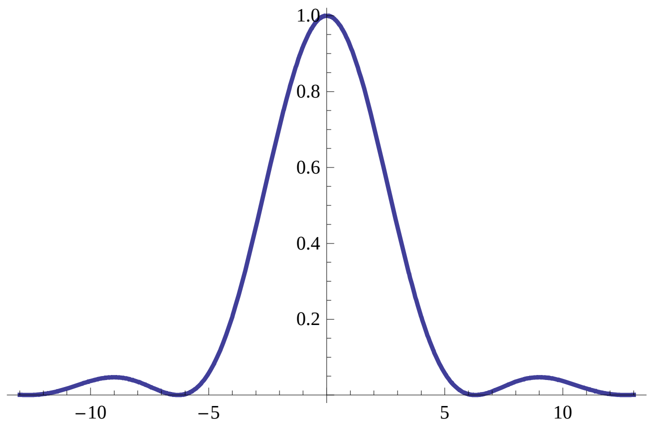


Part d)

Shannon' s sampling theorem says that f(t) can be uniquely reconstructed from the samples f(nT) where T=1/fs, and fs=ω0/Pi, i.e. from the samples f(n*Pi/ω0):

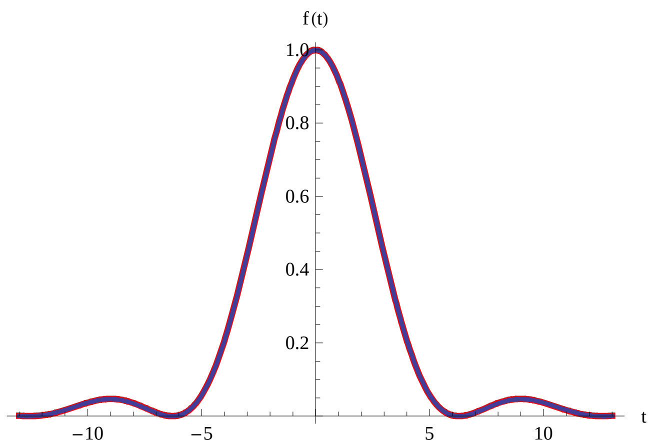
```
fSum[t_] := Sum[foft[n*dt]*Sinc[n*Pi + ω0*t], {n, -5, -1}] +
  Sum[foft[n*dt]*Sinc[n*Pi + ω0*t], {n, 1, 5}] + 1*Sinc[0*Pi + ω0*t]
```

```
plot2 = Plot[fSum[t], {t, -13, 13}, PlotStyle → AbsoluteThickness[2]]
```



As it is shown here, the signal f (t) is recontructed completely by sampling at f (nT)

```
Show[plot1, plot2, Axes → True, AxesLabel → {"t", "f(t)"}]
```



Laplace transforms

Find the Laplace transforms of the following functions.

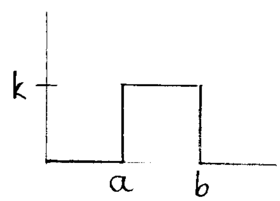
6.1.9 $\quad e^{3a-2bt}$

Sol^{n}

$$\mathcal{L}\{e^{3a-2bt}\} = e^{3a}\mathcal{L}\{e^{-2bt}\}$$

$$= e^{3a}\cdot\frac{1}{s+2b} \quad \text{from Table 6.1} \quad \text{Ans}$$

6.1.14



Sol^{n}

$$f(t) = \begin{cases} k, & a \leqslant t \leqslant b \\ 0, & \text{otherwise} \end{cases}$$

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^a e^{-st}\cancelto{0}{f(t)}\,dt + \int_a^b e^{-st}\cancelto{k}{f(t)}\,dt + \int_b^\infty e^{-st}\cancelto{0}{f(t)}\,dt$$

$$= k\int_a^b e^{-st}\,dt = k\left[\frac{e^{-st}}{-s}\right]_{t=a}^{t=b}$$

$$= \frac{k}{s}\left(e^{-as} - e^{-bs}\right) \quad \text{Ans}$$

Inverse Laplace transforms

Given $F(s) = \mathcal{L}(f)$, find $f(t)$.

6.1.29 $\quad \dfrac{4s - 3\pi}{s^2 + \pi^2}$

Sol^{n}

$$\mathcal{L}^{-1}\left\{\frac{4s-3\pi}{s^2+\pi^2}\right\} = 4\mathcal{L}^{-1}\left\{\frac{s}{s^2+\pi^2}\right\} - 3\mathcal{L}^{-1}\left\{\frac{\pi}{s^2+\pi^2}\right\}$$

From Table 6.1; $\quad = 4\cos(\pi t) - 3\sin(\pi t) \quad$ Ans

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6.1.31 $\dfrac{s^4 - 3s^2 + 12}{s^5}$

Sol^n $\mathcal{L}^{-1}\left\{\dfrac{s^4 - 3s^2 + 12}{s^5}\right\} = \mathcal{L}^{-1}\left\{\dfrac{1}{s}\right\} - 3\mathcal{L}^{-1}\left\{\dfrac{1}{s^3}\right\} + 12\mathcal{L}^{-1}\left\{\dfrac{1}{s^5}\right\}$

From Table 6.1; $= 1 - 3 \cdot \dfrac{t^2}{2} + 12 \cdot \dfrac{t^4}{4!}$

$= 1 - \dfrac{3t^2}{2} + \dfrac{t^4}{2}$ Ans

6.1.34 $\dfrac{20}{(s-1)(s+4)}$

Sol^n $\mathcal{L}^{-1}\left\{\dfrac{20}{(s-1)(s+4)}\right\} = \mathcal{L}^{-1}\left\{\dfrac{4}{s-1} - \dfrac{4}{s+4}\right\}$

$= 4\mathcal{L}^{-1}\left\{\dfrac{1}{s-1}\right\} - 4\mathcal{L}^{-1}\left\{\dfrac{1}{s+4}\right\}$

From Table 6.1; $= 4e^{t} - 4e^{-4t}$ Ans

S-Shifting

Find the transform.

6.1.41 $3.8te^{2.4t}$

Sol^n $\mathcal{L}\left\{3.8te^{2.4t}\right\} = 3.8\mathcal{L}\left\{e^{2.4t}t\right\} = 3.8\,F(s-2.4)$

$= 3.8 \cdot \dfrac{1}{(s-2.4)^2}$ from s-shifting theorem

$= \dfrac{3.8}{(s-2.4)^2}$ Ans

Find the inverse transform

6.1.47 $\dfrac{7}{(s-1)^3}$

Sol^n $\mathcal{L}^{-1}\left\{\dfrac{7}{(s-1)^3}\right\} = 7\mathcal{L}^{-1}\left\{\dfrac{1}{(s-1)^3}\right\}$

From Table 6.1 & s-shifting theorem $\mathcal{L}^{-1}\left\{F(s-a)\right\} = e^{at}f(t)$;

$$\mathcal{L}^{-1}\left\{\frac{7}{(s-1)^3}\right\} = 7 \cdot \frac{e^t t^2}{2!}$$

$$= \frac{7}{2} e^t t^2 \qquad \underline{\text{Ans}}$$

Initial value problems

Solve the following initial value problems by the Laplace Transform.

6.2.11 $y' + \frac{1}{2}y = 17 \sin 2t \quad , \quad y(0) = -1$

Sol^n The subsidiary equation is

$$sY - y(0) + \frac{1}{2}Y = 17 \mathcal{L}\{\sin 2t\}$$

$$sY + 1 + \frac{Y}{2} = 17 \cdot \frac{2}{s^2+2^2}$$

$$\left(s + \frac{1}{2}\right)Y = \frac{34}{s^2+4} - 1$$

$$Y = \frac{34}{(s^2+4)(s+\frac{1}{2})} - \frac{1}{s+\frac{1}{2}}$$

partial fraction expansion; $Y = \frac{-8s+4}{s^2+4} + \frac{8}{s+\frac{1}{2}} - \frac{1}{s+\frac{1}{2}}$

$$y(t) = \mathcal{L}^{-1}\{Y\} = \mathcal{L}^{-1}\left\{\frac{-8s+4}{s^2+4}\right\} + 7\mathcal{L}^{-1}\left\{\frac{1}{s+\frac{1}{2}}\right\}$$

$$= -8\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} + 2\mathcal{L}^{-1}\left\{\frac{2}{s^2+2^2}\right\} + 7\mathcal{L}^{-1}\left\{\frac{1}{s+\frac{1}{2}}\right\}$$

From Table 6.1 & s-shifting theorem; $y(t) = -8\cos 2t + 2\sin 2t + 7e^{-t/2}$ **Ans**

6.2.12 $y'' - y' - 6y = 0 \quad , \quad y(0) = 6 \; , \; y'(0) = 13$

Sol^n 1) The subsidiary eqn. is

$$s^2Y - sy(0) - y'(0) - sY + y(0) - 6Y = 0$$

$$(s^2 - s - 6)Y = sy(0) + y'(0) - y(0) = 6s + 7$$

2) the transfer function $Q = \frac{1}{s^2 - s - 6}$

$$Y = 6sQ + 7Q = \frac{6s+7}{s^2 - s - 6} = \frac{6s+7}{(s-3)(s+2)}$$

partial fraction expansion,

$$Y = \frac{5}{s-3} + \frac{1}{s+2}$$

3) inverse transform

$$y(t) = \mathcal{L}^{-1}\{Y\} = \mathcal{L}^{-1}\left\{\frac{5}{s-3}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}$$

$$= 5e^{3t} + e^{-2t}$$

Ans

6.2.15 $y'' + 2y' + 2y = 0, \quad y(0) = 1, \quad y'(0) = -3$

Solⁿ 1) Subsidiary eqn is

$$s^2Y - sy(0) - y'(0) + 2sY - 2y(0) + 2Y = 0$$

$$(s^2 + 2s + 2)Y = sy(0) + y'(0) + 2y(0)$$

$$= s - 3 + 2 = s - 1$$

2) The transfer function $Q = \frac{1}{s^2 + 2s + 2}$

$$Y = (s+1)Q = \frac{s-1}{s^2 + 2s + 2} = \frac{s-1}{(s+1)^2 + 1}$$

3) inverse transform

$$y(t) = \mathcal{L}^{-1}\{Y\} = \mathcal{L}^{-1}\left\{\frac{s-1}{(s+1)^2+1}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+1}\right\} + \mathcal{L}^{-1}\left\{\frac{-2}{(s+1)^2+1}\right\}$$

By the s-shifting theorem and Table 6.1

$$y(t) = e^{-t}\cos t - 2e^{-t}\sin t$$

$$= e^{-t}(\cos t - 2\sin t)$$

Ans

6.2.18 $y'' + 9y = 10e^{-t}$, $y(0) = 0$, $y'(0) = 0$

Sol^n 1) The subsidiary eqn is

$$s^2 Y - sy(0) - y'(0) + 9Y = \frac{10}{s+1}$$

$$(s^2 + 9)\, Y = \frac{10}{s+1} + \cancel{sy(0)}^{0} + \cancel{y'(0)}^{0}$$

2) The transfer fn. $Q = \dfrac{1}{s^2+9}$

$$Y = \frac{10}{s+1} Q = \frac{10}{(s+1)(s^2+9)}$$

$$= \frac{1}{s+1} + \frac{-s+1}{s^2+9}$$; partial fraction expansion

3) inverse transform

$$y(t) = \mathcal{L}^{-1}\{Y\} = \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} - \mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2+9}\right\}$$

$$= e^{-t} - \cos 3t + \frac{1}{3}\sin 3t$$ Ans

Initial value problems

6.3.27 $y'' + 9y = r(t)$, $r(t) = 8\sin t$ if $0 < t < \pi$ and 0 if $t > \pi$; $y(0) = 0$, $y'(0) = 4$

Sol^n

$$r(t) = \begin{cases} 8\sin t, & 0 < t < \pi \\ 0, & t > \pi \end{cases}$$

$r(t)$ can be written as $r(t) = 8\sin t\,(1 - u(t-\pi))$

$$y'' + 9y = r(t) = 8\sin t\,(1 - u(t-\pi))$$

The subsidiary eqn. is

$$s^2Y - s\overset{0}{\cancel{y(0)}} - y'(0) + 9Y = \mathcal{L}\{8\sin t\,(1 - u(t-\pi))\}$$

$$(s^2+9)Y - 4 = 8\mathcal{L}\{\sin t\} - 8\mathcal{L}\{\sin t \cdot u(t-\pi)\}$$

$$= \frac{8}{s^2+1} - 8e^{-\pi s}\mathcal{L}\{\sin(t+\pi)\}$$

$$\sin(t+\pi) = \sin t \cdot \overset{-1}{\cancel{\cos\pi}} + \cos t \cdot \overset{0}{\cancel{\sin\pi}} = -\sin t$$

$$(s^2+9)Y - 4 = \frac{8}{s^2+1} - 8e^{-\pi s}\mathcal{L}\{-\sin t\}$$

$$= \frac{8}{s^2+1} + \frac{8e^{-\pi s}}{s^2+1}$$

$$Y = \frac{8}{(s^2+1)(s^2+9)} + \frac{8e^{-\pi s}}{(s^2+1)(s^2+9)} + \frac{4}{s^2+9}$$

partial fraction expansion,

$$Y = \frac{1}{s^2+1} - \frac{1}{s^2+9} + \frac{e^{-\pi s}}{s^2+1} - \frac{e^{-\pi s}}{s^2+9} + \frac{4}{s^2+9}$$

$$y(t) = \mathcal{L}^{-1}\{Y\} = \sin t + \sin 3t + \sin(t-\pi)u(t-\pi) - \frac{1}{3}\sin(3(t-\pi))u(t-\pi)$$

$$= \sin t + \sin 3t - \sin t\, u(t-\pi) + \frac{1}{3}\sin(3t)\,u(t-\pi)$$ **Ans**

Note : $\sin(t-\pi) = -\sin t$, $\sin(3t - 3\pi) = -\sin(3t)$

$$y(t) = \begin{cases} \sin t + \sin(3t) & ; \; 0 < t < \pi \\ \frac{4}{3}\sin(3t) & ; \; t > \pi \end{cases}$$ **Ans**

6.3.32 $y'' + 8y' + 15y = r(t)$, $r(t) = 35e^{2t}$ if $0 < t < 2$ and 0 if $t > 2$; $y(0) = 3$, $y'(0) = -8$

Sol^{n}

$$r(t) = \begin{cases} 35e^{2t}, & 0 < t < 2 \\ 0, & t > 2 \end{cases}$$

or $$r(t) = 35e^{2t}(1 - u(t-2))$$

$$y'' + 8y' + 15y = 35e^{2t}(1 - u(t-2))$$

The subsidiary eqn. is

$$s^2Y - sy(0) - y'(0) + 8sY - 8y(0) + 15Y = 35\mathcal{L}\{e^{2t}(1-u(t-2))\}$$

$$(s^2 + 8s + 15)Y - 3s + 8 - 24 = 35\left[\frac{1}{s-2}\right] - 35e^{-2s}\mathcal{L}\{e^{2(t+2)}\}$$

$$(s^2 + 8s + 15)Y - 3s - 16 = \frac{35}{s-2} - \frac{35e^4 \cdot e^{-2s}}{s-2}$$

$$Y = \left[\frac{35}{s-2} - \frac{35e^4 \cdot e^{-2s}}{s-2} + 3s + 16\right] \cdot \frac{1}{s^2 + 8s + 15}$$

$$= \frac{35}{(s-2)(s+3)(s+5)} - \frac{35e^4 \cdot e^{-2s}}{(s-2)(s+3)(s+5)} + \frac{3s}{(s+3)(s+5)} + \frac{16}{(s+3)(s+5)}$$

partial fraction expansion,

$$Y = \frac{1}{s-2} - \cancel{\frac{3.5}{s+3}} + \frac{2.5}{s+5} - e^4 \cdot e^{-2s}\left[\frac{1}{s-2} - \frac{3.5}{s+3} + \frac{2.5}{s+5}\right] - \cancel{\frac{4.5}{s+3}} + \frac{7.5}{s+5}$$
$$+ \cancel{\frac{8}{s+3}} - \frac{8}{s+5}$$

$$= \frac{1}{s-2} + \frac{2}{s+5} - e^4 \cdot e^{-2s}\left[\frac{1}{s-2} - \frac{3.5}{s+3} + \frac{2.5}{s+5}\right]$$

$$y(t) = \mathcal{L}^{-1}\{Y\} = e^{2t} + 2e^{-5t} - e^4 e^{2(t-2)}u(t-2) + 3.5e^4 \cdot e^{-3(t-2)}u(t-2)$$
$$- 2.5e^4 \cdot e^{-5(t-2)}u(t-2)$$

$$= e^{2t} + 2e^{-5t} + u(t-2)\left[-e^{2t} - 2.5e^{-5t+14} + 3.5e^{-3t+10}\right]$$ Ans

or $$y(t) = \begin{cases} e^{2t} + 2e^{-5t}, & 0 < t < 2 \\ e^{-5t}(2 - 2.5e^{14}) + 3.5e^{-3t+10}, & t > 2 \end{cases}$$ Ans

Effect of Delta function on vibrating systems

- Showing the details, find, graph, and discuss the solution.

6.4.1 $y'' + y = \delta(t-2\pi), \quad y(0) = 10, \; y'(0) = 0$

Sol^n The subsidiary eqn. is

$$s^2Y - sy(0) - \cancel{y'(0)}^{\,0} + Y = \mathcal{L}\{\delta(t-2\pi)\}$$

$$(s^2+1)Y - 10s = e^{-2\pi s}$$

$$Y = \frac{e^{-2\pi s}}{s^2+1} + \frac{10s}{s^2+1}$$

$$y(t) = \mathcal{L}^{-1}\{Y\} = \sin(t-2\pi)\cdot u(t-2\pi) + 10\cos t \qquad \text{Ans}$$

$$\sin(t-2\pi) = \sin t\cdot\cancel{\cos(2\pi)}^{\,1} - \cos t\,\cancel{\sin(2\pi)}^{\,0} = \sin t$$

$$\therefore \quad y(t) = \begin{cases} 10\cdot\cos t, & 0 < t < 2\pi \\ \sin t + 10\cos t, & t > 2\pi \end{cases} \qquad \text{Ans}$$

See graph at the end. With the impulse $\delta(t-2\pi)$, the solution seems to have a phase shift to the right but the frequency and the amplitude seem to be unchanged.

6.4.4 $y'' + 3y' + 2y = 10(\sin t + \delta(t-1)), \quad y(0) = 1, \; y'(0) = -1$

Sol^n The subsidiary eqn. is

$$s^2Y - sy(0) - y'(0) + 3sY - 3y(0) + 2Y = 10\,\mathcal{L}\{\sin t + \delta(t-1)\}$$

$$s^2Y - s + 1 + 3sY - 3 + 2Y = 10\left(\frac{1}{s^2+1} + e^{-s}\right)$$

$$(s^2+3s+2)Y = \frac{10}{s^2+1} + 10e^{-s} + s + 2$$

$$Y = \frac{10}{(s^2+1)(s+1)(s+2)} + \frac{10e^{-s}}{(s+1)(s+2)} + \frac{\cancel{s+2}}{(s+1)\cancel{(s+2)}}$$

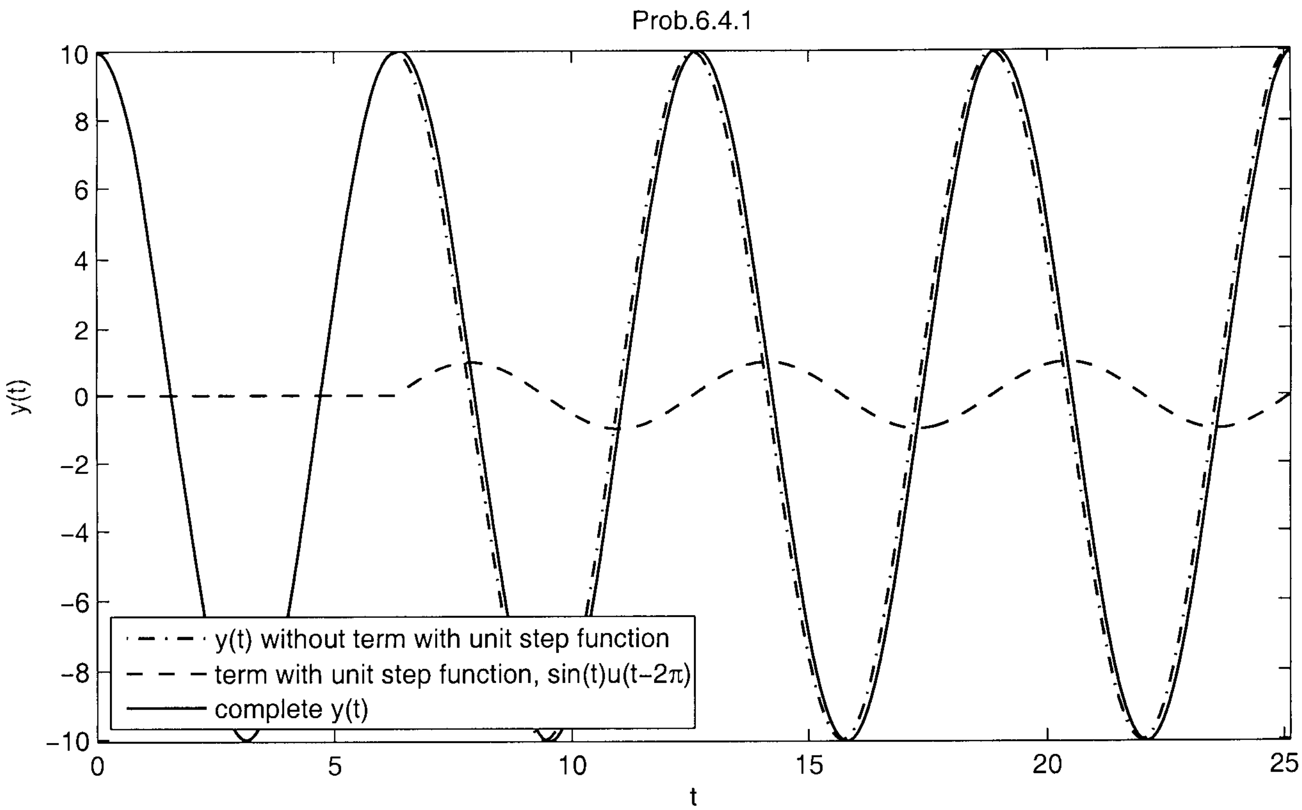
partial fraction expansion

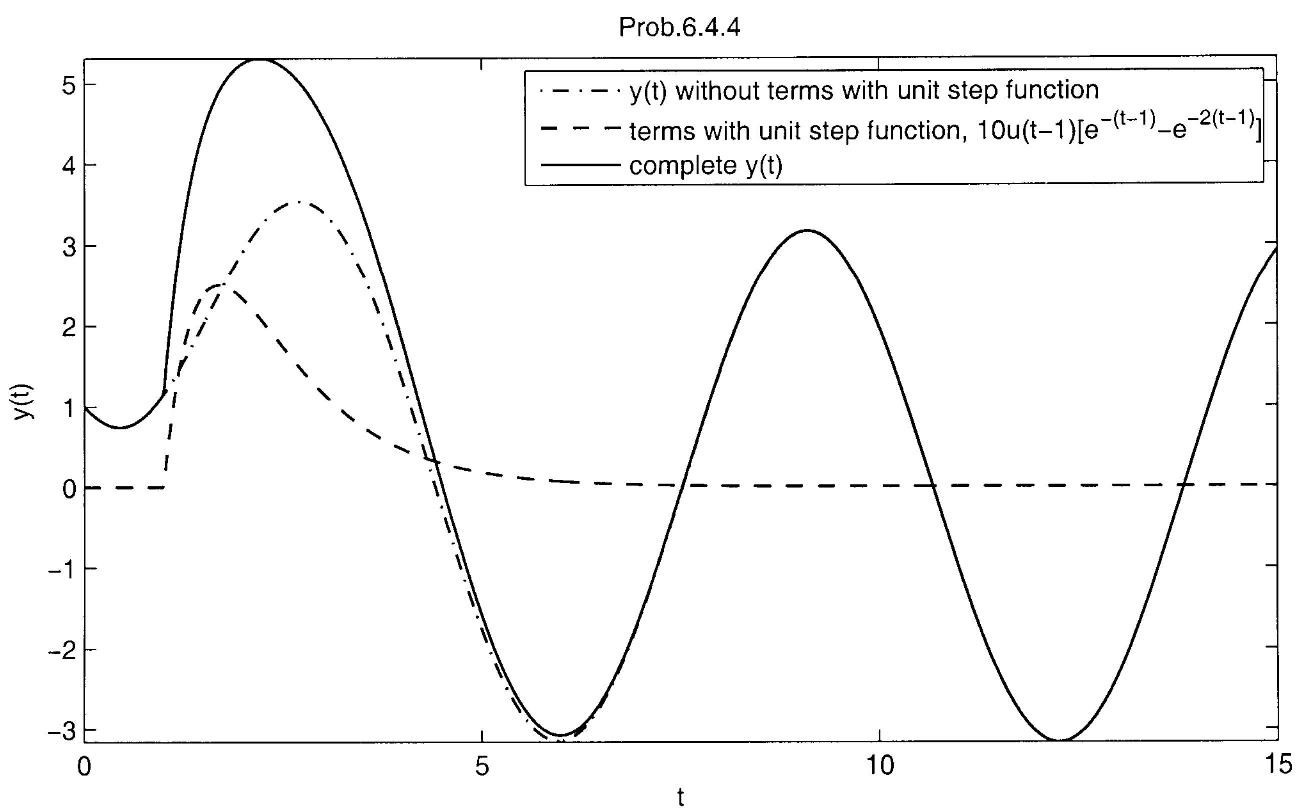
$$Y = \frac{-3s+1}{s^2+1} + \frac{5}{s+1} - \frac{2}{s+2} + e^{-s}\left(\frac{10}{s+1} - \frac{10}{s+2}\right) + \frac{1}{s+1}$$

$$y(t) = \mathcal{L}^{-1}\{Y\} = -3\cos t + \sin t + 6e^{-t} - 2e^{-2t} + 10e^{-(t-1)}\cdot u(t-1) - 10e^{-2(t-1)}\cdot u(t-1) \qquad \text{Ans}$$

$$y(t) = \begin{cases} -3\cos t + \sin t + 6e^{-t} - 2e^{-2t}, & 0 < t < 1 \\ -3\cos t + \sin t + (6 + 10e)e^{-t} - (2 + 10e^2)e^{-2t}, & t > 1 \end{cases}$$ Ans

See plot at the end. Without the δ-term, the solution approaches a harmonic oscillation pretty soon. However, about $t \approx 8$, the effect of the δ-term seems to be damped out and the graphs of $y(t)$ with and without δ-term look coincident.


Prob.6.4.1
10
8
6
4
2
0
−2
−4
−6
−8
−10
y(t)
0
5
10
15
20
25
t
y(t) without term with unit step function
term with unit step function, sin(t)u(t−2π)
complete y(t)


Prob.6.4.4
5
4
3
2
1
0
−1
−2
−3
y(t)
0
5
10
15
t
y(t) without terms with unit step function
terms with unit step function, 10u(t−1)[e^{−(t−1)}−e^{−2(t−1)}]
complete y(t)

Problem 6.4.8

The subsidiary equation is

$$(s^2 + 5s + 6)Y = e^{-\pi s/2} - \frac{s}{s^2 + 1}\, e^{-\pi s}.$$

Its solution is

$$Y = \left(\frac{1}{s+2} - \frac{1}{s+3}\right) e^{-\pi s/2} - \left(-\frac{0.4}{s+2} + \frac{0.3}{s+3} + \frac{0.1(s+1)}{s^2+1}\right) e^{-\pi s}.$$

The inverse transform of Y is

$$y = u(t - \tfrac{1}{2}\pi)\left[e^{-2t+\pi} - e^{-3t+3\pi/2}\right]$$

$$- 0.1u(t - \pi)\left[-4e^{-2t+2\pi} + 3e^{-3t+3\pi} - \cos t - \sin t\right].$$

This solution is zero from 0 to $\frac{1}{2}\pi$ and then increases rapidly. Its first negative half-wave has a smaller maximum amplitude (about 0.1) than the continuation as a harmonic oscillation with maximum amplitude of about 0.15.