ME 565 - HW 3

Section 12.3 #15

$$\frac{\partial^2 u}{\partial t^2} = -c^2 \frac{\partial^4 u}{\partial x^4}, \quad C = \frac{EI}{\rho A}$$

$$u(x,t) = F(x)\cdot G(t) \Rightarrow \begin{cases} u_{xxxx} = \frac{\partial^4}{\partial x^4}[F(x)G(t)] = F^{(4)}G \\ u_{tt} = \frac{\partial^2}{\partial t^2}[F(x)G(t)] = F\ddot{G} \end{cases}$$

$$\frac{\partial^2 u}{\partial t^2} = -c^2 \frac{\partial^4 u}{\partial x^4} \Rightarrow F\ddot{G} = -c^2 F^{(4)} G \Rightarrow \frac{F\ddot{G}}{c^2 FG} = -\frac{c^2 F^{(4)} G}{c^2 FG} \Rightarrow \frac{F^{(4)}}{F} = -\frac{\ddot{G}}{c^2 G} = \beta^4 = \text{const}$$

$$\Rightarrow \begin{cases} F^{(4)} - \beta^4 F = 0 \longrightarrow \lambda^4 - \beta^4 = 0. \Rightarrow \lambda = \beta, -\beta, i\beta, -i\beta \\ \ddot{G} + c^2\beta^4 G = 0 \end{cases}$$

$$\therefore F(x) = C_1 e^{\beta x} + C_2 e^{-\beta x} + C_3 e^{i\beta x} + C_4 e^{-i\beta x}$$

$$= (A\cos\beta x + B\sin\beta x) + (C\cosh\beta x + D\sinh\beta x)$$

$$\downarrow$$

$$\lambda^2 + (c\beta^2)^2 = 0$$

$$\Rightarrow \lambda = ic\beta^2, -ic\beta^2$$

$$\therefore G(t) = C_5 e^{ic\beta^2 t} + C_6 e^{-ic\beta^2 t}$$

$$= a\cos c\beta^2 t + b\sin c\beta^2 t \ \#$$

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16. (Simply supported beam in Fig. 290A) Find solutions $u_n = F_n(x)G_n(t)$ of (21) corresponding to zero initial velocity and satisfying the boundary conditions (see Fig. 290A)

$u(0, t) = 0, \ u(L, t) = 0$

(ends simply supported for all times t),

$u_{xx}(0, t) = 0, \ u_{xx}(L, t) = 0$

(zero moments, hence zero curvature, at the ends).

Solution: For $u = F(x)G(t)$, if $G \equiv 0$, then $u = FG \equiv 0$, which is of no interest, Hence $G \not\equiv 0$, and then

$$\begin{cases} u(0,t) = 0 = F(0)\cdot G(t) \\ u(L,t) = 0 = F(L)\cdot G(t) \end{cases} \Rightarrow \begin{cases} F(0) = 0 \\ F(L) = 0 \end{cases}$$

$$\begin{cases} u_{xx}(0,t) = F(0)'' G(t) \\ u_{xx}(L,t) = F(L)'' G(t) \end{cases} \Rightarrow \begin{cases} F''(0) = 0 \\ F''(L) = 0 \end{cases}$$

$$F(x) = A\cos\beta x + B\sin\beta x + C\cosh\beta x + D\sinh\beta x$$

$$F''(x) = (-A\cos\beta x - B\sin\beta x + C\cosh\beta x + D\sinh\beta x)\beta^2$$

BC 1. At $x = 0$: $\begin{cases} A + C = 0 \\ -A + C = 0 \end{cases} \Rightarrow A = C = 0.$

3c2 At $x = L$:

$$F(L) = A\cos\beta L + B\sin\beta L + C\cosh\beta L + D\sinh\beta L = 0$$

$$F''(L) = (-A\cos\beta x - B\sin\beta L + C\cosh\beta L + D\sinh\beta L)\beta^2 = 0$$

$$\Rightarrow 2B\sin\beta L = 0 \Rightarrow \beta L = n\pi \Rightarrow \beta = \frac{n\pi}{L}$$

If $B = 0$ then $D = 0$. To avoid the trivial solution:

Setting $B \neq 1$, we thus obtain inifinitely many solutions $F(x) = F_n(x)$,

where $F_n(x) = \sin\frac{n\pi}{L}x \quad (n = 1, 2, \cdots)$.

Also, since $\sin\beta L = 0 \rightarrow$ then $D = 0$

$$\ddot{G} + c^2p^4G = 0$$

$$G(t) = A\cos(cp^2t) + B\sin(cp^2t)$$

$$\dot{G}(t) = -cp^2A\sin(cp^2t) + cp^2B\cos(cp^2t)$$

I.C. $\quad \dot{G}(0) = 0 = cp^2B \Rightarrow \underline{B = 0}\ ; (cp^2 \neq 0)$

$$G(t) = A\cos(cp^2t), \quad p = \frac{n\pi}{L}$$

$$G_n(t) = a_n\cos(cn^2\pi^2t/L^2) \quad *$$

$\underline{\text{Ans}}$ $\quad \therefore u_n(x,t) = F_n(x)G_n(t) = b_n\sin\left(\frac{n\pi x}{L}\right)\cdot\cos\left(\frac{cn^2\pi^2t}{L^2}\right), \quad b_n = a_nc_n$

12.3.17 Find the solution of $\frac{\partial^2 u}{\partial t^2} = -c^2\frac{\partial^4 u}{\partial x^4}$ that satisfies the condition in 12.3.16 as well as $u(x,0) = f(x) = x(L-x)$.

$\underline{\text{Sol}^n}$ From 12.3.16,

$$u_n(x,t) = b_n\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{cn^2\pi^2t}{L^2}\right)$$

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} b_n\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{cn^2\pi^2t}{L^2}\right)$$

$$u(x,0) = x(L-x) = \sum_{n=1}^{\infty} b_n\sin\left(\frac{n\pi x}{L}\right)$$

Use Fourier sine series;

$$b_n = \frac{2}{L}\int_0^L x(L-x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{2}{L}\left[\int_0^L xL\sin\left(\frac{n\pi x}{L}\right)dx - \int_0^L x^2\sin\left(\frac{n\pi x}{L}\right)dx\right]$$

$$= \frac{2}{L}\left\{L\left(\frac{L}{n\pi}\right)^2\left[-\frac{n\pi x}{L}\cos\left(\frac{n\pi x}{L}\right)+\sin\left(\frac{n\pi x}{L}\right)\right]_{x=0}^{x=L}\right.$$

$$-\left(\frac{L}{n\pi}\right)^3\left[-\left(\frac{n\pi x}{L}\right)^2\cos\left(\frac{n\pi x}{L}\right)+2\frac{n\pi x}{L}\sin\left(\frac{n\pi x}{L}\right)\right.$$

$$\left.\left.+2\cos\left(\frac{n\pi x}{L}\right)\right]_{x=0}^{x=L}\right\} \quad ; \text{ Use Integration table}$$

$$= \frac{2}{L}\left\{L\left(\frac{L}{n\pi}\right)^2\left[-n\pi\cos(n\pi)\right]-\left(\frac{L}{n\pi}\right)^3\left[-(n\pi)^2\cos(n\pi)\right.\right.$$

$$\left.+2\cos(n\pi)-2\right\}$$

$$= \frac{2L^2}{(n\pi)^3}\left[2-2(-1)^n\right]$$

$$\therefore \quad u(x,t) = \frac{2L^2}{\pi^3}\sum_{n=1}^{\infty}\frac{(2-2(-1)^n)\sin\left(\frac{n\pi x}{L}\right)}{n^3}\cos\left(\frac{cn^2\pi^2 t}{L^2}\right) \quad \underline{\underline{\text{Ans}}}$$

$$= \frac{8L^2}{\pi^3}\left(\cos\left[c\left(\frac{\pi}{L}\right)^2 t\right]\sin\left(\frac{\pi x}{L}\right)+\frac{1}{3^3}\cos\left[c\left(\frac{3\pi}{L}\right)^2 t\right]\sin\left(\frac{3\pi x}{L}\right)\right.$$

$$\left.+\dots\right) \quad \underline{\underline{\text{Ans}}}$$

```
"Problem 12.3 # 17";
Integrate[x * (L - x) * Sin[n * π * x / L], {x, 0, L}, Assumptions → n ∈ Integers]
```

$$-\frac{2\left(-1+(-1)^{n}\right) L^{3}}{n^{3} \pi^{3}}$$

$$\texttt{A[n\_] := } -\frac{2\left(-1+(-1)^{n}\right)}{n^{3} \pi^{3}} * 2 / L$$

```
A[5]
```

$$\frac{8}{125 \, L \, \pi^{3}}$$

```
c = 1;
L = 1;
```

$$\texttt{u}_{\texttt{n\_}}\texttt{[x\_, t\_] := Cos[c * n^2 * π^2 / L^2 * t] * Sin[π * n * x / L]}$$

$$\texttt{u}_{\texttt{n}}\texttt{[x, t]}$$

$$\text{Cos}\left[n^{2} \pi^{2} t\right] \text{Sin}[n \pi x]$$

$$\texttt{u[x\_, t\_, m\_] := Sum[A[n] * }\texttt{u}_{\texttt{n}}\texttt{[x, t], \{n, 1, m\}]}$$

```
"Finding the first three eigenfunctions that are non trivial";
```

```
u[x, t, 5]
```

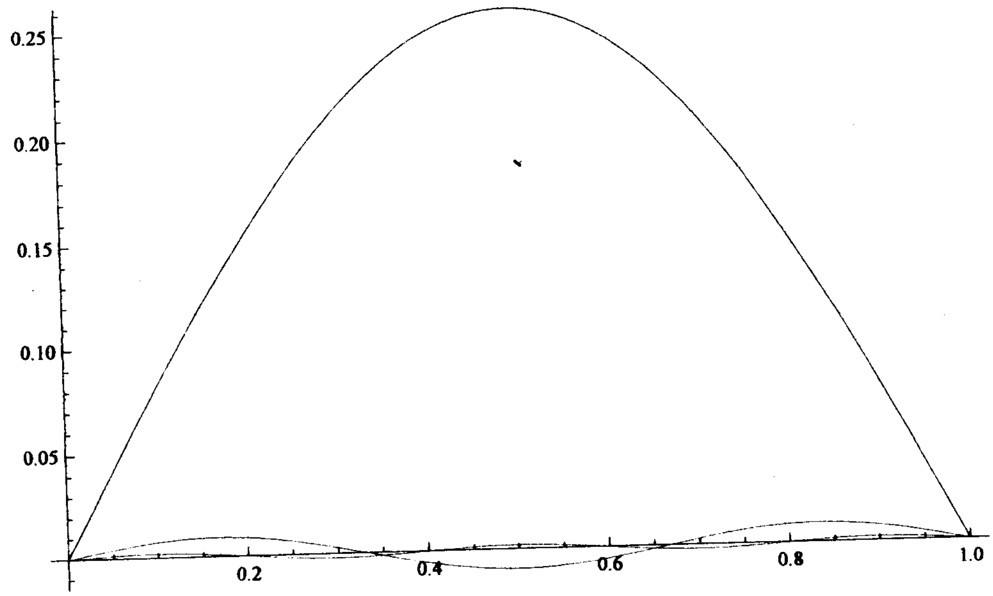
$$\frac{8 \,\text{Cos}\left[\pi^{2} t\right] \text{Sin}[\pi x]}{\pi^{3}}+\frac{8 \,\text{Cos}\left[9 \pi^{2} t\right] \text{Sin}[3 \pi x]}{27 \pi^{3}}+\frac{8 \,\text{Cos}\left[25 \pi^{2} t\right] \text{Sin}[5 \pi x]}{125 \pi^{3}}$$

```
"Finding the first three frequencies, note that the second and fourth frequence
   are irrelevant because the solutions to u[x,t] are 0 for n=2 and n=4 ";
w[n_] := (c * n^2 * π^2 / L^2) / (2 π)
```

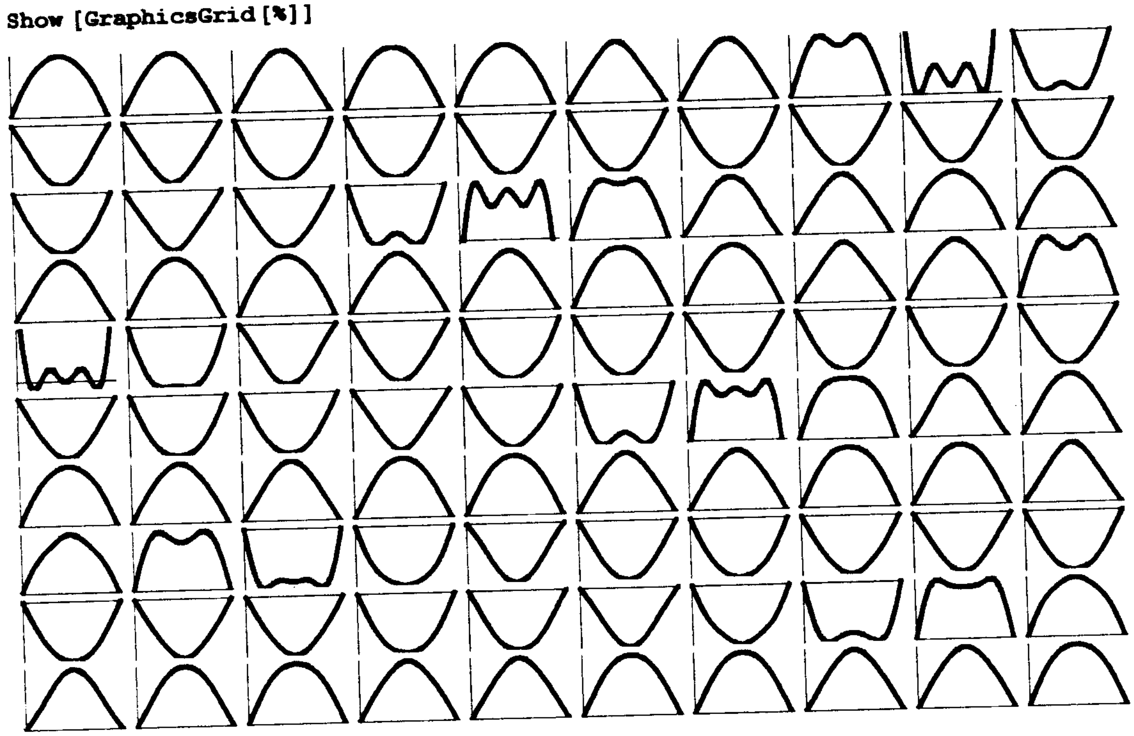
```
freqTable = Table[w[i], {i, 1, 5}]
```

$$\left\{\frac{\pi}{2},\ 2 \pi,\ \frac{9 \pi}{2},\ 8 \pi,\ \frac{25 \pi}{2}\right\}$$

```
"Ploting the first three modes";
Plot[{A[1] * u1[x, 0], A[3] * u3[x, 0], A[5] * u5[x, 0]}, {x, 0, L}]
```




```
Table[
  Plot[u[x, t, 5], {x, 0, L}, PlotStyle → AbsoluteThickness[2], Ticks → False], {t, 0, 2, .02}];
SolnPlots = %;
Partition[SolnPlots, 10];
Show [GraphicsGrid[%]]
```



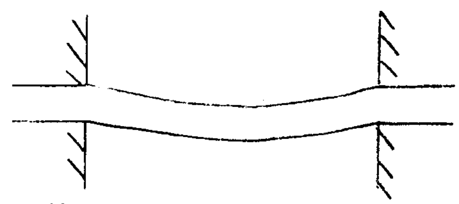
```
"I did indeed view animation to verify the behavior seen above";
```

12.3.19 (Clamped beam) What are the boundary conditions for the clamped beam in Fig. 290 B? Show that F in Prob. 15 satisfies these conditions if βL is a solution of the equation

$$\cosh \beta L \cos \beta L = 1.$$

Determine approximate solutions of (22), for instance, graphically from the intersections of the curves of $\cos \beta L$ and $1/\cosh \beta L$.

Sol$^{\underline{n}}$ Fig. 290B: Clamped beam at both ends.



let $u(x,t)$ is the beam deflection

Since both ends are clamped, no deflection and no movement of the beam at those locations. Thus, the boundary conditions are

$$\left.\begin{aligned} u(0,t) &= 0, & u(L,t) &= 0, \\ u_x(0,t) &= 0, & u_x(L,t) &= 0. \end{aligned}\right\} \quad \underline{\underline{\text{Ans}}}$$

From 12.3.15, $u(x,t) = F(x)G(t)$, where

$$F(x) = A\cos\beta x + B\sin\beta x + C\cosh\beta x + D\sinh\beta x$$

$$F'(x) = -\beta A\sin\beta x + \beta B\cos\beta x + \beta C\sinh\beta x + \beta D\cosh\beta x$$

For $F(x)$ to be a solution of the PDE, $u(x,t) = F(x)G(t)$, $F(x)$ must satisfies the BCs for $u(x,t)$, i.e.,

$$F(0) = 0, \qquad F(L) = 0,$$

$$F'(0) = 0, \qquad F'(L) = 0.$$

$$F(0) = A\cancel{\cos}^{\,1}(0) + B\cancel{\sin}^{\,0}(0) + C\cancel{\cosh}^{\,1}(0) + D\cancel{\sinh}^{\,0}(0)$$

$$0 = A + C \Rightarrow C = -A \quad *$$

$$F'(0) = -\beta A \overset{0}{\cancel{\sin(0)}} + \beta B \overset{1}{\cancel{\cos(0)}} + \beta C \overset{0}{\cancel{\sinh(0)}} + \beta D \overset{1}{\cancel{\cosh(0)}}$$

$$0 = \beta B + \beta D \Rightarrow D = -B \quad ; \beta \neq 0$$

Now,

$$F(x) = A(\cos\beta x - \cosh\beta x) + B(\sin\beta x - \sinh\beta x)$$

$$F'(x) = -\beta A(\sin\beta x + \sinh\beta x) + \beta B(\cos\beta x - \cosh\beta x)$$

$$F(L) = A(\cos\beta L - \cosh\beta L) + B(\sin\beta L - \sinh\beta L) = 0 \quad -(1)$$

$$F'(L) = -\beta A(\sin\beta L + \sinh\beta L) + \beta B(\cos\beta L - \cosh\beta L) = 0 \quad -(2)$$

Write (1) & (2) in a matrix form and divided through (2) with $\beta \to \beta \neq 0$,

$$\begin{bmatrix} \cos\beta L - \cosh\beta L & \sin\beta L - \sinh\beta L \\ -\sin\beta L - \sinh\beta L & \cos\beta L - \cosh\beta L \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

For non-trivial solution, i.e. $A \neq 0, B \neq 0$,

$$\begin{vmatrix} \cos\beta L - \cosh\beta L & \sin\beta L - \sinh\beta L \\ -\sin\beta L - \sinh\beta L & \cos\beta L - \cosh\beta L \end{vmatrix} = 0$$

$$(\cos\beta L - \cosh\beta L)^2 + (\sin\beta L - \sinh\beta L)(\sin\beta L + \sinh\beta L) = 0$$

$$\cos^2\beta L - 2\cos\beta L\cosh\beta L + \cosh^2\beta L + \sin^2\beta L - \sinh^2\beta L = 0$$

$$\underbrace{(\cos^2\beta L + \sin^2\beta L)}_{=1} + \underbrace{(\cosh^2\beta L - \sinh^2\beta L)}_{=1} - 2\cos\beta L\cosh\beta L = 0$$

$$\underset{1}{\cancel{2}} - \underset{1}{\cancel{2}}\cos\beta L\cosh\beta L = 0$$

$$\cos\beta L\cosh\beta L = 1 \quad -(22)$$

Thus, F satisfies the BCs if βL satisfies (22). <u>Ans</u>

See graph of the approximate solutions of (22)

---

part 3 :

Note, Since $\det m = 0$ (for nontrivial solution) the rows of the matrix m are linearly dependent

To solve for A & B, take 1st row :

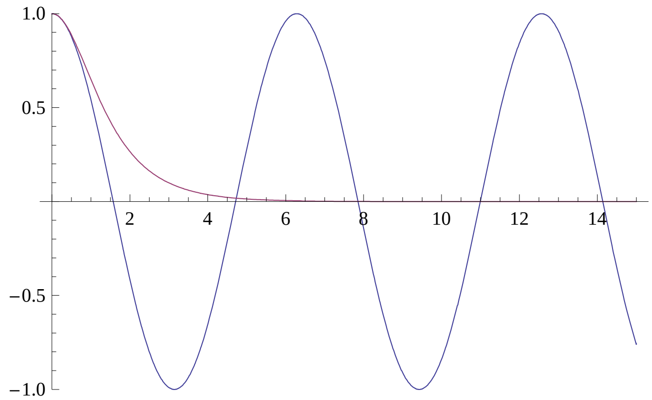
$$A(\cos\beta L - \cosh\beta L) + B(\sin\beta L - \sinh\beta L) = 0$$

$$\Rightarrow B = A \cdot \frac{\cos\beta L - \cosh\beta L}{\sinh\beta L - \sin\beta L}$$

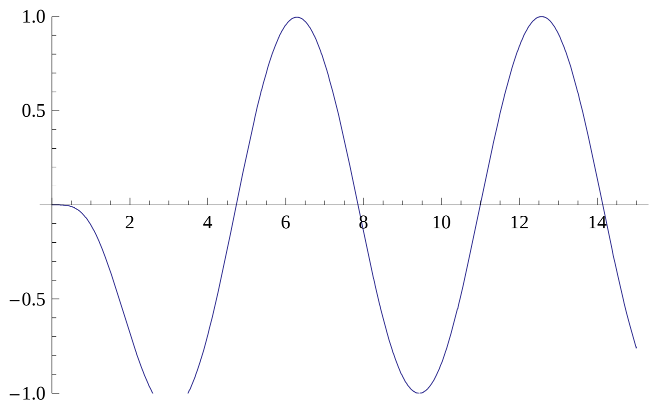
with A arbitrary

Let's choose $A = 1$ $\Rightarrow$

```
Plot[{Cos[x], 1 / Cosh[x]}, {x, 0, 15}, PlotRange → {-1, 1}]
```




```
Plot[Cos[x] - 1 / Cosh[x], {x, 0, 15}, PlotRange → {-1, 1}]
```




```
FindRoot[Cos[x] - 1 / Cosh[x], {x, 4.5, 5.0}]

{x → 4.73004}

FindRoot[Cos[x] - 1 / Cosh[x], {x, 7.5, 8.0}]

{x → 7.8532}

FindRoot[Cos[x] - 1 / Cosh[x], {x, 10.5, 11.5}]

{x → 10.9956}

roots = {4.73, 7.8532, 11.00}

{4.73, 7.8532, 11.}
```

Recall: f[x_, b_] := A* (Cos[b*x] - Cosh[b*x]) + B *(Sin[b*x] - Sinh[b*x])

where B = A* (Cos[b] - Cosh[b])/(Sinh[b] - Sin[b]), with A arbitrary (A=1):

```
f[x_, b_] :=
 Cos[b * x] - Cosh[b * x]  +  (Cos[b] - Cosh[b]) / (Sinh[b] - Sin[b]) * (Sin[b * x] - Sinh[b * x])
```

```
shape = f[x, #] & /@ roots
```

```
{Cos[4.73 x] - Cosh[4.73 x] - 0.982502 (Sin[4.73 x] - Sinh[4.73 x]),
 Cos[7.8532 x] - Cosh[7.8532 x] - 1.00078 (Sin[7.8532 x] - Sinh[7.8532 x]),
 Cos[11. x] - Cosh[11. x] - 0.999966 (Sin[11. x] - Sinh[11. x])}
```

```
Table[Integrate[shape[[i]] * shape[[j]], {x, 0, 1}], {i, 1, 3}, {j, 1, 3}] // Chop
```

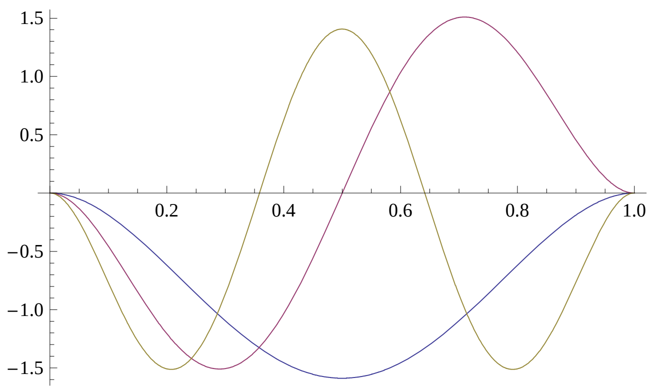
```
{{1.00001, -0.0000134101, 0.000312379},
 {-0.0000134101, 1., -0.00110137}, {0.000312379, -0.00110137, 0.999599}}
```

Note we found (approximately) a unit matrix which confirms Fn are orthogonal

```
MatrixForm[%]
```

$$\begin{pmatrix} 1.00001 & -0.0000134101 & 0.000312379 \\ -0.0000134101 & 1. & -0.00110137 \\ 0.000312379 & -0.00110137 & 0.999599 \end{pmatrix}$$

```
Plot[%, {x, 0, 1}]
```



Let' s find Cn :

```
Table[Integrate[shape[[i]] * x^2 * (1 - x)^2, {x, 0, 1}], {i, 1, 3}]
```

$\{-0.0398366 + 0.\ i,\ -3.81899 \times 10^{-8} + 0.\ i,\ -0.000609261 + 0.\ i\}$

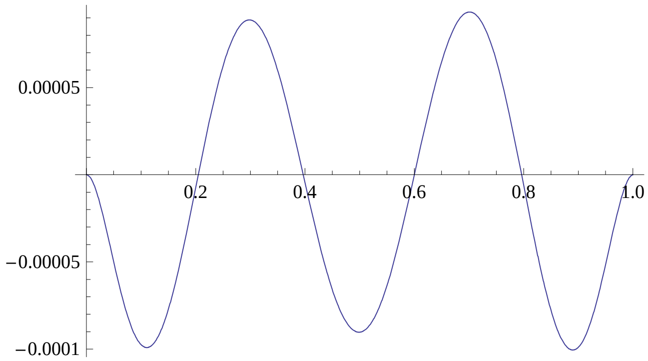
```
Chop[%]
```

$\{-0.0398366,\ -3.81899 \times 10^{-8},\ -0.000609261\}$

```
%.shape
```

$-0.0398366\,(\mathrm{Cos}[4.73\,x] - \mathrm{Cosh}[4.73\,x] - 0.982502\,(\mathrm{Sin}[4.73\,x] - \mathrm{Sinh}[4.73\,x])) - 3.81899 \times 10^{-8}\,(\mathrm{Cos}[7.8532\,x] - \mathrm{Cosh}[7.8532\,x] - 1.00078\,(\mathrm{Sin}[7.8532\,x] - \mathrm{Sinh}[7.8532\,x])) - 0.000609261\,(\mathrm{Cos}[11.\,x] - \mathrm{Cosh}[11.\,x] - 0.999966\,(\mathrm{Sin}[11.\,x] - \mathrm{Sinh}[11.\,x]))$

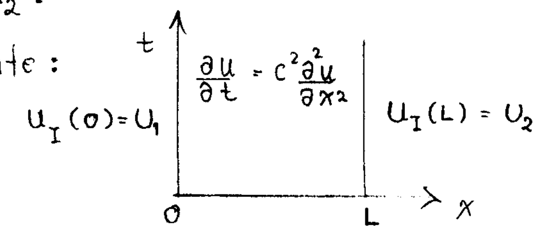
```
Plot[% - x^2 * (1 - x)^2, {x, 0, 1}]
```



12.5.10 If the end $x=0$ and $x=L$ of the bar in the text are kept at constant temperature U_1 and U_2 respectively what is the temperature $u_I(x)$ in the bar after a long time $(t\to\infty)$.

Sol^n Guess : As $t\to\infty$ $u_I(x)$ has a linear profile with $u_I(0)=U_1$ and $u_I(L)=U_2$.

Calculate :



1D Heat Eq^n. $$\frac{\partial u}{\partial t}=c^2\frac{\partial^2 u}{\partial x^2} \qquad -(1)$$

As $t\to\infty$, the temperature distribution in the bar approaches a steady-state conditions, i.e., $\frac{\partial u}{\partial t}=0$. Thus, Eq. (1) becomes

$$\frac{\partial^2 u}{\partial x^2} = 0$$

$$u(x) = Ax+B \qquad ; \quad A, B \text{ are constants.}$$

$$u(0) = U_1 = B \quad *$$

$$u(L) = U_2 = AL+U_1$$

$$A = \frac{U_2-U_1}{L}$$

$$\therefore \quad u_I(x) = \left(\frac{U_2-U_1}{L}\right)x + U_1$$

<u>Ans</u>

12.5.11 In Prob 10, find the temperature at any time

Sol^n Let $$u(x,t) = U_I(x)+U_T(x,t) \qquad -(2)$$

where $U_I(x)$ is the steady-state solution in Prob. 10,

$U_T(x,t)$ is a particular solution of Eq. (1) in Prob. 10 that takes care for the transient temperature.

Since $U_I(x)$ satisfies Eq. 1 as $t \to \infty$, $U_T(x,\infty) = 0$.

Also, $U_I(0) = U_1$ and $U_I(L) = U_2$, thus

$$U_{II}(0,t) = 0, \quad U_{II}(L,t) = 0.$$

Now, Solving

$$\frac{\partial}{\partial t}(U_I + U_{II}) = c^2 \frac{\partial^2}{\partial x^2}(U_I + U_{II})$$

$$\cancel{\frac{\partial U_I}{\partial t}}^{\,0} + \frac{\partial U_{II}}{\partial t} = c^2 \cancel{\frac{\partial^2 U_I}{\partial x^2}}^{\,0} + c^2 \frac{\partial^2 U_{II}}{\partial x^2}$$

$$\frac{\partial U_{II}}{\partial t} = c^2 \frac{\partial^2 U_{II}}{\partial x^2} \qquad \text{— (3)}$$

with $\quad (0,t) = 0, \quad (\;\; (L,t) = 0$

Eq. (3) with the homogeneous BC's can be solved as shown in the textbook, p. 553. The solution is in the form:

$$U_{II}(x,t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L} e^{-\lambda_n^2 t} \quad ; \quad \lambda_n = \frac{cn\pi}{L}$$

If $u(x,0) = f(x)$, Then

$$U_{II}(x,0) = u(x,0) - U_I(x) = f(x) - U_I(x)$$

Use Fourier sine series to determine B_n,

$$B_n = \frac{2}{L}\int_0^L [f(x) - U_I(x)] \sin \frac{n\pi x}{L}\, dx$$

$$\therefore \quad u(x,t) = (U_2 - U_1)\frac{x}{L} + U_1 + \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L} e^{-\left(\frac{cn\pi}{L}\right)^2 t}. \qquad \underline{\underline{\text{Ans}}}$$

where B_n is defined above

Bar- under adiabatic conditions

Find the temperature in Prob. 13 with $L = \pi$, $c = 1$ and

12.5.17 $\quad f(x) = \pi^2 - x^2$

Sol^n From Prob. 12.5.13,

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L} e^{-(cn\pi/L)^2 t}$$

For $L = \pi$, $c = 1$,

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n \cos(nx)\, e^{-n^2 t}$$

From 11.3 (p. 491) and $f(x) = \pi^2 - x^2$

$$A_0 = \frac{1}{L}\int_0^L f(x)\,dx = \frac{1}{\pi}\int_0^{\pi} (\pi^2 - x^2)\,dx$$

$$= \frac{1}{\pi}\left[\pi^2 x - \frac{x^3}{3}\right]_0^{\pi} = \frac{2\pi^2}{3}$$

$$A_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx = \frac{2}{\pi}\int_0^{\pi} (\pi^2 - x^2)\cos(nx)\,dx$$

$$= \frac{2}{\pi}\left\{\left[\frac{\pi^2}{n}\sin(nx)\right]_{x=0}^{x=\pi} - \frac{1}{n^3}\left[(nx)^2\sin(nx) + 2nx\cos(nx) - 2\sin(nx)\right]_{x=0}^{x=\pi}\right\}$$

$$= -\frac{4\cos(n\pi)}{n^2} = \frac{4(-1)^{n+1}}{n^2}$$

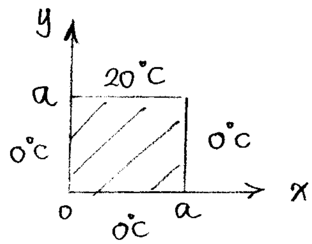
$$u(x,t) = \frac{2\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}\cos(nx)\,e^{-n^2 t}$$

$$= \frac{2\pi^2}{3} + 4\left(\cos x\, e^{-t} - \frac{1}{4}\cos(2x)\,e^{-4t} + \frac{1}{9}\cos(3x)\,e^{-9t} + \dots\right) \quad \underline{\underline{\text{Ans}}}$$

See plot at the end

<u>12.5.31</u> (Heat flow in a plate). The faces of the thin square plate with side $a = 24$ are perfectly insulated. The upper side is kept at $20\,^\circ$C and the other side are kept at $0\,^\circ$C. Find the steady-state temperature $u(x,y)$ in the plate.

<u>Sol$^{\underline{n}}$</u>



$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

This problem and boundary conditions are similar to what is shown on p. 558, with $f(x) = 20$ on the upper side of the plate and $a = b$. Thus, the solution of this problem is in the form of Eq. (17) p. 560, i.e.,

$$u(x,y) = \sum_{n=1}^{\infty} A_n^* \sin\frac{n\pi x}{a}\sinh\frac{n\pi y}{a}$$

$$= \sum_{n=1}^{\infty} A_n^* \sin\frac{n\pi x}{24}\sinh\frac{n\pi y}{24}$$

where $$A_n^* = \frac{\cancel{2}^{\,1}}{\underset{\scriptstyle 12\ 3}{\cancel{24}}\sinh(n\pi)}\int_0^{24} \overset{5}{\cancel{20}}\sin\frac{n\pi x}{24}\,dx$$

$$= \frac{5}{3\sinh(n\pi)}\left[-\frac{24}{n\pi}\cos\frac{n\pi x}{24}\right]_{x=0}^{x=24}$$

$$= \frac{5}{3\sinh(n\pi)}\left[-\frac{24}{n\pi}\cos(n\pi)+\frac{24}{n\pi}\right]$$

$$= \frac{40\,(1-(-1)^n)}{n\pi\sinh(n\pi)}$$

$$= \begin{cases} 0 & \text{for } n \text{ even} \\ \dfrac{80}{n\pi\sinh(n\pi)} & \text{for } n \text{ odd} \end{cases}$$

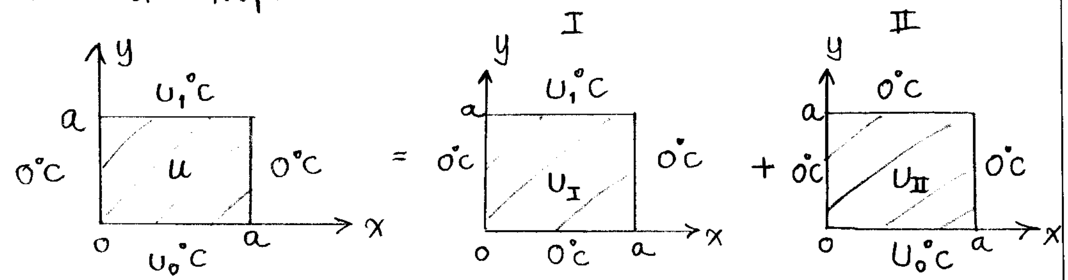
$$= \frac{80}{\pi(2n-1)\sinh(2n-1)\pi} \qquad n = 1, 2, 3, \dots$$

Ans $$u(x,y) = \frac{80}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)\sinh(2n-1)\pi}\cdot\sin\frac{(2n-1)\pi x}{24}\cdot\sinh\frac{(2n-1)\pi y}{24}$$

See plot at the end

12.5.32 Find the steady-state temperature in the plate in Prob. 31 if the lower side is kept at U_0°C, the upper side at U_1°C, and the other sides are kept at 0°C.

Sol^n



Let $$u(x,y) = u_I(x,y) + u_{II}(x,y)$$

Steady-state 2D heat eq. : $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$\frac{\partial^2 (u_I + u_{II})}{\partial x^2} + \frac{\partial^2 (u_I + u_{II})}{\partial y^2} = 0$$

$$\frac{\partial^2 u_I}{\partial x^2} + \frac{\partial^2 u_I}{\partial y^2} + \frac{\partial^2}{\partial x^2} u_{II} + \frac{\partial^2}{\partial y^2} u_{II} = 0$$

$$\frac{\partial^2 u_I}{\partial x^2} + \frac{\partial^2 u_I}{\partial y^2} = 0 \quad \text{and} \quad \frac{\partial^2 u_{II}}{\partial x^2} + \frac{\partial^2 u_{II}}{\partial y^2} = 0$$

$u_I(0,y) = 0$	$u_{II}(0,y) = 0$
$u_I(a,y) = 0$	$u_{II}(a,y) = 0$
$u_I(x,0) = 0$	$u_{II}(x,0) = U_0$
$u_I(x,a) = U_1$	$u_{II}(x,a) = 0$

Part I is similar to prob in fig. 293 (p. 558). Thus, the solution follows Eq. (17) p. 560, i.e. with $a = 24$ and $f(x) = U_1$,

$$u_I(x,y) = \sum_{n=1}^{\infty} A_n^* \sin \frac{n\pi x}{24} \cdot \sinh \frac{n\pi y}{24}$$

$$A_n^* = \frac{2}{24 \sinh(n\pi)} \int_0^{24} U_1 \sin \frac{n\pi x}{24}\, dx$$

$$= \frac{2U_1}{\cancel{24} \sinh(n\pi)} \left[-\frac{\cancel{24}}{n\pi} \cos \frac{n\pi x}{24} \right]_{x=0}^{x=24}$$

$$= \frac{2U_1}{n\pi \cdot \sinh(n\pi)} \left[-\cos(n\pi) + 1 \right]$$

$$= \frac{2U_1}{n\pi \sinh(n\pi)} (1 - (-1)^n) = \begin{cases} 0 & ; n \text{ even} \\ \dfrac{4U_1}{n\pi \sinh(n\pi)} & ; n \text{ odd} \end{cases}$$

$$= \frac{4U_1}{(2n-1)\pi \sinh(2n-1)\pi} \; ; n = 1, 2, 3, \ldots$$

$$u_I(x,y) = \frac{4U_1}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)\sinh(2n-1)\pi} \cdot \sin \frac{(2n-1)\pi x}{24} \cdot \sinh \frac{(2n-1)\pi y}{24} \; *$$

Part II; for $u_{II}(x,y) = F(x)\, G(a-y)$, following the same derivation on p. 559, we get $F_n(x) = \sin \frac{n\pi x}{a} = \sin \frac{n\pi x}{24}$

and $$G(a-y) = A_n e^{n\pi(a-y)/a} + B_n e^{-n\pi(a-y)/a}$$

Note that : for $u_{II}(x,y) = F(x)\,G(a-y)$

$$\frac{\partial u_{II}}{\partial y} = -F(x)\frac{dG}{dy} \; , \quad \frac{\partial^2 u_{II}}{\partial y^2} = F(x)\frac{d^2G}{dy^2}$$

$$G(a-y) = A_n e^{n\pi(a-y)/a} + B_n e^{-n\pi(a-y)/a}$$

$$y=a; \quad G(0) = 0 = A_n + B_n \Rightarrow A_n = -B_n$$

$$G(a-y) = A_n\left(e^{n\pi(a-y)/a} - e^{-n\pi(a-y)/a}\right)$$

$$= 2A_n \sinh\frac{n\pi(a-y)}{a} = A_n^* \sinh\frac{n\pi(a-y)}{a} \quad ; A_n^* = 2A_n$$

$$u_{II\,n}(x,y) = A_n^* \sin\frac{n\pi x}{a}\sinh\frac{n\pi(a-y)}{a}$$

$$u_{II}(x,y) = \sum_{n=1}^{\infty} A_n^* \sin\frac{n\pi x}{a}\cdot\sinh\frac{n\pi(a-y)}{a}$$

B.C. ; $\quad u_{II}(x,0) = U_0 = \sum_{n=1}^{\infty} A_n^* \sinh(n\pi)\cdot\sin\frac{n\pi x}{a}$

Fourier sine series; $\quad A_n^*\sinh(n\pi) = \frac{2}{a}\int_0^a U_0 \sin\frac{n\pi x}{a}\,dx$

$$A_n^* = \frac{2U_0}{a\sinh(n\pi)}\int_0^a \sin\frac{n\pi x}{a}\,dx$$

$$= \frac{2U_0}{\cancel{a}\sinh(n\pi)}\left[-\frac{\cancel{a}}{n\pi}\cos\frac{n\pi x}{a}\right]_{x=0}^{x=a}$$

$$= \frac{2U_0}{n\pi\sinh(n\pi)}\left[-\cos(n\pi)+1\right] = \begin{cases} 0 & ; n \text{ even} \\ \dfrac{4U_0}{n\pi\sinh(n\pi)} & ; n \text{ odd} \end{cases}$$

$$= \frac{4U_0}{(2n-1)\pi\sinh(2n-1)\pi} \quad ; n = 1,2,3,\ldots$$

$$u_{II}(x,y) = \frac{4U_0}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)\sinh[(2n-1)\pi]}\cdot\sin\left[\frac{(2n-1)\pi x}{a}\right]\cdot\sinh\left[\frac{(2n-1)\pi(a-y)}{a}\right]$$

with $a = 24$,

$u(x,y) = u_I(x,y) + u_{II}(x,y)$ Ans

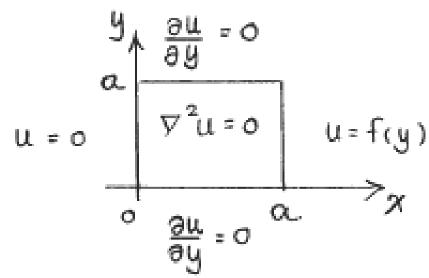
where
$$u_I(x,y) = \frac{4U_1}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)}\cdot\sin\left[\frac{(2n-1)\pi x}{24}\right]\cdot\frac{\sinh[(2n-1)\pi y/24]}{\sinh[(2n-1)\pi]}$$

and
$$u_{II}(x,y) = \frac{4U_0}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)}\cdot\sin\left[\frac{(2n-1)\pi x}{24}\right]\cdot\frac{\sinh[(2n-1)\pi(1-y/24)]}{\sinh[(2n-1)\pi]}$$

See plot at the end

12.5.33 Find the steady-state temperature in the plate in 12.5.31 with the upper and lower sides perfectly insulated, the left side kept at 0°C and the right side kept at $f(y)$°C. Also find the solution for $f(y) = 20$.

Sol^{n}



2D steady-state heat equation in the plate : $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ —(1)

Let $u(x,y) = F(x)\,G(y)$, Eq.(1) becomes,

$$G\frac{d^2F}{dx^2} + F\frac{d^2G}{dy^2} = 0$$

$$\frac{1}{F}\frac{d^2F}{dx^2} = \frac{-1}{G}\frac{d^2G}{dy^2} = k^2 \quad \text{—(2)}; \quad k = \text{const.}$$

B.C.'s $\quad \frac{\partial u}{\partial y} = F(x)\frac{dG}{dy}$

$$\frac{\partial u}{\partial y}(x,0) = 0 = F(x)\frac{dG}{dy}(0) \Rightarrow \frac{dG}{dy}(0) = 0$$

$$\frac{\partial u}{\partial y}(x,a) = 0 = F(x)\frac{dG}{dy}(a) \Rightarrow \frac{dG}{dy}(a) = 0$$

$$u(0,y) = 0 = F(0)\,G(y) \Rightarrow F(0) = 0$$

From (2); $\quad \frac{d^2G}{dy^2} + k^2 G = 0$

$$G(y) = A\cos(ky) + B\sin(ky)$$

$$\frac{dG}{dy} = -Ak\sin(ky) + Bk\cos(ky)$$

For $k \neq 0$

Apply B.C.; $\quad \frac{dG}{dy}(0) = 0 = Bk \Rightarrow \underline{B = 0}$

$$\frac{dG}{dy}(a) = 0 = -Ak\sin(ka) \Rightarrow \underline{k = \frac{n\pi}{a}}$$

$$\therefore \quad G_n(y) = A_n \cos\left(\frac{n\pi}{a} y\right)$$

From (2) ; $$\frac{d^2F}{dx^2} - k^2F = 0$$

$$F(x) = \tilde{C}\sinh(kx) + \tilde{D}\cosh(kx)$$

Apply BC. ; $$F(0) = \underline{0 = \tilde{D}}$$

$$\therefore \quad F_n(x) = \tilde{C}_n \sinh\left(\frac{n\pi}{a}x\right) \quad ; \quad k_n = \frac{n\pi}{a}$$

$$u_n(x,y) = F_n(x)G_n(y)$$

(3) — $$u(x,y) = \sum_{n=1}^{\infty} u_n(x,y) = \sum_{n=1}^{\infty} C_n \sinh\left(\frac{n\pi x}{a}\right)\cos\left(\frac{n\pi y}{a}\right) \quad ; C_n = A_n\tilde{C}_n$$

For k=0

$$\frac{d^2G_0}{dy^2} = 0$$

$$G_0(y) = A_1 y + A_2 \quad , \quad \frac{dG}{dy} = A_1 \quad ; \ A_1, A_2 \equiv \text{constants}$$

B.Cs; $$\frac{dG_0(0)}{dy} = \underline{0 = A_1} = \frac{dG}{dy}(a)$$

$$G_0(y) = A_2$$

and $$\frac{d^2F_0}{dx^2} = 0$$

$$F_0(x) = B_1 x + B_2 \quad ; \quad B_1, B_2 \equiv \text{constants}$$

BC. $$F_0(0) = \underline{0 = B_2} \Rightarrow F_0(x) = B_1 x$$

$$u_0(x,y) = G_0F_0 = C_0 x \quad ; \quad C_0 = B_1A_2 \quad \text{— (4)}$$

Combine (3) + (4);

$$u(x,y) = C_0 x + \sum_{n=1}^{\infty} C_n \sinh\left(\frac{n\pi x}{a}\right)\cos\left(\frac{n\pi y}{a}\right)$$

Apply B.C.; $u(a,y) = f(y) = C_0 a + \sum_{n=1}^{\infty} C_n \sinh(n\pi)\cos\left(\frac{n\pi y}{a}\right)$

Fourier cosine series; (see p. 491)

$$C_0 a = \frac{1}{a}\int_0^a f(y)\,dy$$

$$C_0 = \frac{1}{a^2}\int_0^a f(y)\,dy$$

$$C_n \sinh(n\pi) = \frac{2}{a}\int_0^a f(y)\cos\left(\frac{n\pi y}{a}\right)dy$$

$$C_n = \frac{2}{a\sinh(n\pi)}\int_0^a f(y)\cos\left(\frac{n\pi y}{a}\right)dy$$

For $a = 24$;

$$u(x,y) = a_0 x + \sum_{n=1}^{\infty} a_n \frac{\sinh(n\pi x/24)}{\sinh(n\pi)}\cos\left(\frac{n\pi y}{24}\right)$$

where $a_0 = \frac{1}{24^2}\int_0^{24} f(y)\,dy$

$$a_n = \frac{1}{12}\int_0^{24} f(y)\cos\left(\frac{n\pi y}{24}\right)dy$$

Ans

Problem 12.5.17

```
In[1]:= A0 = 1 / Pi * Integrate[Pi^2 - x^2, {x, 0, Pi}]
```

In[29]:= $\frac{2\pi^2}{3}$

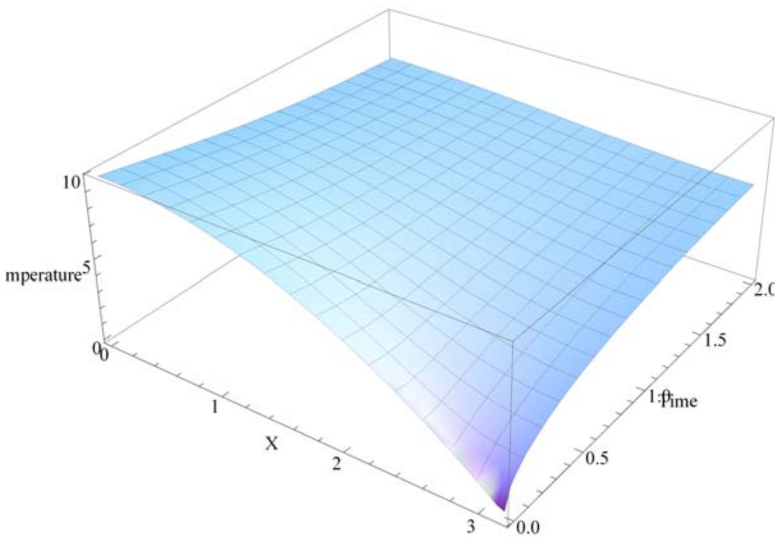
```
In[4]:= An = 2 / Pi * Integrate[(Pi^2 - x^2) * Cos[n x], {x, 0, Pi}, Assumptions → n ∈ Integers]
```

Out[4]= $-\frac{4(-1)^n}{n^2}$

```
In[34]:= u[x_, t_] := A0 + Sum[An * Cos[n x] Exp[-n^2 * t], {n, 1, 30}]

In[36]:= Plot3D[u[x, t], {x, 0, Pi}, {t, 0, 2}, AxesLabel → {"X", "Time", "Temperature"}]
```

Out[36]=



Problem 12.5.31

```
In[5]:= Anstar[n_] :=
         2 / 24 / Sinh[n Pi] Integrate[20 Sin[n Pi x / 24], {x, 0, 24}, Assumptions → n ∈ Integers]

In[6]:= Anstar[n]
```

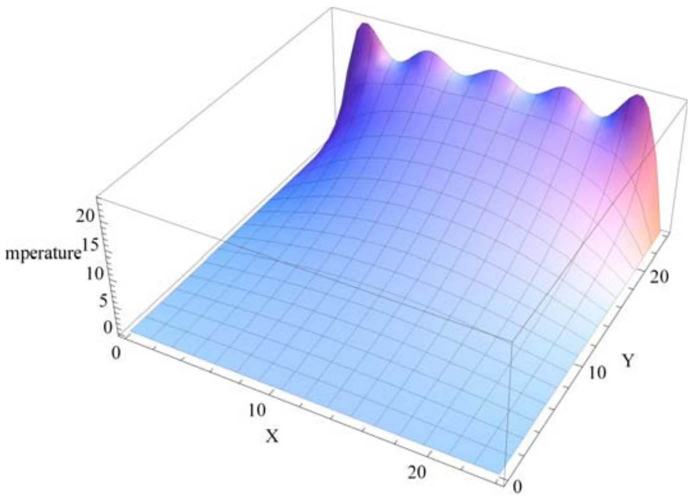
Out[6]= $-\frac{40\left(-1+(-1)^n\right)\,\mathrm{Csch}[n\,\pi]}{n\,\pi}$

```
In[10]:= u[x_, y_] := Sum[Anstar[n] * Sin[n Pi x / 24] * Sinh[n Pi y / 24], {n, 1, 10}]
```

In[13]:= **u[x, y]**

Out[13]=
$$\frac{80\,\mathrm{Csch}[\pi]\,\mathrm{Sin}\left[\frac{\pi x}{24}\right]\,\mathrm{Sinh}\left[\frac{\pi y}{24}\right]}{\pi}+\frac{80\,\mathrm{Csch}[3\pi]\,\mathrm{Sin}\left[\frac{\pi x}{8}\right]\,\mathrm{Sinh}\left[\frac{\pi y}{8}\right]}{3\pi}+\frac{16\,\mathrm{Csch}[5\pi]\,\mathrm{Sin}\left[\frac{5\pi x}{24}\right]\,\mathrm{Sinh}\left[\frac{5\pi y}{24}\right]}{\pi}+\frac{80\,\mathrm{Csch}[7\pi]\,\mathrm{Sin}\left[\frac{7\pi x}{24}\right]\,\mathrm{Sinh}\left[\frac{7\pi y}{24}\right]}{7\pi}+\frac{80\,\mathrm{Csch}[9\pi]\,\mathrm{Sin}\left[\frac{3\pi x}{8}\right]\,\mathrm{Sinh}\left[\frac{3\pi y}{8}\right]}{9\pi}$$

In[12]:= **Plot3D[u[x, y], {x, 0, 24}, {y, 0, 24}, AxesLabel → {"X", "Y", "Temperature"}]**

Out[12]=


Problem 12.5.32

In[32]:= **U0 = 30**

Out[32]= 30

In[33]:= **U1 = 50**

Out[33]= 50

In[34]:= **Anstar1[n_] :=**
2 / 24 / Sinh[n Pi] Integrate[U1 Sin[n Pi x / 24], {x, 0, 24}, Assumptions → n ∈ Integers]

In[35]:= **Anstar1[n]**

Out[35]=
$$-\frac{100\left(-1+(-1)^{n}\right)\,\mathrm{Csch}[n\,\pi]}{n\,\pi}$$

In[36]:= **uI[x_, y_] := Sum[Anstar1[n] * Sin[n Pi x / 24] * Sinh[n Pi y / 24], {n, 1, 10}]**

In[37]:= **a = 24**

Out[37]= 24

In[38]:= **Anstar2[n_] := 2 / a / Sinh[n Pi] Integrate[U0 Sin[n Pi x / a], {x, 0, a}, Assumptions → n ∈ Integers]**

In[39]:= `Anstar2[n]`

Out[39]= $-\frac{60\left(-1+(-1)^{n}\right)\mathrm{Csch}[n\,\pi]}{n\,\pi}$

In[40]:= `uII[x_, y_] := Sum[Anstar2[n] * Sin[n Pi x / a] * Sinh[n Pi (a - y) / a], {n, 1, 10}]`

In[41]:= `utot[x_, y_] := uI[x, y] + uII[x, y]`

In[44]:= `utot[x, y]`

Out[44]=
$$\frac{120\,\mathrm{Csch}[\pi]\,\mathrm{Sin}\left[\frac{\pi x}{24}\right]\mathrm{Sinh}\left[\frac{1}{24}\pi(24-y)\right]}{\pi}+\frac{40\,\mathrm{Csch}[3\pi]\,\mathrm{Sin}\left[\frac{\pi x}{8}\right]\mathrm{Sinh}\left[\frac{1}{8}\pi(24-y)\right]}{\pi}+$$
$$\frac{24\,\mathrm{Csch}[5\pi]\,\mathrm{Sin}\left[\frac{5\pi x}{24}\right]\mathrm{Sinh}\left[\frac{5}{24}\pi(24-y)\right]}{\pi}+\frac{120\,\mathrm{Csch}[7\pi]\,\mathrm{Sin}\left[\frac{7\pi x}{24}\right]\mathrm{Sinh}\left[\frac{7}{24}\pi(24-y)\right]}{7\pi}+$$
$$\frac{40\,\mathrm{Csch}[9\pi]\,\mathrm{Sin}\left[\frac{3\pi x}{8}\right]\mathrm{Sinh}\left[\frac{3}{8}\pi(24-y)\right]}{3\pi}+\frac{200\,\mathrm{Csch}[\pi]\,\mathrm{Sin}\left[\frac{\pi x}{24}\right]\mathrm{Sinh}\left[\frac{\pi y}{24}\right]}{\pi}+$$
$$\frac{200\,\mathrm{Csch}[3\pi]\,\mathrm{Sin}\left[\frac{\pi x}{8}\right]\mathrm{Sinh}\left[\frac{\pi y}{8}\right]}{3\pi}+\frac{40\,\mathrm{Csch}[5\pi]\,\mathrm{Sin}\left[\frac{5\pi x}{24}\right]\mathrm{Sinh}\left[\frac{5\pi y}{24}\right]}{\pi}+$$
$$\frac{200\,\mathrm{Csch}[7\pi]\,\mathrm{Sin}\left[\frac{7\pi x}{24}\right]\mathrm{Sinh}\left[\frac{7\pi y}{24}\right]}{7\pi}+\frac{200\,\mathrm{Csch}[9\pi]\,\mathrm{Sin}\left[\frac{3\pi x}{8}\right]\mathrm{Sinh}\left[\frac{3\pi y}{8}\right]}{9\pi}$$

In[43]:= `Plot3D[utot[x, y], {x, 0, 24}, {y, 0, a}, AxesLabel → {"X", "Y", "Temperature"}]`

Out[43]=