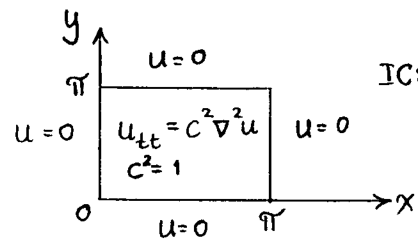
ME 565 – HW 4 – Wi 09

<u>Deflection</u>

Find the deflection $u(x,y,t)$ of the square membrane of side π and $c^2=1$ if the initial velocity is 0 and the initial deflection is

12.8.11 $\quad k\sin 2x \sin 5y$

Sol^n



ICs : $u(x,y,0) = k\sin 2x \sin 5y$

$u_t(x,y,0) = 0$

The given conditions are similar to what derived in sec. 12.8, thus, from Eq.(14) p. 576,

$$u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(B_{mn}\cos\lambda_{mn}t + B^*_{mn}\sin\lambda_{mn}t\right)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

For this given problem with $u_t(x,y,0)=0 \Rightarrow B^*_{mn}=0$ (Eq.(19) p. 577)

and $a=\pi$, $b=\pi$, $c^2=1$, $u(x,y,0) = f(x,y) = k\sin 2x\sin 5y$

$$u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{mn}\cos(\lambda_{mn}t)\sin(mx)\sin(ny) \quad , \quad \lambda_{mn} = \sqrt{n^2+m^2}$$

where $$B_{mn} = \frac{4}{\pi^2}\int_0^{\pi}\int_0^{\pi} k\sin(2x)\sin(5y)\sin(mx)\sin(ny)\,dx\,dy$$

From Mathematica,

$$B_{mn} = \begin{cases} \dfrac{4k}{\pi^2}\left[\dfrac{-10\sin(m\pi)\sin(n\pi)}{(m^2-4)(n^2-25)}\right] & , \quad m\neq 2,\ n\neq 5 \\[2ex] \dfrac{4k}{\pi^2}\left[\dfrac{\pi^2}{4}\right] & , \quad m=2,\ n=5 \end{cases}$$

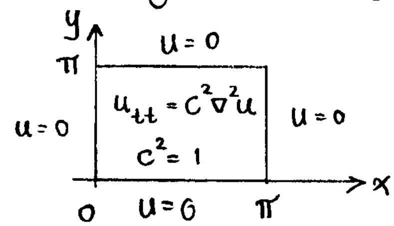
$$B_{mn} = \begin{cases} 0 & ; \quad m\neq 2,\ n\neq 5 \quad \Rightarrow \sin(\ell\pi)=0 \text{ for } \ell \equiv \text{integer} \\ k & ; \quad m=2,\ n=5 \end{cases}$$

<u>Ans</u> $\quad u(x,y,t) = k\cos(\sqrt{29}\,t)\sin(2x)\sin(5y) \quad ; \quad \lambda_{mn} = \sqrt{5^2+2^2} = \sqrt{29}$

See plot at the end

12.8.13 $0.1xy(\pi-x)(\pi-y)$

Sol^n



IC's; $u(x,y,0) = F(x,y) = 0.1xy(\pi-x)(\pi-y)$

$u_t(x,y,0) = 0$

Similar to 12.8.11, with the given conditions

$$u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{mn}\cos(\lambda_{mn}t)\sin(mx)\sin(ny) \quad ; \quad \lambda_{mn} = \sqrt{m^2+n^2}$$

where $$B_{mn} = \frac{4}{\pi^2}\int_0^{\pi}\int_0^{\pi} 0.1\,xy(\pi-x)(\pi-y)\sin(mx)\sin(ny)\,dx\,dy$$

By using Mathematica,

$$B_{mn} = \frac{0.4}{\pi^2}\left[\frac{4}{m^3n^3}(\cos(m\pi)-1)(\cos(n\pi)-1)\right]$$

$$B_{mn} = \begin{cases} 0 & ; \ m \text{ or } n \text{ is even} \\ \dfrac{6.4}{\pi^2 m^3 n^3} & ; \ m \text{ and } n \text{ are odd} \end{cases}$$

$$u(x,y,t) = \sum_{\substack{m=1 \\ m,n \text{ odd}}}^{\infty}\sum_{n=1}^{\infty} \frac{6.4}{\pi^2}\left(\frac{1}{m^3n^3}\right)\cos\left(\sqrt{m^2+n^2}\,t\right)\sin(mx)\sin(ny). \quad \text{Ans}$$

See plot at the end

12.9.6

a) Show that $u_n = r^n\cos n\theta$, $u_n = r^n \sin n\theta$, $n = 0, 1, 2, \dots$ are solutions of $\nabla^2 u = 0$. (What would u_n be in Cartesian coordinate?)

Sol^n From eqn. (5) p. 580,

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} \quad \text{—(1)}$$

For $u_n = r^n\cos n\theta$;

$$\frac{\partial u_n}{\partial r} = nr^{n-1}\cos n\theta \ , \quad \frac{\partial^2 u_n}{\partial r^2} = n(n-1)r^{n-2}\cos n\theta$$

$$\frac{\partial u_n}{\partial \theta} = -nr^n \sin n\theta \ , \quad \frac{\partial^2 u_n}{\partial \theta^2} = -n^2 r^n\cos n\theta$$

$$\nabla^2 u_n = n(n-1)r^{n-2}\cos n\theta + \frac{1}{r}nr^{n-1}\cos n\theta - \frac{1}{r^2}n^2r^n\cos n\theta$$

$$\nabla^2 u_n = \cancel{n^2 r^{n-2}\cos n\theta} - \cancel{n r^{n-2}\cos n\theta} + \cancel{n r^{n-2}\cos n\theta} - \cancel{n^2 r^{n-2}\cos n\theta}$$

$$= 0$$

$\therefore$ $u_n = r^n \cos n\theta$ is a solution of $\nabla^2 u_n = 0$. **Ans**

For $u_n = r^n \sin n\theta$

$$\frac{\partial u}{\partial r} = n r^{n-1}\sin n\theta \quad , \quad \frac{\partial^2 u}{\partial r^2} = n(n-1) r^{n-2}\sin n\theta$$

$$\frac{\partial u}{\partial \theta} = n r^n \cos n\theta \quad , \quad \frac{\partial^2 u}{\partial \theta^2} = -n^2 r^n \sin n\theta$$

$$\nabla^2 u_n = n(n-1) r^{n-2}\sin n\theta + \frac{1}{r} n r^{n-1}\sin n\theta + \frac{1}{r^2}(-n^2 r^n \sin n\theta)$$

$$= \cancel{n^2 r^{n-2}\sin n\theta} - \cancel{n r^{n-2}\sin n\theta} + \cancel{n r^{n-2}\sin n\theta} - \cancel{n^2 r^{n-2}\sin n\theta}$$

$$= 0$$

$\therefore$ $u_n = r^n \sin n\theta$ is a solution of $\nabla^2 u_n = 0$. **Ans**

Try small n, e.g. $n = 2$

$$r^2 \cos 2\theta = r^2(\cos^2\theta - \sin^2\theta)$$

$$= (r\cos\theta)^2 - (r\sin\theta)^2 = x^2 - y^2 \leftarrow \text{In Cartesian Coordinate}$$ **Ans**

$$r^2 \sin 2\theta = r^2(2\sin\theta\cos\theta) = 2 r\sin\theta \cdot r\cos\theta$$

$$= 2xy \leftarrow \text{In Cartesian coordinate}$$ **Ans**

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b) Show that a solution of the Laplace equation in the disk $r < R$ satisfying the B.C. $u(R,\theta) = f(\theta)$ is

$$u(r,\theta) = a_0 + \sum_{n=1}^{\infty}\left[a_n\left(\frac{r}{R}\right)^n \cos n\theta + b_n\left(\frac{r}{R}\right)^n \sin n\theta\right] \quad -(1)$$

Sol^n Ck B.C.

$$u(R,\theta) = a_0 + \sum_{n=1}^{\infty}\left[a_n \cos n\theta + b_n \sin n\theta\right]$$

$$= f(\theta)$$

If a_0, a_n, b_n are the Fourier coefficients of $f(\theta)$, $u(r,\theta)$ shown in Eq.(1) satisfies the B.C. $u(R,\theta) = f(\theta)$. **Ans**

Ck. if $u(r,\theta)$ satisfy $\nabla^2 u = 0$,

$$\frac{\partial u}{\partial r} = \sum_{n=1}^{\infty}\left[a_n \cdot \frac{n}{R}\left(\frac{r}{R}\right)^{n-1}\cos n\theta + b_n \cdot \frac{n}{R}\left(\frac{r}{R}\right)^{n-1}\sin n\theta\right]$$

$$\frac{\partial^2 u}{\partial r^2} = \sum_{n=1}^{\infty}\left[a_n \frac{n(n-1)}{R^2}\left(\frac{r}{R}\right)^{n-2}\cos n\theta + b_n \frac{n(n-1)}{R^2}\left(\frac{r}{R}\right)^{n-2}\sin n\theta\right]$$

$$\frac{\partial u}{\partial \theta} = \sum_{n=1}^{\infty}\left[-a_n n\left(\frac{r}{R}\right)^{n}\sin n\theta + b_n n\left(\frac{r}{R}\right)^{n}\cos n\theta\right]$$

$$\frac{\partial^2 u}{\partial \theta^2} = \sum_{n=1}^{\infty}\left[-n^2 a_n\left(\frac{r}{R}\right)^{n}\cos n\theta - n^2 b_n\left(\frac{r}{R}\right)^{n}\sin n\theta\right]$$

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

$$= \sum_{n=1}^{\infty}\left[a_n \frac{n(n-1)}{R^2}\left(\frac{r}{R}\right)^{n-2}\cos n\theta + b_n \frac{n(n-1)}{R^2}\left(\frac{r}{R}\right)^{n-2}\sin n\theta\right.$$

$$+ \frac{1}{r}a_n \frac{n}{R}\left(\frac{r}{R}\right)^{n-1}\cos n\theta + \frac{1}{r}b_n \frac{n}{R}\left(\frac{r}{R}\right)^{n-1}\sin n\theta$$

$$\left. - \frac{1}{r^2}n^2 a_n\left(\frac{r}{R}\right)^{n}\cos n\theta - \frac{1}{r^2}n^2 b_n\left(\frac{r}{R}\right)^{n}\sin n\theta\right]$$

$$= \sum_{n=1}^{\infty}\left[\cancel{\frac{a_n n^2}{R^n} r^{n-2}}\cos n\theta - \cancel{\frac{a_n n}{R^n} r^{n-2}\cos n\theta} + \cancel{\frac{b_n n^2}{R^n} r^{n-2}}\sin n\theta\right.$$

$$- \frac{b_n n}{R^n} r^{n-2}\sin n\theta + \cancel{\frac{a_n n}{R^n} r^{n-2}\cos n\theta} + \frac{b_n n}{R^n} r^{n-2}\sin n\theta$$

$$\left. - \cancel{\frac{a_n n^2}{R^n} r^{n-2}}\cos n\theta - \cancel{\frac{b_n n^2}{R^n} r^{n-2}}\sin n\theta\right]$$

$$= \sum_{n=1}^{\infty} 0 \qquad = 0$$

$\therefore$ $u(r,\theta)$ given in Eq. (1) satisfies $\nabla^2 u = 0$. <u>Ans</u>

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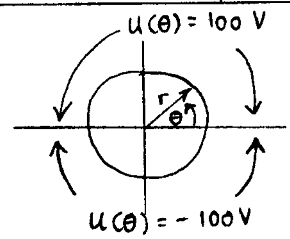
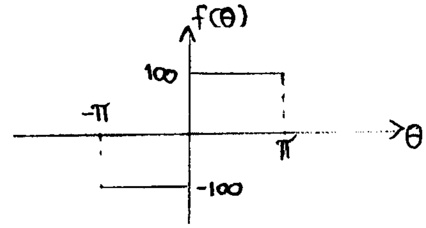
c) Solve the Dirichlet problem using b) if $R = 1$, $u(\theta) = -100$ Volts if $-\pi < \theta < 0$, $u(\theta) = 100$ Volts if $0 < \theta < \pi$ (Sketch this disk, indicate the boundary values.)

Sol^n





$f(\theta)$ is an odd function. $\Rightarrow a_0 = 0$, $a_n = 0$ (See p. 491)

$R = 1$; $\quad u(r,\theta) = \sum_{n=1}^{\infty} b_n r^n \sin n\theta$

At the boundary $r = R = 1$,

$$u(R,\theta) = \sum_{n=1}^{\infty} b_n \sin n\theta = f(\theta)$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(\theta)\sin n\theta \, d\theta$$

$$= \frac{2}{\pi}\int_0^{\pi} 100 \sin(n\theta)d\theta = \frac{200}{\pi}\left[\frac{-\cos n\theta}{n}\right]_{\theta=0}^{\theta=\pi}$$

$$= \frac{200}{n\pi}\left[-\cos n\pi + 1\right] = \frac{200}{n\pi}\left[1-(-1)^n\right]$$

$$= \begin{cases} 0 & ; \; n \text{ even} \\ \frac{400}{n\pi} & ; \; n \text{ odd} \end{cases}$$

$$\left.\begin{aligned} u(r,\theta) &= \sum_{n=1}^{\infty} \frac{200}{n\pi}\left[1-(-1)^n\right] r^n \sin n\theta \\ &= \frac{400}{\pi}\left[r\sin\theta + \frac{r^3}{3}\sin 3\theta + \frac{r^5}{5}\sin 5\theta + \dots \right] \end{aligned}\right\} \underline{\text{Ans}}$$

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d) Show that the solution of the Neumann problem $\nabla^2 u = 0$ if $r < R$, $u_N(R,\theta) = f(\theta)$ (where $u_N = \partial u/\partial N$ is the directional derivative in the direction of the outer normal) is

$$u(r,\theta) = A_0 + \sum_{n=1}^{\infty} r^n (A_n \cos n\theta + B_n \sin n\theta) \qquad \text{— (1)}$$

with $$A_n = \frac{1}{\pi n R^{n-1}}\int_{-\pi}^{\pi} f(\theta)\cos n\theta \, d\theta,$$

$$B_n = \frac{1}{\pi n R^{n-1}}\int_{-\pi}^{\pi} f(\theta)\sin n\theta \, d\theta$$

<u>Sol^{n}</u> $\nabla^2 u = 0 = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$ Laplace eqn.

from (1)

$$\frac{\partial u}{\partial r} = \sum_{n=1}^{\infty} n r^{n-1}(A_n \cos n\theta + B_n \sin n\theta) \quad \text{—(2)}$$

$$\frac{\partial^2 u}{\partial r^2} = \sum_{n=1}^{\infty} n(n-1) r^{n-2}(A_n \cos n\theta + B_n \sin n\theta)$$

$$\frac{\partial u}{\partial \theta} = \sum_{n=1}^{\infty} r^n(-nA_n \sin n\theta + nB_n \cos n\theta)$$

$$\frac{\partial^2 u}{\partial \theta^2} = \sum_{n=1}^{\infty} r^n(-n^2A_n \cos n\theta - n^2B_n \sin n\theta)$$

$$\nabla^2 u = \sum_{n=1}^{\infty} \Big[n(n-1) r^{n-2}(A_n \cos n\theta + B_n \sin n\theta) + \frac{1}{r} n r^{n-1}(A_n \cos n\theta + B_n \sin n\theta) + \frac{1}{r^2} r^n(-n^2A_n \cos n\theta - n^2 B_n \sin n\theta)\Big]$$

$$= \sum_{n=1}^{\infty} \Big[\cancel{n^2 r^{n-2} A_n \cos n\theta} - \cancel{n r^{n-2} A_n \cos n\theta} + n^2 r^{n-2} B_n \sin n\theta - n r^{n-2} B_n \sin n\theta + \cancel{n r^{n-2} A_n \cos n\theta} + n r^{n-2} B_n \sin n\theta - \cancel{n^2 r^{n-2} A_n \cos n\theta} - n^2 r^{n-2} B_n \sin n\theta \Big]$$

$$= 0 \quad *$$

$\therefore$ $u(r,\theta)$ given in (1) satisfies the Laplace eqn, $\nabla^2 u = 0$. <u>Ans</u>

<u>B.C.</u> $u_N(R,\theta) = \frac{\partial u}{\partial N}(R,\theta) = f(\theta)$

For this problem in polar coordinate $\frac{\partial u}{\partial N} = \frac{\partial u}{\partial r}$

From (2); $\frac{\partial u}{\partial r}(R,\theta) = \sum_{n=1}^{\infty} n R^{n-1}(A_n \cos n\theta + B_n \sin n\theta) = f(\theta)$

$$\sum_{n=1}^{\infty} \left[n R^{n-1} A_n \cos n\theta + n R^{n-1} B_n \sin n\theta \right] = f(\theta)$$

Using Fourier series to compute for A_n and B_n

$$n R^{n-1} A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta) \cos n\theta \, d\theta$$

$$A_n = \frac{1}{\pi n R^{n-1}}\int_{-\pi}^{\pi} f(\theta) \cos n\theta \, d\theta \quad *$$

and $$nR^{n-1}B_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin n\theta \, d\theta$$

$$B_n = \frac{1}{\pi n R^{n-1}}\int_{-\pi}^{\pi} f(\theta)\sin n\theta \, d\theta \quad *$$

$\therefore$ $u(r,\theta)$ given in (1) with arbitrary A_0 and the given A_n and B_n satisfies $\frac{\partial u}{\partial N}(R,\theta) = f(\theta)$. <u>Ans</u>

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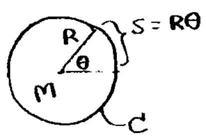
e) Show that (9), sec. 10.4, imposes on $f(\theta)$ in d) the "compatibility condition" $\int_{-\pi}^{\pi} f(\theta)\,d\theta = 0$.

<u>Sol</u>ⁿ

<u>Solⁿ</u> Eq.(9), sec. 10.4, is $$\iint_M \nabla^2 u \, dx\,dy = \oint_C \frac{\partial u}{\partial N}\, ds \quad —(1)$$

Here M is a closed bounded region with boundary C and s is the arclength of C. For the problem in d) with polar coordinate

$s = R\theta$



and $ds = R\,d\theta$

from (1); $$\iint_M \underbrace{\nabla^2 u}_{\substack{=0 \\ u \text{ satisfies } \nabla^2 u = 0}} dx\,dy = \oint_C \underbrace{\frac{\partial u}{\partial N}}_{= f(\theta) = u_N(R,\theta)} ds$$

$$0 = \oint_C f(\theta)\, R\,d\theta \quad —(2)$$

$R \neq 0$ and $\oint_C (\;)\,d\theta = \int_{-\pi}^{\pi} (\;)\,d\theta = \int_0^{2\pi} (\;)\,d\theta$, etc.

$(2) \times \frac{1}{R}$; $$\oint_C f(\theta)\frac{\cancel{R}\,d\theta}{\cancel{R}} = \int_{-\pi}^{\pi} f(\theta)\,d\theta = 0$$ <u>Ans</u>

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f) Solve $\nabla^2 u = 0$ in the annulus $1 < r < 3$ if $u_r(1,\theta) = \sin\theta$, $u_r(3,\theta) = 0$.

<u>Sol^n</u> $\quad \nabla^2 u = \dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2} = 0 \qquad$ — (1)

Let $u(r,\theta) = R(r)\Theta(\theta)$

Eq. (1) becomes; $\quad R''\Theta + \dfrac{1}{r}R'\Theta + \dfrac{1}{r^2}\Theta''R = 0$

Divided through by $\dfrac{R\Theta}{r^2}$;

$$r^2\frac{R''}{R} + r\frac{R'}{R} + \frac{\Theta''}{\Theta} = 0$$

$$r^2\frac{R''}{R} + r\frac{R'}{R} = -\frac{\Theta''}{\Theta} = n^2$$

R-equation; $\quad r^2R'' + rR' - n^2R = 0 \Rightarrow$ Euler-Cauchy Eq.

auxiliary eq. $\quad m^2 + (1-1)m - n^2 = 0$

$$m = \pm n$$

$$R_n(r) = C_1 r^n + C_2 r^{-n}$$

$$R_n'(r) = C_1 n r^{n-1} - C_2 n r^{-n-1} = C_1 n\left(r^{n-1} - \frac{C_2}{C_1} r^{-n-1}\right)$$

B.C. $u_r(3,\theta) = R'(3)\Theta(\theta) = 0 \Rightarrow R'(3) = 0$ for nontrivial sol^n

$$R_n'(3) = 0 = C_1 n\left(3^{n-1} - \frac{C_2}{C_1} 3^{-n-1}\right)$$

$$\frac{C_2}{C_1} = 3^{n-1}\cdot 3^{n+1} = 3^{2n} = 9^n \quad *$$

$$\therefore \quad R_n(r) = C_1\left(r^n + 9r^{-n}\right)$$

Θ-equation; $\quad \Theta'' + n^2\Theta = 0$

$$\Theta_n(\theta) = \tilde{A}\sin n\theta + \tilde{B}\cos n\theta$$

$$u(r,\theta) = \sum_{n=1}^{\infty} R_n(r)\Theta_n(\theta) = \sum_{n=1}^{\infty}\left(r^n + 9r^{-n}\right)(A_n\sin n\theta + B_n\cos n\theta)\ ; \quad A_n = C_1\tilde{A},\ B_n = C_1\tilde{B}$$

$$u_r(r,\theta) = \sum_{n=1}^{\infty}\left(nr^{n-1} - 9nr^{-n-1}\right)(A_n\sin n\theta + B_n\cos n\theta)$$

$$u_r(1,\theta) = \sin\theta = \sum_{n=1}^{\infty}(n - 9n)(A_n\sin n\theta + B_n\cos n\theta)$$

Since $\sin\theta$ is an odd fn $\rightarrow B_n = 0$

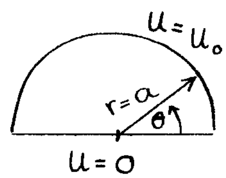
$$A_n = \frac{1}{\pi(n-9n)}\int_{-\pi}^{\pi}\sin n\theta \sin\theta \, d\theta$$

$$= \begin{cases} 0 & ; \ n \neq 1 \\ \frac{1}{1-9} = \frac{-1}{8} & ; \ n = 1 \end{cases}$$

$$\therefore \quad u(r,\theta) = -(r + 9/r)\frac{\sin\theta}{8} \qquad \text{Ans}$$

12.9.15 Find the steady-state temperature in a semicircular thin plate $r < a$, $0 < \theta < \pi$ with the semicircle $r = a$ kept at constant temperature u_0 and the segment $-a < x < a$ at 0.

Sol^n



Steady-state temperature, u

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2} = 0$$

Let $u(r,\theta) = R(r)\Theta(\theta)$

B.C. $u(a,\theta) = u_0$

$u(r,0) = 0 = R(r)\Theta(0) \Rightarrow \Theta(0) = 0$

$u(r,\pi) = 0 = R(r)\Theta(\pi) \Rightarrow \Theta(\pi) = 0$

Follow the similar step as in 12.9.6, we get

$$R_n(r) = c_1 r^n + c_2 r^{-n}$$

$$\Theta_n(\theta) = \tilde{A}\sin n\theta + \tilde{B}\cos n\theta$$

at the $r = 0$, $u(r,\theta)$ must still be well behaved, thus $c_2 = 0$ and

$$R_n(r) = c_1 r^n \quad *$$

With $\Theta_n(0) = 0 = \tilde{B}\overset{1}{\cancel{\cos 0}} \Rightarrow \tilde{B} = 0$, [$\Theta_n(\pi) = 0$ gave the same result, $\tilde{B} = 0$]

$$\Theta_n(\theta) = \tilde{A}\sin n\theta \quad *$$

BC @ $r = a$; $u(a,\theta) = \sum_{n=1}^{\infty} A_n a^n \sin n\theta = u_0$; $A_n = c_1\tilde{A}$

Fourier series ; $A_n = \frac{2}{a^n\pi}\int_0^{\pi} u_0 \sin n\theta \, d\theta = \frac{2u_0}{a^n\pi}\left[\frac{-\cos n\theta}{n}\right]_{\theta=0}^{\theta=\pi}$

$$A_n = \frac{2u_0}{a^n\pi}\left[-\frac{\cos n\pi}{n} + \frac{1}{n}\right] = \frac{2u_0}{a^n\pi n}\left[1-(-1)^n\right]$$

$$= \begin{cases} 0 & , \; n \text{ even} \\ \dfrac{4u_0}{a^n\pi n} & , \; n \text{ odd} \end{cases}$$

$$\therefore \quad u(r,\theta) = \frac{4u_0}{\pi}\sum_{\substack{n=1\\ n\,\text{odd}}}^{\infty} \frac{1}{n}\left(\frac{r}{a}\right)^n \sin n\theta \qquad \underline{\underline{\text{Ans}}}$$

$$= \frac{4u_0}{\pi}\left[\frac{r}{a}\sin\theta + \frac{1}{3}\left(\frac{r}{a}\right)^3 \sin 3\theta + \frac{1}{5}\left(\frac{r}{a}\right)^5 \sin 5\theta + \cdots\right]$$

See plot at the end

Sect. 12.11

6. $W = \mathscr{L}\{w\}$, $W_{xx} = (100s^2 + 100s + 25)W = (10s + 5)^2W$. The solution of this ODE is

$$W = c_1(s)e^{-(10s+5)x} + c_2(s)e^{(10s+5)x}$$

with $c_2(s) = 0$, so that the solution is bounded. $c_1(s)$ follows from

$$W(0, s) = \mathscr{L}\{w(0, t)\} = \mathscr{L}\{\sin t\} = \frac{1}{s^2 + 1} = c_1(s).$$

Hence

$$W = \frac{1}{s^2 + 1}\, e^{-(10s+5)x}.$$

The inverse Laplace transform (the solution of our problem) is

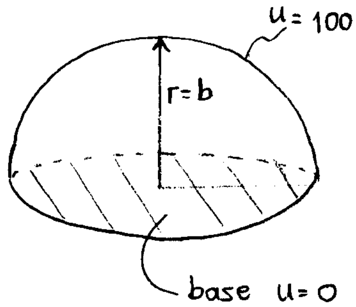
$$w = \mathscr{L}^{-1}\{W\} = e^{-5x}u(t - 10x)\sin(t - 10x),$$

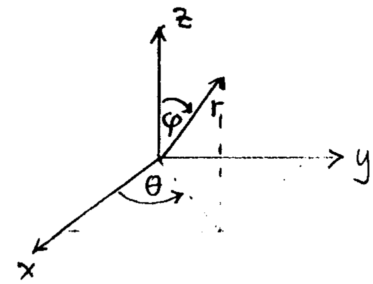
a traveling wave decaying with x. Here u is the unit step function (the Heaviside function).

Problem 1

Find the steady-state temperature in a solid hemisphere ($0 \le r \le b$, $0 \le \varphi \le \pi/2$, $0 \le \theta \le 2\pi$) if the planare base is held at zero temperature while the hemispherical portion is kept at constant temperature 100.

Sol$^{\underline{n}}$





Steady-state temperature in a solid hemisphere $\Rightarrow$ Laplacian eq. in a spherical coordinate;

$$\nabla^2 u = \frac{1}{r^2}\left[(r^2 u_r)_r + \frac{1}{\sin\varphi}(\sin\varphi\, u_\varphi)_\varphi + \frac{1}{\sin^2\varphi} u_{\theta\theta} \right] = 0$$

Due to symmetry in θ, the above eq. becomes,

$$(r^2 u_r)_r + \frac{1}{\sin\varphi}(\sin\varphi\, u_\varphi)_\varphi = 0 \qquad \text{—(1)}$$

Let $u(r,\varphi) = R(r)\Phi(\varphi)$

Eq. (1) becomes, $\quad r^2 R''\Phi + 2rR'\Phi + \dfrac{R}{\sin\varphi}(\sin\varphi\,\Phi')' = 0$

Divide by $R\Phi$;

$$\underbrace{r^2\frac{R''}{R} + 2r\frac{R'}{R}}_{\text{fn. of } r \text{ only}} = \underbrace{-\frac{1}{\sin\varphi}\frac{(\sin\varphi\,\Phi')'}{\Phi}}_{\text{fn. of } \varphi \text{ only}} = K$$

R-equation $\quad r^2R'' + 2rR' - KR = 0 \quad \Rightarrow$ Euler-Cauchy eqn

auxiliary eq $\quad m^2 + (2-1)m - K = 0$

Let $K = n(n+1)$; $\quad m^2 + m - n(n+1) = 0$

$$(m-n)(m+(n+1)) = 0 \quad \Rightarrow \quad m = n, -(n+1)$$

$$R_n(r) = c_1 r^n + c_2 r^{-(n+1)}$$

Since $R(r)$ must be finite at $r = 0 \Rightarrow c_2 = 0$; $\quad R_n(r) = c_1 r^n$ *

φ-equation $\quad \dfrac{1}{\sin\varphi}\dfrac{(\sin\varphi\,\Phi')'}{\Phi} = -K = -n(n+1) \xrightarrow{(2)} \Rightarrow$ (from R-eqn)

Let $\cos\varphi = w$ and $\sin^2\varphi = 1-\cos^2\varphi = 1-w^2$

$$\frac{d}{d\varphi} = \frac{d}{dw}\frac{dw}{d\varphi} = -\sin\varphi\frac{d}{dw}$$

thus Eq. (2) becomes; $\dfrac{d}{dw}\left[(1-w^2)\dfrac{d\Phi}{dw}\right] + n(n+1)\Phi = 0$

$$(1-w^2)\frac{d^2\Phi}{dw^2} - 2w\frac{d\Phi}{dw} + n(n+1)\Phi = 0 \Rightarrow \text{Legendre's equation}$$

The solution of Legendre's equation are (see p. 590)

$$\Phi_n(\varphi) \sim P_n(\cos\varphi) \qquad \text{— (3)}$$

$$u_n(r,\varphi) = R_n(r)\,\Phi_n(\varphi) = C_n r^n P_n(\cos\varphi)$$

The boundary condition on the spherical base is $u(r,\frac{\pi}{2}) = 0$

$$u(r,\tfrac{\pi}{2}) = R(r)\,\Phi(\tfrac{\pi}{2}) = 0 \Rightarrow \Phi(\tfrac{\pi}{2}) = 0$$

From (3); $\Phi_n(\frac{\pi}{2}) \sim P_n(\cos\frac{\pi}{2}) = P_n(0) = 0$

We need to choose P_n such that $P_n(0) = 0$.

From p. 180, we found that

$P_0(x) = 1$, $\qquad P_1(x) = x$

$P_2(x) = \frac{1}{2}(3x^2-1)$, $\qquad P_3(x) = \frac{1}{2}(5x^3-3x)$

$P_4(x) = \frac{1}{8}(35x^4-30x^2+3)$, $\qquad P_5(x) = \frac{1}{8}(63x^5-70x^3+15x)$

$\vdots \qquad\qquad \vdots$

Only n odd give $P_n(0) = 0$. Thus, the restriction for P_n to satisfy B.C. at the sphere base is that n must be an odd number, i.e, P_{2n+1} ; $n = 0, 1, 2, 3, \ldots$

$$u(r,\varphi) = \sum_{n=0}^{\infty} C_{2n+1}\, r^{2n+1} P_{2n+1}(\cos\varphi)$$

B.C. on the spherical portion,

$$u(b,\varphi) = 100 = \sum_{n=0}^{\infty} C_{2n+1}\, b^{2n+1} P_{2n+1}(\cos\varphi) = f(\varphi)$$

$$C_{2n+1}\, b^{2n+1} = \frac{\langle f(\varphi),\, P_{2n+1}(\cos\varphi)\rangle}{\langle P_{2n+1}(\cos\varphi),\, P_{2n+1}(\cos\varphi)\rangle}$$

$$C_{2n+1} = \frac{1}{b^{2n+1}} \, \frac{\int_0^{\pi/2} 100\, P_{2n+1}(\cos\varphi)\sin\varphi\, d\varphi}{\int_0^{\pi/2} P_{2n+1}^2(\cos\varphi)\sin\varphi\, d\varphi}$$

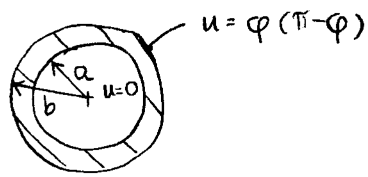
$$\left.\begin{aligned} u(r,\varphi) = \sum_{n=0}^{\infty} C_{2n+1}\, r^{2n+1} P_{2n+1}(\cos\varphi) \end{aligned}\right\} \quad \underline{\underline{\text{Ans.}}}$$

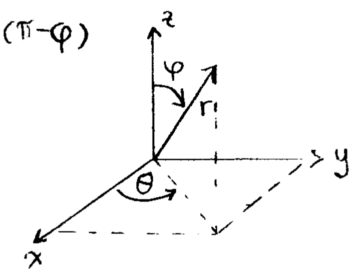
with C_{2n+1} as defined above. (See plot at the end)

Problem 2

Find the steady-state temperature distribution in a spherical shell $(a \leq r \leq b)$ if the inner surface is held at 0 temperature while the outer surface has temperature distribution $u(b,\varphi) = \varphi(\pi-\varphi)$.

<u>Sol$^{\underline{n}}$</u>





Steady-state temperature in a spherical shell

$=0$ due to symmetry in θ-dir.

$$\nabla^2 u = \frac{1}{r^2}\left[(r^2 u_r)_r + \frac{1}{\sin\varphi}(\sin\varphi\, u_\varphi)_\varphi + \cancel{\frac{1}{\sin^2\varphi} u_{\theta\theta}}\right] = 0$$

$$(r^2 u_r)_r + \frac{1}{\sin\varphi}(\sin\varphi\, u_\varphi)_\varphi = 0$$

Following similar steps in Prob. 2), we have

$$R_n(r) = c_1 r^n + C_2 r^{-(n+1)}$$

and $\quad \Phi(\varphi) \sim P_n(\cos\varphi)$

B.C. $r = a$, $\quad u(a,\varphi) = R(a)\Phi(\varphi) = 0 \Rightarrow R(a) = 0$

$$R(a) = C_1\left(a^n + \frac{C_2}{C_1 a^{n+1}}\right) = 0$$

For non-trivial sol$^{\underline{n}}$, $C_1 \neq 0$; $\quad \frac{C_2}{C_1} = -a^n \cdot a^{n+1} = -a^{2n+1}$

$$* \quad R_n(r) = c_1\left(r^n - \frac{a^{2n+1}}{r^{n+1}}\right)$$

$$u(r,\varphi) = \sum_{n=0}^{\infty} \tilde{A}_n R_n \bar{\Phi}_n = \sum_{n=0}^{\infty} A_n\left(r^n - \frac{a^{2n+1}}{r^{n+1}}\right) P_n(\cos\varphi)$$

B.C at the outer shell ; $u(b,\varphi) = f(\varphi) = \varphi(\pi-\varphi)$

$$u(b,\varphi) = f(\varphi) = \sum_{n=0}^{\infty} A_n\left(b^n - \frac{a^{2n+1}}{b^{n+1}}\right) P_n(\cos\varphi)$$

using Fourie- Legendre series , (see p. 590)

$$A_n = \frac{1}{\left(b^n - \frac{a^{2n+1}}{b^{n+1}}\right)} \cdot \frac{2n+1}{2} \int_0^{\pi} \varphi(\pi-\varphi) P_n(\cos\varphi) \sin\varphi \, d\varphi$$

$$\therefore \quad u(r,\varphi) = \sum_{n=0}^{\infty} A_n\left(r^n - \frac{a^{2n+1}}{r^{n+1}}\right) P_n(\cos\varphi)$$

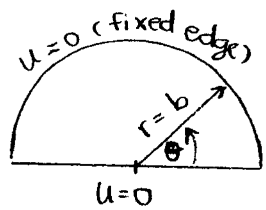
with A_n as shown above } Ans

See plot at the end

Problem 3

Determine the vibration of a semi-circular membrane ($0 \leq r \leq b$, $0 \leq \theta \leq \pi$) with initial conditions $u(r, \theta, 0) = \theta(\pi-\theta)\, r(b-r)$, $u_t(r,\theta,0) = 0$. Determine mode shapes, nodal lines and natural frequencies for at least the first 5 modes.

<u>Sol</u>



IC's $u(r,\theta,0) = \theta(\pi-\theta)\sin\left(\frac{\pi r}{b}\right)$

$u_t(r,\theta,0) = 0$

2D wave eqn. in polar coordinate :

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right) \qquad \text{— (1)}$$

Let $u(r,\theta,t) = R(r)\Theta(\theta)T(t)$

Eq.(1) becomes, $R\Theta T'' = c^2\left(R''\Theta T + \frac{1}{r}R'\Theta T + \frac{1}{r^2}R\Theta'' T\right)$

Divided through by $c^2 R\Theta T$;

$$\underbrace{\frac{T''}{c^2 T}}_{\text{fn. of } t \text{ only}} = \underbrace{\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta}}_{\text{fn. of } r,\theta \text{ only}} = -K_1^2$$

T-equation; $T'' + c^2K^2T = 0$

$$* \quad T(t) = C_1\sin(\omega t) + C_2\cos(\omega t) \; ; \; \omega = \sqrt{c^2K^2} \qquad \text{— (2)}$$

B.C. fixed edge membrane $\Rightarrow$ $u(b,\theta,t) = 0 = R(b)\Theta(\theta)T(t) \Rightarrow R(b)=0$

$\Rightarrow$ $u(r,0,t) = u(r,\pi,t) = 0 \Rightarrow \Theta(0) = \Theta(\pi)=0$

I.C $\quad u_t(r,\theta,0) = R(r)\Theta(\theta)T'(0) = 0 \Rightarrow T'(0) = 0$

$$T'(t) = C_1 c K_1 \cos(cK_1 t) - C_2 c K_1 \sin(cK_1 t)$$

$$T'(0) = C_1 c K_1 \quad 0 \Rightarrow \underline{C_1 = 0} \Rightarrow$$

Note: K_1 is not always 0 see R-eq. below

$$T(t) = C_2 \cos(cK_1 t)$$

R-Θ equation;
$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + K_1^2 = \frac{-1}{r^2}\frac{\Theta''}{\Theta}$$

$$r^2\frac{R''}{R} + r\frac{R'}{R} + K_1^2 r^2 = -\frac{\Theta''}{\Theta} = K_2^2$$

R-equation; $\quad r^2R'' + rR' + (K_1^2r^2 - K_2^2)R = 0 \quad$ —(3)

standard form of Bessel's eq.; $\quad x^2y'' + xy' + (x^2-\nu^2)y = 0 \quad$ —(4)

Compare (3) & (4); $\quad K_2 \to \nu, \; K_1 r \to x, \; y \to R$

$$y' = \frac{dy}{dx} = \frac{dR}{dx} = \underbrace{\frac{dR}{dr}\frac{dr}{dx}} = R'\frac{1}{K_1}$$

$$\frac{d(K_1 r)}{dx} = \frac{dx}{dx} \Rightarrow \frac{dr}{dx} = \frac{1}{K_1}$$

$$x^2y'' + xy' + (x^2-\nu^2)y = (K_1 r)^2\frac{R''}{K_1^2} + K_1 r\frac{R'}{K_1} + (K_1^2r^2 - K_2^2)R = 0$$

$\therefore$ Eq (3) is in a form of Bessel's equation.

Θ-equation; $\quad \Theta'' + K_2^2\Theta = 0$

$$\Theta(\theta) = C_3\sin(K_2\theta) + C_4\cos(K_2\theta)$$

Apply B.C.;
$$\Theta(0) = C_4 = 0 \Rightarrow \underline{C_4 = 0}$$
$$\Theta(\pi) = C_3\sin(K_2\pi) = 0$$

For nontrivial sol^n, $C_3 \neq 0 \Rightarrow \sin(K_2\pi) = 0 \Rightarrow K_2\pi = n\pi$ where n = integer

$$* \quad \Theta_n(\theta) = C_3\sin(n\theta) \quad \text{—(5)}$$

Go back to R-equation, the solution to the Bessel's equation is (p.202)

$$R(r) = C_5 J_{K_2}(K_1 r) + C_6 Y_{K_2}(K_1 r) \quad ; \quad \nu = K_2, \; x = K_1 r$$

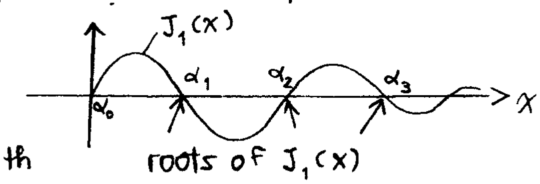
with $K_2 = n$ from Θ-eq.; $\quad R(r) = C_5 J_n(K_1 r) + C_6 Y_n(K_1 r)$

Since $R(r)$ have to be finite at the center $(r=0)$ but $Y_n(K_1 r) \to \infty$ as $r \to 0$. Thus, $C_6 = 0$ and

$$R(r) = C_5 J_n(K_1 r)$$

Apply B.C. $R(b) = C_5 J_n(K_1 b) = 0$

For non-trivial solution, $C_5 \neq 0 \Rightarrow J_n(K_1 b) = 0 \Rightarrow K_1 b$ must be a root of nth Bessel's function e.g.



Let α_{mn} is the m^{th} root of the n^{th} Bessel's Function, e.g α_{11} is the 1^{st} root of $J_1(x)$.

$$K_1 b = \alpha_{mn} \Rightarrow K_1 = \frac{\alpha_{mn}}{b}$$

$$* \quad R_{mn}(r) = C_5 J_n\left(\frac{\alpha_{mn}}{b} r\right) \qquad (6)$$

Thus $$* \quad T_{mn}(t) = C_2 \cos\left(\frac{c\alpha_{mn} t}{b}\right) \qquad (7)$$

$$u_{mn}(r,\theta,t) = R_{mn}(r)\,\Theta_n(\theta)\,T_{mn}(t)$$

$$u(r,\theta,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} C_{mn} J_n\left(\frac{\alpha_{mn} r}{b}\right) \sin(n\theta) \cos\left(\frac{c\alpha_{mn}}{b} t\right)$$

next page →

Apply I.C. : $u(r,\theta,0) = \theta(\pi-\theta)\, r(b-r) = f(r,\theta)$

$$= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} C_{mn}\, \underbrace{J_n\left(\frac{\alpha_{mn} r}{b}\right)\sin(n\theta)}_{}$$

these functions are orthogonal with respect to the weighting function r (see Mathematica file for verification)

Let's find C_{mn} by projecting $f(r,\theta)$ onto ~~the~~ orthogonal set of functions :

$$C_{mn} = \frac{f(r,\theta)\cdot J_n\left(\frac{\alpha_{mn} r}{b}\right)\sin(n\theta)}{J_n\left(\frac{\alpha_{mn} r}{b}\right)\sin(n\theta)\cdot J_n\left(\frac{\alpha_{mn} r}{b}\right)\sin(n\theta)} =$$

weighting function

$$= \frac{\int_0^1\int_0^{\pi} r\, f(r,\theta)\, J_n\left(\frac{\alpha_{mn} r}{b}\right)\sin(n\theta)\, d\theta\, dr}{\int_0^1\int_0^{\pi} r\left[J_n\left(\frac{\alpha_{mn} r}{b}\right)\sin(n\theta)\right]^2 d\theta\, dr} = \text{See Mathematica file}$$

$$= \frac{4}{\pi b^2 J_{n+1}^2(\alpha_{mn})}\int_0^b\int_0^{\pi} r\, f(r,\theta)\, J_n\left(\frac{\alpha_{mn} r}{b}\right)\sin(n\theta)\, d\theta\, dr$$

$\rightarrow f(r,\theta) = \theta(\pi-\theta)\, r(b-r)$

Solution :

$$u(r,\theta,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} C_{mn}\, J_n\left(\frac{\alpha_{mn} r}{b}\right)\sin(n\theta)\cos\left(\frac{c\,\alpha_{mn} t}{b}\right)$$

where C_{mn} is given above

See plot at the end

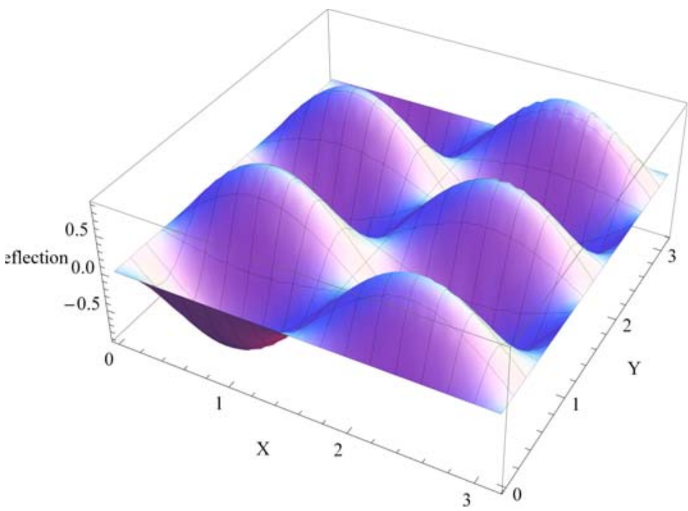
Ex. 12.8 .11 : Deflection of square membrane

```
In[2]:= k = 1;

In[3]:= f[x_, y_] := k Sin[2 x] Sin[5 y]

In[6]:= Bmn = 4 / Pi^2 Integrate[f[x, y] Sin[m x] Sin[n y], {x, 0, Pi}, {y, 0, Pi}]
```

Out[6]= $-\frac{40\,\mathrm{Sin}[m\,\pi]\,\mathrm{Sin}[n\,\pi]}{(-4+m^2)\,(-25+n^2)\,\pi^2}$

```
In[11]:= u[x_, y_, t_] := k Cos[Sqrt[29] t] Sin[2 x] Sin[5 y]

In[14]:= Plot3D[u[x, y, 10], {x, 0, Pi}, {y, 0, Pi}, AxesLabel -> {"X", "Y", "Deflection"}]
```

Out[14]=



Ex. 12.8 .13 : Deflection of square membrane

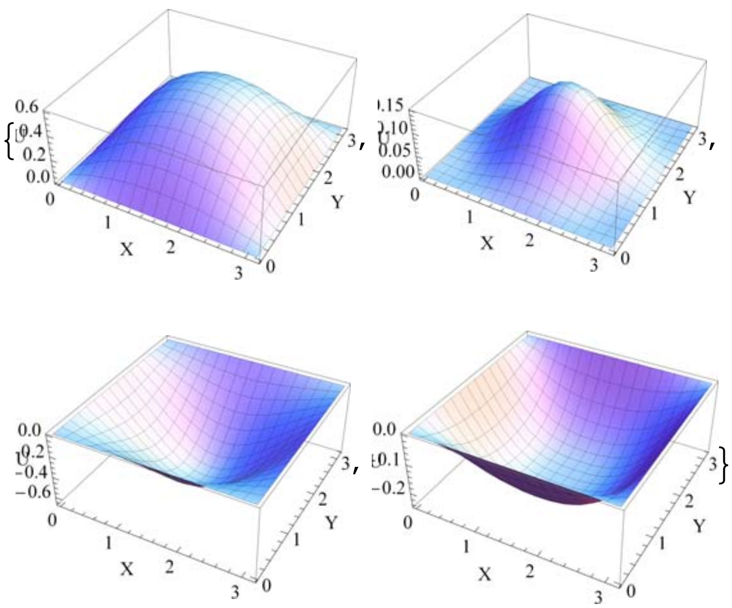
```
In[11]:= f[x_, y_] := 0.1 x y (Pi - x) (Pi - y)

In[12]:= Bmn[m_, n_] := 4 / Pi^2
           Integrate[f[x, y] Sin[m x] Sin[n y], {x, 0, Pi}, {y, 0, Pi}, Assumptions → {n, m} ∈ Integers]

In[13]:= λmn = Sqrt[m^2 + n^2];

In[14]:= u[x_, y_, t_] := Sum[Sum[Bmn[m, n] Cos[λmn t] Sin[m x] Sin[n y], {n, 1, 4}], {m, 1, 4}]

In[15]:= u[x, y, t];
```

In[16]:=
```
uu = %
```

Out[16]=
$$0.648456\,\text{Cos}\left[\sqrt{2}\,t\right]\text{Sin}[x]\,\text{Sin}[y]+0.\,\text{Cos}\left[\sqrt{5}\,t\right]\text{Sin}[2x]\,\text{Sin}[y]+0.0240169\,\text{Cos}\left[\sqrt{10}\,t\right]\text{Sin}[3x]\,\text{Sin}[y]+0.\,\text{Cos}\left[\sqrt{17}\,t\right]\text{Sin}[4x]\,\text{Sin}[y]+0.\,\text{Cos}\left[\sqrt{5}\,t\right]\text{Sin}[x]\,\text{Sin}[2y]+0.\,\text{Cos}\left[2\sqrt{2}\,t\right]\text{Sin}[2x]\,\text{Sin}[2y]+0.\,\text{Cos}\left[\sqrt{13}\,t\right]\text{Sin}[3x]\,\text{Sin}[2y]+0.\,\text{Cos}\left[2\sqrt{5}\,t\right]\text{Sin}[4x]\,\text{Sin}[2y]+0.0240169\,\text{Cos}\left[\sqrt{10}\,t\right]\text{Sin}[x]\,\text{Sin}[3y]-1.68734\times10^{-17}\,\text{Cos}\left[\sqrt{13}\,t\right]\text{Sin}[2x]\,\text{Sin}[3y]+0.000889514\,\text{Cos}\left[3\sqrt{2}\,t\right]\text{Sin}[3x]\,\text{Sin}[3y]-8.43668\times10^{-18}\,\text{Cos}[5t]\,\text{Sin}[4x]\,\text{Sin}[3y]+0.\,\text{Cos}\left[\sqrt{17}\,t\right]\text{Sin}[x]\,\text{Sin}[4y]+0.\,\text{Cos}\left[2\sqrt{5}\,t\right]\text{Sin}[2x]\,\text{Sin}[4y]+0.\,\text{Cos}[5t]\,\text{Sin}[3x]\,\text{Sin}[4y]+0.\,\text{Cos}\left[4\sqrt{2}\,t\right]\text{Sin}[4x]\,\text{Sin}[4y]$$

In[18]:=
```
plots = Table[Plot3D[uu, {x, 0, Pi}, {y, 0, Pi}, AxesLabel -> {"X", "Y", "U"}], {t, 0, 3}]
```

In[19]:=



Ex. 12.9 .15 : SS Temperature in semicircular thin plate

In[2]:=
```
An[n_] := 2 / a^n / Pi * Integrate[u0 Sin[n θ], {θ, 0, Pi}, Assumptions → n ∈ Integers]
```

In[3]:=
```
An[n]
```

Out[3]=
$$-\frac{2\left(-1+(-1)^{n}\right)a^{-n}\,u0}{n\,\pi}$$

In[11]:=
```
coeffs = Table[An[n], {n, 1, 5}]
```

Out[11]=
$$\left\{\frac{280}{\pi},\,0,\,\frac{280}{3\pi},\,0,\,\frac{56}{\pi}\right\}$$

In[12]:=
```
u[r_, θ_] := Sum[An[n] r^n Sin[n θ], {n, 1, 10}]
```

In[13]:=
```
u0 = 70; a = 1;
```

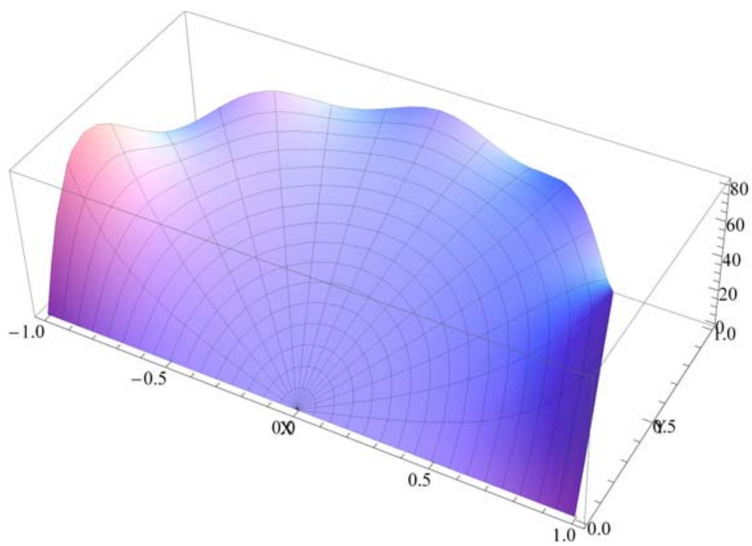
In[14]:=
```
uu = u[r, θ]
```

Out[14]=
$$\frac{280\, r\, \mathrm{Sin}[\theta]}{\pi} + \frac{280\, r^3\, \mathrm{Sin}[3\,\theta]}{3\,\pi} + \frac{56\, r^5\, \mathrm{Sin}[5\,\theta]}{\pi} + \frac{40\, r^7\, \mathrm{Sin}[7\,\theta]}{\pi} + \frac{280\, r^9\, \mathrm{Sin}[9\,\theta]}{9\,\pi}$$

In[15]:=
```
u[0.5, Pi / 2] // N
```

Out[15]= 41.3267

In[16]:=
```
RevolutionPlot3D[uu, {r, 0, a}, {θ, 0, Pi}, AxesLabel -> {"X", "Y"}]
```



Problem 1: SS Temperature in a solid hemisphere

In[1]:=
```
c[n_] := Integrate[100 * LegendreP[2 n - 1, Cos[φ]] * Sin[φ], {φ, 0, Pi / 2}] /
   Integrate[(LegendreP[2 n - 1, Cos[φ]]) ^2 * Sin[φ], {φ, 0, Pi / 2}]
```

In[2]:=
```
c[n]
```

Out[2]=
$$\frac{50\sqrt{\pi}}{\mathrm{Gamma}\left[\frac{3}{2}-n\right]\,\mathrm{Gamma}[1+n]\int_0^{\frac{\pi}{2}}\mathrm{LegendreP}[-1+2\,n,\,\mathrm{Cos}[\varphi]]^2\,\mathrm{Sin}[\varphi]\,d\varphi}$$

In[3]:=
```
coeffs = Table[c[n], {n, 1, 5}]  (* Note: these are the c(2n-1) coeff.) *)
```

Out[3]=
$$\left\{150,\ -\frac{175}{2},\ \frac{275}{4},\ -\frac{1875}{32},\ \frac{3325}{64}\right\}$$

In[12]:=
```
u[r_, φ_] := Sum[c[n] r^(2 n - 1) LegendreP[2 n - 1, Cos[φ]], {n, 1, 6}]
```

In[13]:=
```
uu = u[r, φ]
```

Out[13]=
$$150\, r\, \mathrm{Cos}[\varphi] - \frac{175}{4} r^3 \left(-3\,\mathrm{Cos}[\varphi] + 5\,\mathrm{Cos}[\varphi]^3\right) + \frac{275}{32} r^5 \left(15\,\mathrm{Cos}[\varphi] - 70\,\mathrm{Cos}[\varphi]^3 + 63\,\mathrm{Cos}[\varphi]^5\right) -$$
$$\frac{1875}{512} r^7 \left(-35\,\mathrm{Cos}[\varphi] + 315\,\mathrm{Cos}[\varphi]^3 - 693\,\mathrm{Cos}[\varphi]^5 + 429\,\mathrm{Cos}[\varphi]^7\right) +$$
$$\frac{3325\, r^9 \left(315\,\mathrm{Cos}[\varphi] - 4620\,\mathrm{Cos}[\varphi]^3 + 18\,018\,\mathrm{Cos}[\varphi]^5 - 25\,740\,\mathrm{Cos}[\varphi]^7 + 12\,155\,\mathrm{Cos}[\varphi]^9\right)}{8192} - \frac{1}{65\,536} 12\,075\, r^{11}$$
$$\left(-693\,\mathrm{Cos}[\varphi] + 15\,015\,\mathrm{Cos}[\varphi]^3 - 90\,090\,\mathrm{Cos}[\varphi]^5 + 218\,790\,\mathrm{Cos}[\varphi]^7 - 230\,945\,\mathrm{Cos}[\varphi]^9 + 88\,179\,\mathrm{Cos}[\varphi]^{11}\right)$$

In[14]:=
```
u[0.5, Pi/4] // N
```
Out[14]= 54.1327

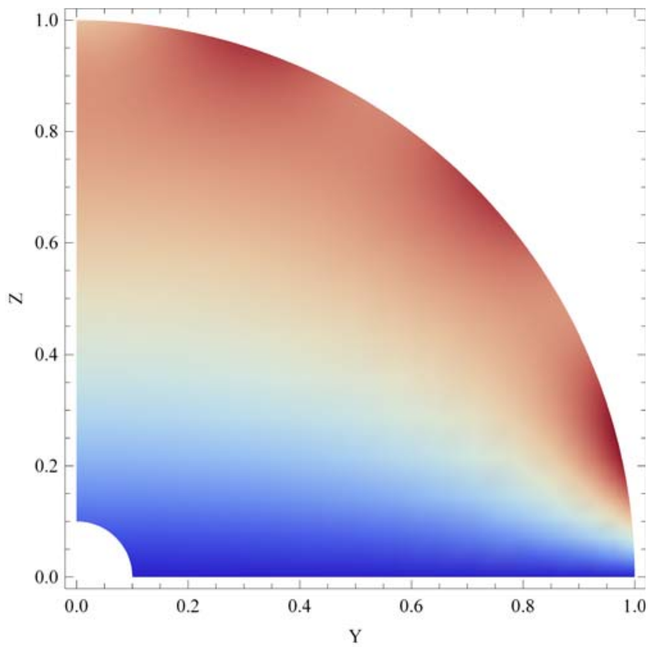
In[15]:=
```
u[0, 0] // N
```
Out[15]= 0.

```
Table[u[r, Pi/4], {r, 0, 1}] // N (* check some more values *)
```
Out[16]= {0., 108.012}

$$\texttt{temp[y\_, z\_] := u}\left[\sqrt{y^2+z^2}\texttt{, ArcTan[z, y]}\right]$$

```
DensityPlot[temp[y, z], {y, 0, 1}, {z, 0, 1},
 AspectRatio -> 1, RegionFunction -> (0.01 < #1^2 + #2^2 < 1 &),
 FrameLabel -> {"Y", "Z"}, ColorFunction -> "ThermometerColors"]
```

To save computing time, only half of the temperature distribution (at a certain θ) is displayed
Note: the origin had to be removed in the plotting

Out[18]=


Problem 2: SS Temperature in a spherical shell

In[1]:=
$$\texttt{aA[n\_] :=} \frac{2n+1}{2}\,\frac{1}{b^n-\frac{a^{2n+1}}{b^{n+1}}}$$

```
Integrate[φ (Pi - φ) LegendreP[n, Cos[φ]] * Sin[φ], {φ, 0, Pi}]
```

In[2]:= `aA[n]`

Out[2]= $$\frac{(1+2\,n)\int_0^{\pi}(\pi-\varphi)\,\varphi\,\text{LegendreP}[n,\text{Cos}[\varphi]]\,\text{Sin}[\varphi]\,d\varphi}{2\left(-a^{1+2\,n}\,b^{-1-n}+b^{n}\right)}$$

In[18]:= `Table[aA[n], {n, 1, 8}]`

Out[18]= $$\left\{0,\ -\frac{80}{279},\ 0,\ -\frac{256}{12\,775},\ 0,\ -\frac{212\,992}{90\,305\,775},\ 0,\ -\frac{4\,456\,448}{13\,005\,519\,975}\right\}$$

In[19]:= $$u[r\_,\varphi\_] := \text{Sum}\left[\text{aA}[n]\left(r\wedge n-\frac{a^{2\,n+1}}{r^{n+1}}\right)\text{LegendreP}[n,\text{Cos}[\varphi]],\{n,0,8\}\right]$$

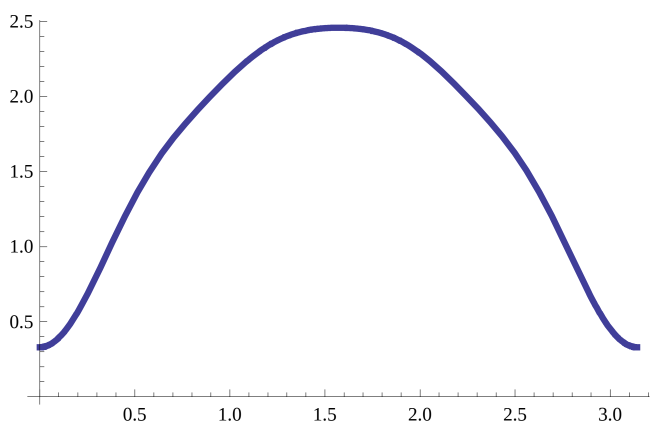
In[5]:= `a = 1; b = 2;`

In[20]:= `u[r, φ]`

Out[20]= $$4\left(1-\frac{1}{r}\right)-\frac{40}{279}\left(-\frac{1}{r^3}+r^2\right)\left(-1+3\,\text{Cos}[\varphi]^2\right)-\frac{32\left(-\frac{1}{r^5}+r^4\right)\left(3-30\,\text{Cos}[\varphi]^2+35\,\text{Cos}[\varphi]^4\right)}{12\,775}-$$
$$\frac{13\,312\left(-\frac{1}{r^7}+r^6\right)\left(-5+105\,\text{Cos}[\varphi]^2-315\,\text{Cos}[\varphi]^4+231\,\text{Cos}[\varphi]^6\right)}{90\,305\,775}-$$
$$\frac{34\,816\left(-\frac{1}{r^9}+r^8\right)\left(35-1260\,\text{Cos}[\varphi]^2+6930\,\text{Cos}[\varphi]^4-12\,012\,\text{Cos}[\varphi]^6+6435\,\text{Cos}[\varphi]^8\right)}{13\,005\,519\,975}$$

Verify the initial condition :

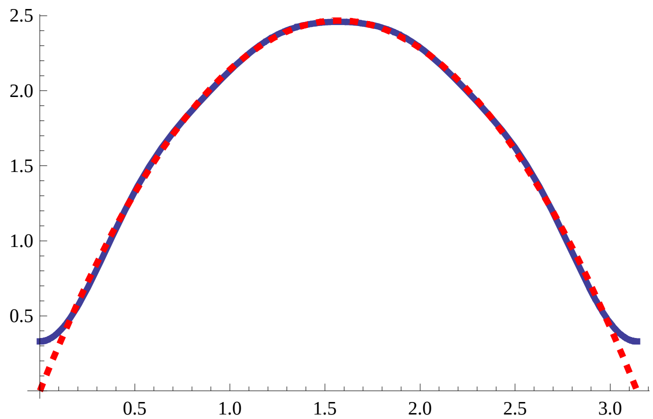
In[22]:= `p1 = Plot[u [b, φ], {φ, 0, Pi}, PlotStyle → AbsoluteThickness[3]]`

Out[22]=


`p2 = Plot[φ (Pi - φ), {φ, 0, Pi}, PlotStyle → {Red, Dashed, AbsoluteThickness[3]}];`

In[27]:=
```
Show[p1, p2]
```

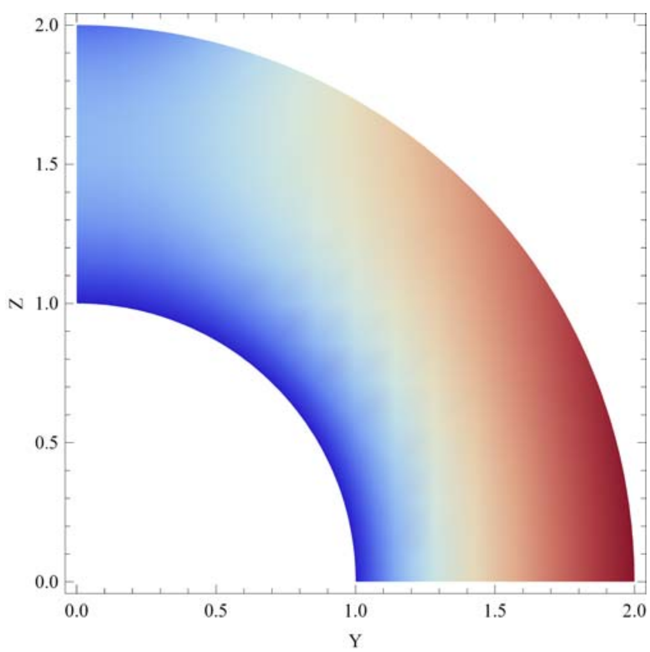
Out[27]=



In[28]:=
```
temp[y_, z_] := u[Sqrt[y^2 + z^2], ArcTan[z, y]]
```

In[30]:=
```
DensityPlot[temp[y, z], {y, 0, b}, {z, 0, b},
 AspectRatio -> 1, RegionFunction -> (a^2 < #1^2 + #2^2 < b^2 &),
 FrameLabel -> {"Y", "Z"}, ColorFunction -> "ThermometerColors"]
```

Out[30]=



Problem 3: Vibration of a semi-circular membrane

```
α[n_, m_] = BesselJZero[n, m]  (* m-th root of the n-th Bessel function Jn*)
```

```
BesselJZero[n, m]
```

```
b = 1;
```

```
f[r_, θ_] = θ * (π - θ) * r * (b - r)
```

```
(1 - r) r (π - θ) θ
```

```
orthogFn[n_, m_] := BesselJ[n, α[n, m] * r / b] * Sin[n θ]
```

```
nn = 1;
```

Check that the functions are orthogonal with respect to the weighting function r :

```
Table[
  Integrate[r * orthogFn[nn, m1] * orthogFn[nn, m2], {r, 0, b}, {θ, 0, Pi}],
  {m1, 1, 4}, {m2, 1, 4}];
```

```
MatrixForm[Chop[%]] // N
```

$$\begin{pmatrix} 0.127403 & 0. & 0. & 0. \\ 0. & 0.0707404 & 0. & 0. \\ 0. & 0. & 0.0489716 & 0. \\ 0. & 0. & 0. & 0.0374484 \end{pmatrix}$$

Yes, it' s a diagonal matrix. The functions are orthogonal !

```
coeffAA[m_, n_] := Integrate[r * orthogFn[n, m] * orthogFn[n, m],
  {r, 0, b}, {θ, 0, Pi}, Assumptions → {n, m} ∈ Integers]
```

```
coeff[m_, n_] := Integrate[r * f[r, θ] * orthogFn[n, m], {r, 0, b}, {θ, 0, Pi}] / coeffAA[m, n]
```

```
coeffAA[1, 1] // N
```

```
0.127403
```

```
checkAA = Pi * (b * BesselJ[1 + 1, α[1, 1]])^2 / 4 // N
```

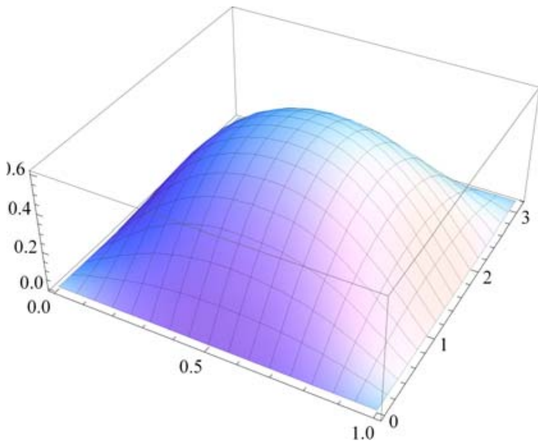
```
0.127403
```

```
Table[coeff[m, n], {n, 1, 2}, {m, 1, 2}] // N
```

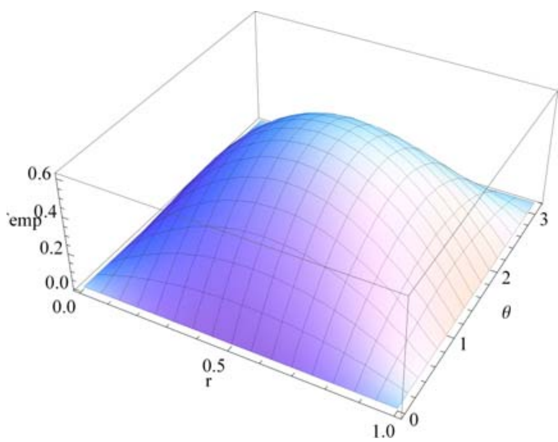
```
{{1.15156, -0.0802614}, {0., 0.}}
```

```
u[r_, θ_, t_] := Sum[Sum[coeff[m, n] * orthogFn[n, m] * Cos[α[n, m] * t / b], {m, 1, 3}], {n, 1, 3}]
```

```
Plot3D[Evaluate[u[r, θ, 0]], {r, 0, 1}, {θ, 0, Pi}]    (* Verify the initial condition *)
```

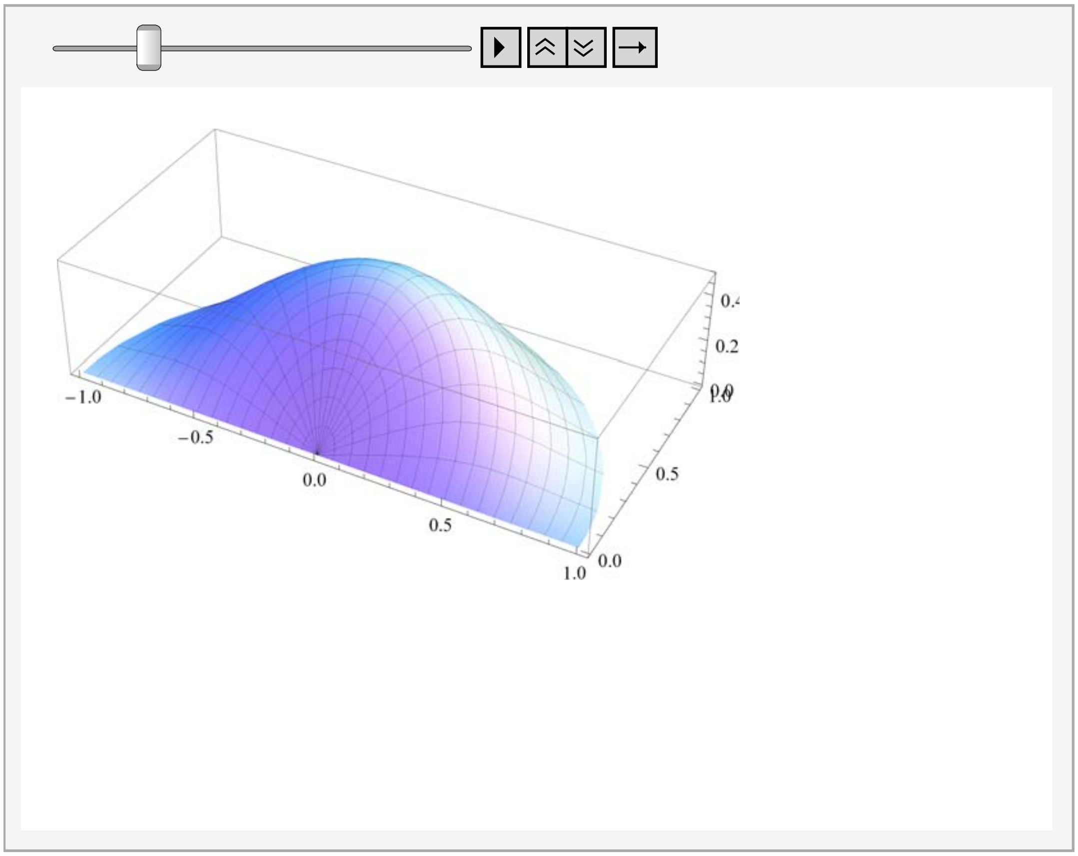



```
Plot3D[f[r, θ], {r, 0, 1}, {θ, 0, Pi}, AxesLabel -> {"r", "θ", "Temp"}]
(*    Yes, the above solution corresponds to f(r,θ)= θ*(π-θ)*r*(b-r)      *)
```




```
plot = Table[ ParametricPlot3D[
    Evaluate[{r * Cos[θ], r * Sin[θ], u[r, θ, t]}],
    {r, 0, 1}, {θ, 0, Pi}],
   {t, 0, 1, .2}];
```

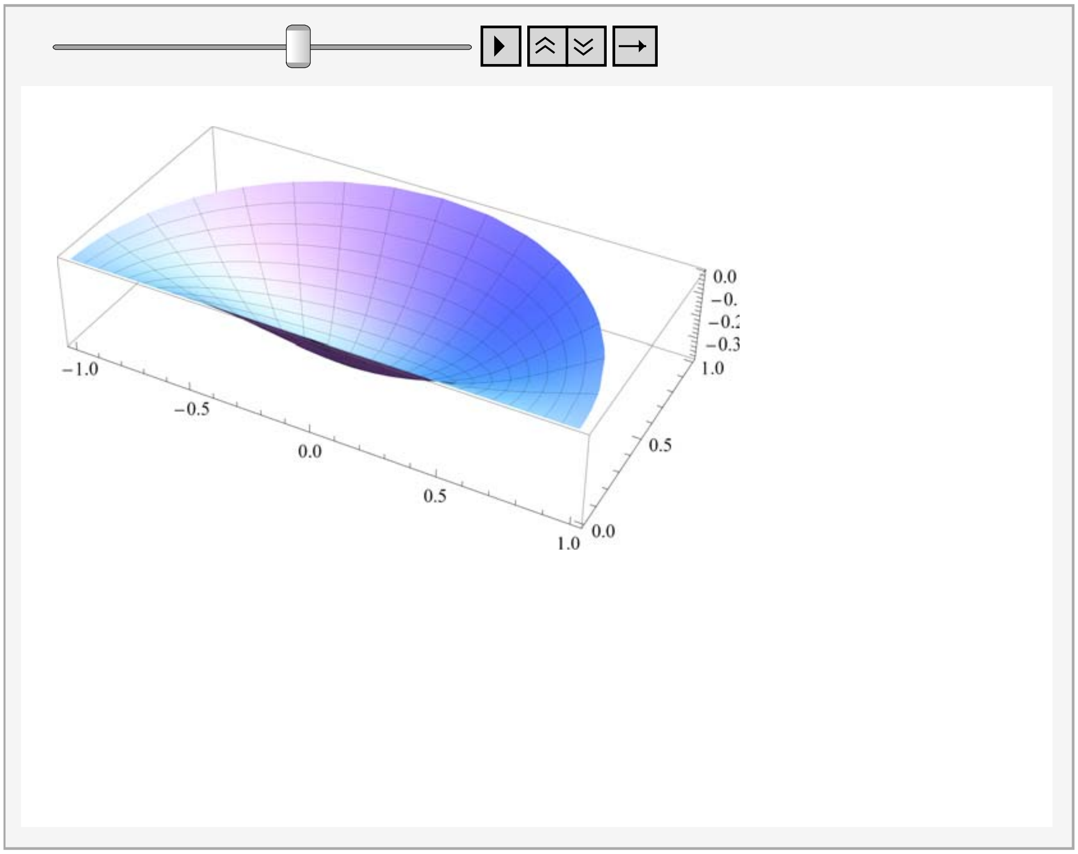
```
ListAnimate[plot]
```



Mode n=1,m=1

```
mode11 = Table[
   ParametricPlot3D[
    Evaluate[{r * Cos[θ], r * Sin[θ], orthogFn[1, 1] * Cos[α[1, 1] * t / b]}],
    {r, 0, 1}, {θ, 0, Pi}], {t, 0, 1, .2}];
```

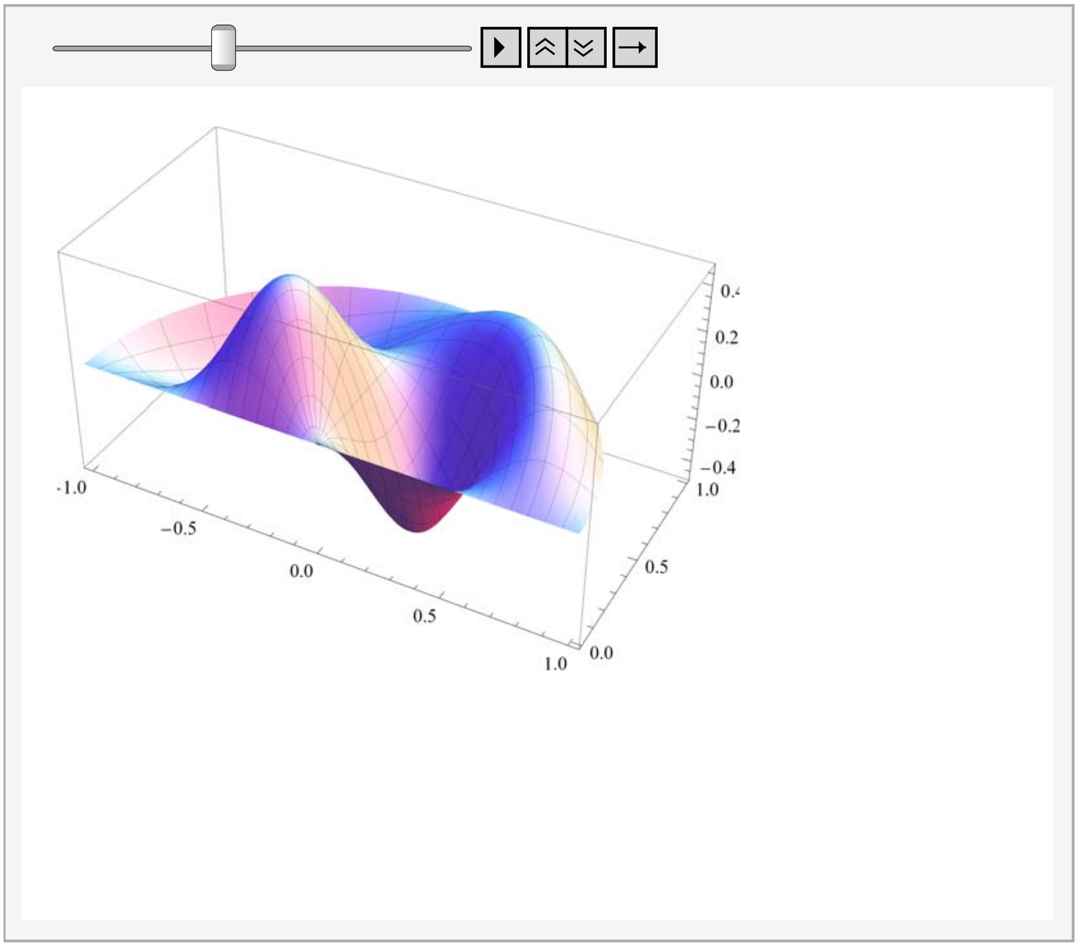
```
ListAnimate[mode11]
```



- **Mode n=2,m=2**

```
mode22 = Table[
   ParametricPlot3D[
    Evaluate[{r * Cos[θ], r * Sin[θ], orthogFn[2, 2] * Cos[α[2, 2] * t / b]}],
    {r, 0, 1}, {θ, 0, Pi}], {t, 0, 1, .2}];
```

```
ListAnimate[mode22]
```



Similarly for mode n = 1, m = 2, etc ...