ME 565, hw 5, Wi 09

13.1.19

$$\frac{1}{\bar{z}^2} = \frac{1}{(x-iy)(x-iy)} \cdot \frac{(x+iy)(x+iy)}{(x+iy)(x+iy)} = \frac{(x+iy)^2}{(x^2+y^2)^2} = \frac{x^2-y^2+2ixy}{(x^2+y^2)^2}$$

$$Re\left(\frac{1}{\bar{z}^2}\right) = \frac{x^2-y^2}{(x^2+y^2)^2}$$

13.2.24

Find and graph all roots in the complex plane

$$w = \sqrt[3]{3+4i} = \sqrt[3]{re^{i(\theta+2\pi k)}} = \sqrt[3]{r} \cdot e^{i\left(\frac{\theta}{3}+\frac{2\pi k}{3}\right)} =$$

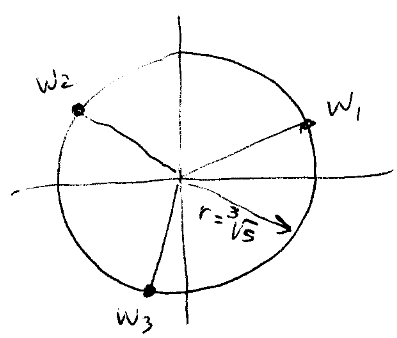
$$= \sqrt[3]{5}\left[\cos\left(\frac{\theta}{3}+\frac{2\pi k}{3}\right) + i\sin\left(\frac{\theta}{3}+\frac{2\pi k}{3}\right)\right]$$

where $\theta = \arctan\left(\frac{4}{3}\right)$

$$w_1 = \sqrt[3]{5}\left(\cos\frac{\theta}{3} + i\sin\frac{\theta}{3}\right) = \sqrt[3]{5}\,(0.9526 + i\,0.3042)$$

$$w_2 = \sqrt[3]{5}\left[\cos\left(\frac{\theta}{3}+\frac{2\pi}{3}\right) + i\sin\left(\frac{\theta}{3}+\frac{2\pi}{3}\right)\right] = \sqrt[3]{5}\,(-0.7397 + i\,0.6729)$$

$$w_3 = \sqrt[3]{5}\left[\cos\left(\frac{\theta}{3}+\frac{4\pi}{3}\right) + i\sin\left(\frac{\theta}{3}+\frac{4\pi}{3}\right)\right] = \sqrt[3]{5}\,(-0.2129 - i\,0.9771)$$



Curves and region of practical interest

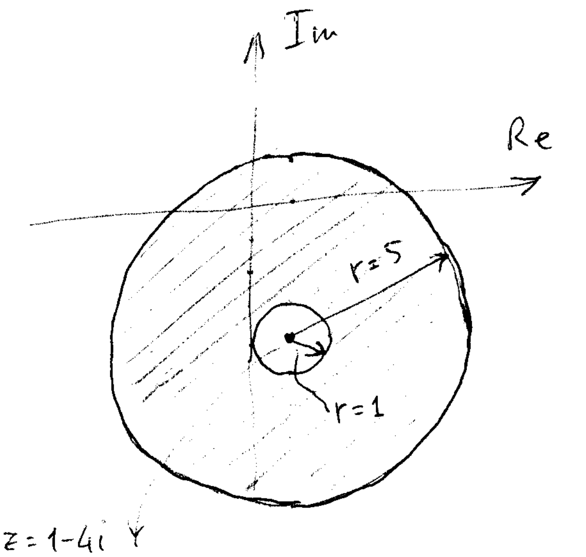
Find & sketch or graph the sets in the complex plane given by

**13.3.2** $1 \le |z-1+4i| < 5$

$1 \le |z-1+4i| \Rightarrow$ exterior of a circle centered at $1-4i$ w/ radius 1 plus the circle itself

$|z-1+4i| \le 5 \Rightarrow$ interior of a circle centered at $1-4i$ w/ radius 5 plus the circle itself

$1 \le |z-1+4i| \le 5 \Rightarrow$ area between two circles.




**13.4.17**

$u = x^3 - 3xy^2$ : is this function harmonic? if yes, find analytic function $f(z) = u(x,y) + i\,v(x,y)$

$u_x = 3x^2 - 3y^2 \to u_{xx} = 6x$

$u_y = -6xy \to u_{yy} = -6x$

$\Rightarrow \nabla^2 u = 0$ ✓ Harmonic function

From Cauchy-Riemann eq. a conjugate $v$ of $u$ must satisfy:

$v_y = u_x = 3x^2 - 3y^2$ and $v_x = -u_y = 6xy$

$v = 3x^2y - y^3 + k(x)$

$v_x = 6xy + k'(x) = 6xy \Rightarrow k'(x) = 0 \Rightarrow k = c$

$v = 3x^2y - y^3 + c$

the corresponding analytic function is

$f(z) = u + iv = (x^3 - 3xy^2) + i(3x^2y - y^3 + c)$

for $c = 0 \to f(z) = x^3 - iy^3 + i3x^2y - 3xy^2 =$

$= (x+iy)^3 = z^3$

Equations.

Find all solutions of the following equations

13.6.19 $\cos z = 2i$

Sol$^{n}$

$$\cos z = \frac{1}{2}(e^{iz}+e^{-iz}) = 2i$$

$$e^{iz}+e^{-iz} = 4i$$

$$e^{2iz} - 4ie^{iz} + 1 = 0$$

let $m = e^{iz}$; $\quad m^2 - 4im + 1 = 0$

$$m = \frac{4i \pm \sqrt{(-4i)^2 - 4\cdot1\cdot1}}{2\cdot1} = \frac{4i \pm \sqrt{-20}}{2} = 2i \pm \sqrt{5}\,i$$

$$\therefore \quad e^{iz} = e^{i(x+iy)} = e^{-y+ix} = 2i \pm \sqrt{5}\,i$$

$$e^{-y}(\cos x + i\sin x) = (2 \pm \sqrt{5})\,i$$

$$e^{-y}\cos x = 0 \qquad \text{and} \qquad e^{-y}\sin x = 2 \pm \sqrt{5}$$

Since $e^{-y} \neq 0$ (unless $y = \infty$) $\Rightarrow \cos x = 0 \Rightarrow \underline{x = (2n+1)\frac{\pi}{2}}$, $n = 0, 1, 2, \dots$

$$e^{-y}\sin x = 4.236,\ -0.236$$

If $x = (2n+1)\frac{\pi}{2}$, $\sin x = \begin{cases} 1, & n \text{ even} \\ -1, & n \text{ odd} \end{cases}$

n-even, $e^{-y}\,\cancel{\sin x}^{\,1} = 4.236,\ -0.236$

$$-y = \ln(4.236),\ \ln(\cancel{-0.236}) \quad \text{not define}$$

$$y = -1.4436 \quad *$$

n-odd, $e^{-y}\,\cancel{\sin x}^{\,-1} = 4.236,\ -0.236$

$$-y = \ln(\cancel{-4.236}),\ \ln(0.236) \quad \text{(not define)}$$

$$y = 1.4436 \quad *$$

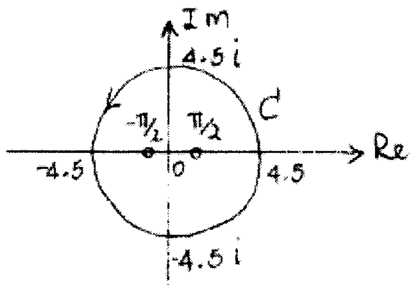
$\therefore$ for $\cos z = 2i$, $z = x + iy = \frac{(2n+1)\pi}{2} - (-1)^n\, 1.4436\, i$, $n = 0, 1, 2, \dots$ Ans

## Residue integration

Evaluate (counterclockwise).

16 3.19 $\quad \oint_C \frac{e^z}{\cos z}\,dz \;, \quad C: |z| = 4.5$

**Sol$^n$**




$\frac{e^z}{\cos z}$ has two singular points inside the integration path $C: |z| = 4.5$ at $z = \frac{\pi}{2}, -\frac{\pi}{2}$.

From the Residue Theorem (p. 715),

$$\oint_C f(z)\,dz = 2\pi i \sum_{j=1}^{k} \operatorname*{Res}_{z=z_j} f(z)$$

$$\oint_C \frac{e^z}{\cos z}\,dz = 2\pi i \left[ \operatorname*{Res}_{z=\frac{\pi}{2}} \left(\frac{e^z}{\cos z}\right) + \operatorname*{Res}_{z=-\frac{\pi}{2}} \left(\frac{e^z}{\cos z}\right) \right]$$

$$= 2\pi i \left[ \frac{e^{\pi/2}}{-\sin(\pi/2)} + \frac{e^{-\pi/2}}{-\sin(-\pi/2)} \right]$$

$$= 2\pi i \left( e^{-\pi/2} - e^{\pi/2} \right)$$

$$= 2\pi i \left( -2\sinh\left(\frac{\pi}{2}\right) \right)$$

$$= -4\pi i \sinh\left(\frac{\pi}{2}\right) \qquad \text{Ans}$$

## General Powers

Showing the details of your work, find the principal value of

**13.7.25** $(1+i)^{1-i}$

**Sol$^{n}$** from Eq. 7 p 632 ; $z^c = e^{c \ln z}$

Thus $(1+i)^{1-i} = \exp\left[(1-i)\ln(1+i)\right]$

For principal value, $(1+i)^{1-i} = \exp\left[(1-i)\,\mathrm{Ln}(1+i)\right]$

$= \exp\left[(1-i)\left\{\ln\sqrt{2} + \frac{\pi}{4}i\right\}\right]$

Note that $\mathrm{Ln}\, z = \ln r + i\,\mathrm{Arg}\, z$ , $r = \sqrt{x^2+y^2}$ and $\mathrm{Arg}\, z = \arctan\frac{y}{x}$

Here $z = 1+i$ , $r = \sqrt{1^2+1^2}$ , $\mathrm{Arg}\, z = \arctan\frac{1}{1} = \frac{\pi}{4}$

$$(1+i)^{1-i} = \exp\left[\ln\sqrt{2} + \frac{\pi}{4} - i\ln\sqrt{2} + i\frac{\pi}{4}\right]$$

$$= e^{\ln\sqrt{2}} \cdot e^{\pi/4} \cdot e^{(\pi/4 - \ln\sqrt{2})i}$$

$$= \sqrt{2}\, e^{\pi/4}\left[\cos\left(\frac{\pi}{4} - \frac{1}{2}\ln 2\right) + i\sin\left(\frac{\pi}{4} - \frac{1}{2}\ln 2\right)\right]$$

(14.2.7 next page!)

$$= 2.808 + 1.318\, i$$ **Ans**

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$\boxed{14.2.14}$ For what contour C will it follow from Cauchy's integr. thm. that:

a) $\oint_C \frac{dz}{z} = 0$ : singularity $z=0$
C : anywhere except $z=0$

b) $\oint \frac{\cos z}{z^6 - z^2}\, dz = 0$ : C anywhere except for $z = 0, \pm 1, \pm i$

e) $\oint \frac{e^{1/z}}{z^2+9}\, dz = 0$ : C anywhere except for $z = \pm 3i$
and $z = 0$ (for $e^{1/z}$)

Cauchy's integral theorem applicable?

Integrate $f(z)$ counterclockwise around the unit circle, indicating whether Cauchy's integral theorem applies.

**14.2.7** $\quad f(z) = \dfrac{1}{(z^8 - 1.2)}$

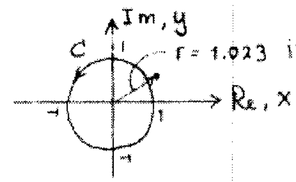
Sol$^n$ $\quad$ Since $f(z)$ is not analytic when $z^8 - 1.2 = 0$ or

$$z^8 = 1.2 = 1.2\,(\cos 0 + i \sin 0)$$

$$z = \sqrt[8]{1.2}\left(\cos \frac{2k\pi}{8} + i \sin \frac{2k\pi}{8}\right), \quad k = 0, 1, \ldots, 7 \quad \text{(See p.611)}$$

$$= 1.023\left(\cos \frac{2k\pi}{8} + i \sin \frac{2k\pi}{8}\right), \quad k = 0, 1, \ldots, 7$$

Since $f(z)$ is integrated around the unit circle (centered at $(0,0)$), thus, all $z$ value that makes $f(z)$ not analytical are outside the integrating domain. From Theorem 1 (p.640)




is outside the unit circle.

$$\int_{z_0}^{z_1} f(z)\,dz = F(z_1) - F(z_0)$$

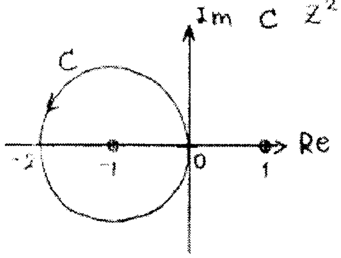
$$\therefore \oint_{c:\text{ unit circle}} f(z)\,dz = \oint_C \frac{1}{(z^8 - 1.2)}\,dz = \underbrace{F(z_0) - F(z_0)}_{\text{starting \& ending points are the same for } \oint_c} = 0 \quad \underline{\text{Ans}}$$

$\oint_C \dfrac{1}{z^8 - 1.2}\,dz = 0$ and $\dfrac{1}{z^8 - 1.2}$ is analyticle in the integrating domain. Therefore, Cauchy's integral theorem applies. $\quad$ Ans

## Contour integration

Using Cauchy's integral formula, integrate counterclockwise.

**14.3.9** $\oint_C \frac{dz}{z^2-1}$ , $C: |z+1| = 1$




$\frac{1}{z^2-1} = \frac{1}{(z-1)(z+1)}$ is not analytical at $z = \pm 1$

However, only $z = -1$ is enclose in C.

$$\oint_C \frac{dz}{z^2-1} = \oint_C \frac{1/(z-1)}{z+1}\,dz = 2\pi i \cdot \frac{1}{(-1-1)}$$ ; Cauchy's integral formula

$$= -\pi i$$ Ans

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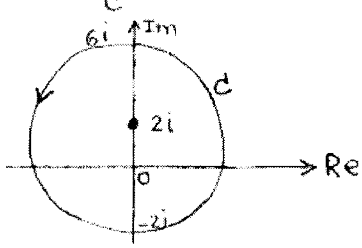
**14.3.10** $\oint_C \frac{e^z}{z-2i}\,dz$ , $C: |z-2i| = 4$

Sol$^n$




$\frac{e^z}{z-2i}$ is not analytical at $z = 2i$ which lies inside of C.

$$\oint_C \frac{e^z}{z-2i}\,dz = 2\pi i \cdot e^{2i}$$ Cauchy's integral formula

$$= 2\pi i(\cos 2 + i \sin 2)$$

$$= -2\pi \sin 2 + 2\pi i \cos 2$$

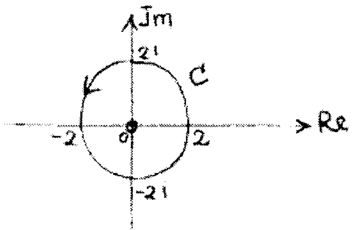
$$= -5.7133 - 2.6147 i$$ Ans

## Contour integration

Integrate counterclockwise around the circle $|z| = 2$.

**14.4.1** $\dfrac{\cosh 3z}{z^5}$

**Sol$^n$**




$\dfrac{\cosh 3z}{z^5}$ is not analytical at $z=0$ which lies inside of C.

By theorem 1 : Derivative of an analytic function

$$\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz = \frac{2\pi i}{n!} f^{(n)}(z_0)$$

$$\therefore \oint_C \frac{\cosh 3z}{z^5}\,dz = \frac{2\pi i}{4!}\left[\cosh^{(4)}(3z)\right]_{z=0}$$

$$= \frac{\cancelto{1}{2}\pi i}{4\cdot\cancel{3}\cdot\cancelto{1}{2}\cdot 1}\cdot\cancelto{27}{81}\,\cancelto{1}{\cosh}(3\cdot 0) \quad ; \quad \cosh^{(4)}(3z) = 81\cosh(3z)$$

$$= \frac{27\pi i}{4} \qquad \underline{\underline{\text{Ans}}}$$

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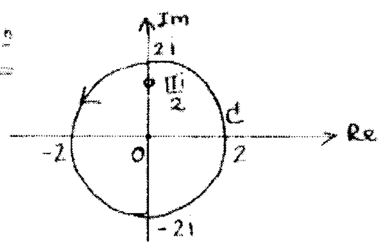
**14.4.2** $\dfrac{\sin z}{(z-\pi i/2)^4}$

**Sol$^n$**




$\dfrac{\sin z}{(z-\pi i/2)^4}$ is not analytical at $z = \dfrac{\pi i}{2}$ which is inside C.

By theorem 1 : Derivative of an analytic function

$$\oint_C \frac{\sin z}{(z-\pi i/2)^4}\,dz = \frac{2\pi i}{3!}\left[\sin''' z\right]_{z=\pi i/2}$$

$$= \frac{\cancel{2}\pi i}{3\cdot\cancel{2}\cdot 1}\left(-\cos\left(\frac{\pi i}{2}\right)\right)$$

$$= -\frac{\pi i}{3}\cosh\frac{\pi}{2} \qquad \underline{\underline{\text{Ans}}}$$

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15.4.6 $\frac{1}{z}, 1$

Sol$^n$

$$\frac{1}{z} = \frac{1}{1-(-z+1)}$$

$$= \sum_{n=0}^{\infty} (-z+1)^n \qquad \text{Geometric Series ( p. 687)}$$

$$= 1 + (-z+1) + (-z+1)^2 + (-z+1)^3 + \cdots$$

$$= 1 - (z-1) + [-(z-1)]^2 + [-(z-1)]^3 + \cdots$$

$$= 1 - (z-1) + (z-1)^2 - (z-1)^3 + \cdots \qquad \text{Ans}$$

The Radius of Convergence, $R = 1$, which is the distance from $z_0 = 1$ to the nearest singular point $z = 0$. Ans

**15.4.7** $\quad \frac{1}{1-z}$ , $i$

<u>Sol$^n$</u>

$$\frac{1}{1-z} = \frac{1}{1-i-(z-i)} = \frac{1}{(1-i)\left(1-\frac{z-i}{1-i}\right)}$$

$$= \frac{1}{1-i}\sum_{n=0}^{\infty}\left(\frac{z-i}{1-i}\right)^n$$

$$= \frac{1+i}{(1-i)(1+i)}\sum_{n=0}^{\infty}\left(\frac{z-i}{1-i}\right)^n = \frac{(1+i)}{2}\sum_{n=0}^{\infty}\left(\frac{z-i}{1-i}\right)^n$$

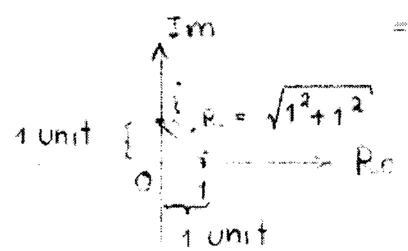
$$= \frac{(1+i)}{2}\left[1+\left(\frac{z-i}{1-i}\right)+\left(\frac{z-i}{1-i}\right)^2+\left(\frac{z-i}{1-i}\right)^3+\cdots\right]$$

$$= \frac{1}{2}+\frac{i}{2}+\frac{(1+i)}{2}\underbrace{\frac{(z-i)}{(1-i)}\cdot\frac{(1+i)}{(1+i)}}_{(1-i)(1+i)=2}+\frac{(1+i)}{2}\underbrace{\frac{(z-i)^2(1+i)^2}{(1-i)^2(1+i)^2}}_{(1-i)^2(1+i)^2=4}+\cdots$$

Since $(1+i)^2 = 1^2+2i+i^2 = 2i$

$$\frac{1}{1-z} = \frac{1}{2}+\frac{i}{2}+\frac{\cancel{2}i}{\cancel{4}_{2}}(z-i)+\frac{(1+i)\cancel{2}i}{\cancel{2}_{1}}\frac{(z-i)^2}{4}+\cdots$$

$$= \frac{1}{2}+\frac{i}{2}+\frac{i}{2}(z-i)+\left(\frac{i}{4}-\frac{1}{4}\right)(z-i)^2+\cdots \quad \underline{\underline{\text{Ans}}}$$




The nearest singular point from $z_0 = i$ is $z = 1$. Thus, the Radius of Convergence, $R = \sqrt{2}$. ( $R = \sqrt{1^2+1^2}$ ) $\underline{\underline{\text{Ans}}}$

## Laurent series near a singularity at 0

Expand the given function in a Laurent series that converges for $0 < |z| < R$ and determine the precise region of convergence.

**16.1.1**

$$\frac{1}{z^4 - z^5} = \frac{1}{z^4(1-z)}$$

$$= \sum_{n=0}^{\infty} z^{n-4} \qquad \text{Geometric series } |z| < 1$$

$$= \frac{1}{z^4} + \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{z} + 1 + z + z^2 + \cdots \qquad \text{Ans}$$

$\frac{1}{z^4 - z^5}$ has singular points at $z = 0, 1$. For a Laurent series that converges for $0 < |z| < R$, we got $R = 1$ (distance from 0 to the nearest singular point.) Ans

## Taylor and Laurent series

Find all Taylor and Laurent series with center $z = z_0$ and determine the precise regions of convergence

**16.1.15** $\frac{1}{1-z^3}$, $z_0 = 0$

**Sol^n**

$$\frac{1}{1-z^3} = \sum_{n=0}^{\infty} z^{3n}, \quad |z| < 1; \text{Geometric series}$$

$$\frac{1}{1-z^3} = \frac{-1}{z^3\left(1 - \frac{1}{z^3}\right)} \qquad \text{use Geometric series}$$

$$= -\sum_{n=0}^{\infty} \frac{1}{z^3} \cdot z^{-3n} \; ; \quad \left|\frac{1}{z^3}\right| < 1 \text{ or } |z| > 1$$

$$= -\sum_{n=0}^{\infty} \frac{1}{z^{3n+3}} \; ; \quad |z| > 1 \qquad \text{Ans}$$

Find all the singular points and the corresponding residues.

**16.3.3** $\dfrac{1}{4+z^2}$

**Sol$^n$** $\dfrac{1}{4+z^2} = \dfrac{1}{(z-2i)(z+2i)}$

Thus, $\dfrac{1}{4+z^2}$ has singular points at $z = 2i, -2i$ **Ans**

$$\operatorname*{Res}_{z=2i}\left(\frac{1}{4+z^2}\right) = \left[\frac{1}{z+2i}\right]_{z=2i} = \frac{1}{4i} = \frac{-i}{4} \quad \textbf{Ans}$$

$$\operatorname*{Res}_{z=-2i}\left(\frac{1}{4+z^2}\right) = \left[\frac{1}{z-2i}\right]_{z=-2i} = \frac{1}{-4i} = \frac{i}{4} \quad \textbf{Ans}$$

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**16.3.9** $\dfrac{1}{(z^2-1)^2}$

**Sol$^n$** $\dfrac{1}{(z^2-1)^2} = \dfrac{1}{(z^2-1)(z^2-1)} = \dfrac{1}{(z-1)^2(z+1)^2}$

Thus, $\dfrac{1}{(z^2-1)^2}$ has singular points at $z = 1, -1$. **Ans**

$$\operatorname*{Res}_{z=1}\left[\frac{1}{(z^2-1)^2}\right] = \lim_{z\to 1}\frac{d}{dz}\left[(z-1)^2\cdot\frac{1}{(z^2-1)^2}\right]$$

$$= \lim_{z\to 1}\frac{d}{dz}\left[\frac{1}{(z+1)^2}\right] = \lim_{z\to 1}\left[\frac{-2}{(z+1)^3}\right]$$

$$= \frac{-1}{4} \quad \textbf{Ans}$$

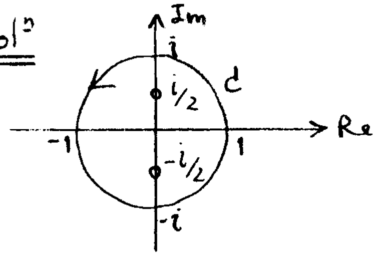
$$\operatorname*{Res}_{z=-1}\left[\frac{1}{(z^2-1)^2}\right] = \lim_{z\to -1}\frac{d}{dz}\left[(z+1)^2\cdot\frac{1}{(z^2-1)^2}\right]$$

$$= \lim_{z\to -1}\frac{d}{dz}\left[\frac{1}{(z-1)^2}\right] = \lim_{z\to -1}\left[\frac{-2}{(z-1)^3}\right]$$

$$= \frac{1}{4} \quad \textbf{Ans}$$

16.3.24 $\displaystyle\oint_C \frac{1-4z+6z^2}{(z^2+1/4)(2-z)}\,dz$ , $C: |z| = 1$

Sol$^n$




$$\frac{1-4z+6z^2}{(z^2+1/4)(2-z)} = \frac{1-4z+6z^2}{(z-\frac{i}{2})(z+\frac{i}{2})(2-z)}$$

Thus, $\dfrac{1-4z+6z^2}{(z^2+\frac{1}{4})(2-z)}$ has three singular points at $z = \frac{i}{2}, \frac{-i}{2}, 2$. However, only $z = \frac{i}{2}, -\frac{i}{2}$ are inside the integrating path $C$.

From Residue theorem

$$\oint_C \frac{1-4z+6z^2}{(z^2+\frac{1}{4})(2-z)}\,dz = 2\pi i\left[\operatorname*{Res}_{z=\frac{i}{2}}\left[\frac{1-4z+6z^2}{(z^2+\frac{1}{4})(2-z)}\right] + \operatorname*{Res}_{z=-\frac{i}{2}}\left[\frac{1-4z+6z^2}{(z^2+\frac{1}{4})(2-z)}\right]\right]$$

$$= 2\pi i\left[\left(\frac{1-4z+6z^2}{(z+\frac{i}{2})(2-z)}\right)_{z=\frac{i}{2}} + \left(\frac{1-4z+6z^2}{(z-\frac{i}{2})(2-z)}\right)_{z=-\frac{i}{2}}\right]$$

$$= 2\pi i\left[\frac{2-4i-3}{4i+1} + \frac{2+4i-3}{-4i+1}\right]$$

$$= 2\pi i\left[\frac{-4i-1}{4i+1} + \frac{4i-1}{-4i+1}\right]$$

$$= 2\pi i\left[-\frac{(\cancel{4i+1})}{\cancel{4i+1}} + \frac{\cancel{4i-1}}{-(\cancel{4i-1})}\right] = 2\pi i\left[-1-1\right]$$

$$= -4\pi i \qquad \underline{\underline{\text{Ans}}}$$

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## Integral involving cosine and sine

Evaluate the following integrals

16.4.1 $\displaystyle\int_0^{2\pi} \frac{d\theta}{7+6\cos\theta}$

Sol$^n$

$$\int_0^{2\pi}\frac{d\theta}{7+6\cos\theta} = \oint_{C:|z|=1}\frac{dz/iz}{7+6\cdot\frac{1}{2}\left(z+\frac{1}{z}\right)} \quad ; \text{(see p. 718)}$$

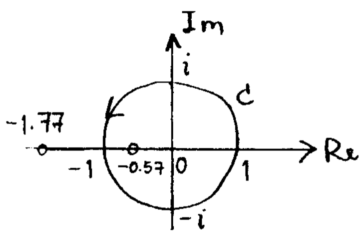
$$= \oint_C \frac{dz}{i\cancel{z}\left[\frac{7z+3z^2+3}{\cancel{z}}\right]}$$

$$= \frac{1}{i}\oint_C \frac{dz}{3(z^2+\frac{7}{3}z+1)}$$

$$\int_0^{2\pi} \frac{d\theta}{7+6\cos\theta} = \frac{1}{3i}\oint_C \frac{dz}{z^2+\frac{7}{3}z+1}$$

$\dfrac{1}{z^2+\frac{7}{3}z+1}$ has singular points at $z = -\frac{7}{6}+\frac{\sqrt{13}}{6}$, $-\frac{7}{6}-\frac{\sqrt{13}}{6}$ (points where $z^2+\frac{7}{3}z+1=0$). Only $z = \frac{-7}{6}+\frac{\sqrt{13}}{6} \approx -0.566$ is inside the integration path, $C: |z|=1$. Thus, from the Residue theorem,

$$\int_0^{2\pi} \frac{d\theta}{7+6\cos\theta} = \frac{1}{3i}\left[2\pi i \operatorname*{Res}_{z=-\frac{7}{6}+\frac{\sqrt{13}}{6}}\left(\frac{1}{z^2+\frac{7}{3}z+1}\right)\right]$$




$$= \frac{2\pi}{3}\left[\frac{1}{z-\left(-\frac{7}{6}-\frac{\sqrt{13}}{6}\right)}\right]_{z=-\frac{7}{6}+\frac{\sqrt{13}}{6}}$$

$$= \frac{2\pi}{3}\left[\frac{1}{\cancel{-\frac{7}{6}}+\frac{\sqrt{13}}{6}+\cancel{\frac{7}{6}}+\frac{\sqrt{13}}{6}}\right]$$

$$= \frac{\cancel{2}\pi}{3}\cdot\frac{6}{\cancel{2}\sqrt{13}} = \frac{2\pi}{\sqrt{13}} \qquad \underline{\underline{\text{Ans}}}$$

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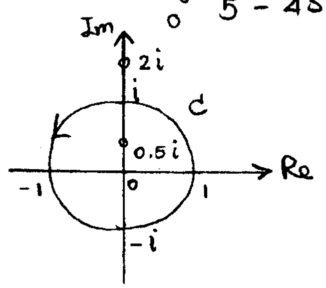
**16.4.5** $\displaystyle\int_0^{2\pi} \frac{d\theta}{5-4\sin\theta}$

Sol$^n$

$$\int_0^{2\pi} \frac{d\theta}{5-4\sin\theta} = \oint_{C:|z|=1} \frac{dz/iz}{5-4\cdot\frac{1}{2i}\left(z-\frac{1}{z}\right)}$$




$$= \oint_C \frac{dz}{\cancel{iz}\left[\frac{5zi-2z^2+2}{\cancel{zi}}\right]}$$

$$= \oint_C \frac{dz}{-2\left(z^2-\frac{5}{2}zi-1\right)} = \frac{-1}{2}\oint_C \frac{dz}{z^2-\frac{5}{2}iz-1}$$

$\dfrac{1}{z^2-\frac{5}{2}iz-1}$ has singular points at $z = 2i, 0.5i$ and only $z=0.5i$ lies in the integration path, $C: |z|=1$. From the Residue theorem,

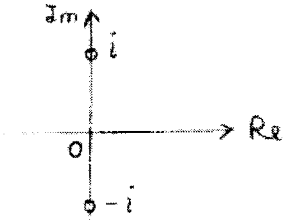
$$\int_0^{2\pi} \frac{d\theta}{5-4\sin\theta} = \frac{-1}{2}\left[2\pi i \operatorname*{Res}_{z=0.5i}\left(\frac{1}{z^2-\frac{5}{2}iz-1}\right)\right]$$

$$= -\pi i\left[\frac{1}{z-2i}\right]_{z=0.5i} = -\pi i\left[\frac{1}{0.5i-2i}\right]$$

$$= \cancel{-}\pi \cancel{i}\cdot\frac{1}{\cancel{-}1.5\cancel{i}} = \frac{2\pi}{3} \qquad \underline{\underline{\text{Ans}}}$$

**16.4.9** $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^2+1}$

**Sol$^n$** let $f(z) = \dfrac{1}{z^2+1}$. $f(z)$ has two simple poles at $z_1 = i$, $z_2 = -i$. Only $z_1 = i$ is in the upper half plane. Since




$f(z)$ does not have poles on the real axis and its denominator is of degree two units higher than the degree of the numerator, we can use Eq. (7) on p. 720 to evaluate the given integral, i.e,

$$\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \sum \operatorname{Res} f(z)$$ ; Res $f(z)$ here accounts for Res $f(z)$ on the upper plane.

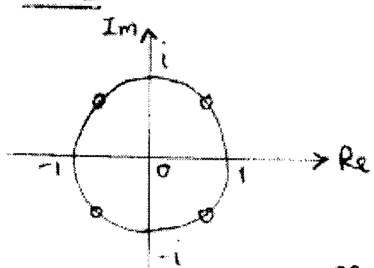
$$\int_{-\infty}^{\infty} \frac{1}{x^2+1}\,dx = 2\pi i \left[ \operatorname*{Res}_{z=i} \frac{1}{z^2+1} \right]$$

$$= 2\pi i \left[ \frac{1}{z+i} \right]_{z=i}$$

$$= 2\pi i \left( \frac{1}{2i} \right) = \pi$$ **Ans**

**16.1.20** $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x}{x^4+1}\,dx$

**Sol$^n$** let $f(z) = \dfrac{1}{z^4+1}$. $f(z)$ has 4 simple poles at




$z = e^{\pi i/4}, e^{3\pi i/4}, e^{5\pi i/4}, e^{7\pi i/4}$

Only $z = e^{\pi i/4}, e^{3\pi i/4}$ are in the upper plane

From Eq. (10) p. 721,

$$\int_{-\infty}^{\infty} f(x)\cos sx\,dx = -2\pi \sum \operatorname{Im} \operatorname{Res} \left[ f(z) e^{isz} \right]$$

Thus, $$\int_{-\infty}^{\infty} \frac{\cos x}{x^4+1}\,dx = -2\pi \sum \operatorname{Im} \operatorname{Res} \left[ \frac{e^{iz}}{z^4+1} \right]$$

$$= -2\pi \left[ \operatorname{Im} \left[ \operatorname*{Res}_{z=e^{\pi i/4}} \frac{e^{iz}}{z^4+1} \right] + \operatorname{Im} \left[ \operatorname*{Res}_{z=e^{3\pi i/4}} \frac{e^{iz}}{z^4+1} \right] \right]$$

$$= -2\pi \left[ \operatorname{Im} \left( \frac{e^{iz}}{4z^3} \right)_{z=e^{\pi i/4}} + \operatorname{Im} \left( \frac{e^{iz}}{4z^3} \right)_{z=e^{3\pi i/4}} \right]$$

$$\text{Conside}\left(\frac{e^{iz}}{4z^3}\right)_{z=e^{\pi i/4}} = \frac{e^{ie^{\pi i/4}}}{4e^{3\pi i/4}} = \frac{e^{ie^{\pi i/4}}}{4\underbrace{e^{\pi i}}_{=-1}e^{-\frac{\pi i}{4}}}$$

$$= -\frac{1}{4}e^{i(1+i)/\sqrt{2}}\cdot\frac{(1+i)}{\sqrt{2}} \quad ; \quad e^{\pi i/4} = \frac{1+i}{\sqrt{2}}$$

$$= -\frac{1}{4}e^{i/\sqrt{2}-1/\sqrt{2}}\frac{(1+i)}{\sqrt{2}}$$

$$= -\frac{1}{4}e^{-1/\sqrt{2}}\left(\cos\frac{1}{\sqrt{2}}+i\sin\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right)$$

$$\operatorname{Im}\left(\frac{e^{iz}}{4z^3}\right)_{z=e^{\pi i/4}} = -\frac{1}{4}e^{-1/\sqrt{2}}\left[\frac{1}{\sqrt{2}}\left(\sin\frac{1}{\sqrt{2}}+\cos\frac{1}{\sqrt{2}}\right)\right] \quad \text{—}(*)$$

$$\text{For}\left(\frac{e^{iz}}{4z^3}\right)_{z=e^{3\pi i/4}} = \frac{e^{ie^{3\pi i/4}}}{4e^{9\pi i/4}} = \frac{e^{i\overset{-1}{\cancel{e^{\pi i}}}\cdot e^{-\pi i/4}}}{4\underbrace{e^{2\pi i}}_{=1}\cdot e^{\pi i/4}}$$

$$= \frac{e^{-i(1-i)/\sqrt{2}}}{4}\cdot\frac{(1-i)}{\sqrt{2}} \quad ; \quad e^{-\pi i/4} = \frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}}$$

$$= \frac{1}{4}e^{-i/\sqrt{2}}\cdot e^{-1/\sqrt{2}}\cdot\frac{(1-i)}{\sqrt{2}}$$

$$= \frac{1}{4}e^{-1/\sqrt{2}}\left(\cos\frac{1}{\sqrt{2}}-i\sin\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}}\right)$$

$$\operatorname{Im}\left(\frac{e^{iz}}{4z^3}\right)_{z=e^{3\pi i/4}} = \frac{1}{4}e^{-1/\sqrt{2}}\left[-\frac{1}{\sqrt{2}}\left(\sin\frac{1}{\sqrt{2}}+\cos\frac{1}{\sqrt{2}}\right)\right]$$

$$= -\frac{1}{4}e^{-1/\sqrt{2}}\left[\frac{1}{\sqrt{2}}\left(\sin\frac{1}{\sqrt{2}}+\cos\frac{1}{\sqrt{2}}\right)\right] \quad \text{—}(**)$$

$$\therefore \int_{-\infty}^{\infty}\frac{\cos x}{x^4+1}\,dx = -\cancel{2}\pi\left\{-\frac{e^{-1/\sqrt{2}}}{\cancel{4}\sqrt{2}}\left(\cancel{2}\sin\frac{1}{\sqrt{2}}+2\cos\frac{1}{\sqrt{2}}\right)\right\}$$

$$= \frac{\pi e^{-1/\sqrt{2}}}{\sqrt{2}}\left(\sin\frac{1}{\sqrt{2}}+\cos\frac{1}{\sqrt{2}}\right)$$

$$= 1.544$$

Ans