**Mechanical Engineering 230—Dynamics**
Midterm Exam
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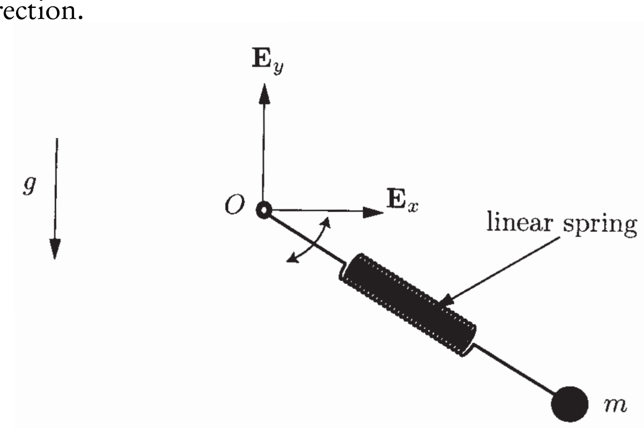
Directions: 50 minutes, calculator, open notes, closed book. Use your own paper, work neatly, and clearly mark your answers. Partial credit may be given. Do no work on this sheet.

**Problem 1** (30 points)**.** The figure shows a particle of mass $m$ attached to a fixed point $O$ by a linearly elastic spring. The spring has stiffness $K$ and unstretched length $L$. The motion of the particle is in the $\mathbf{E}_x - \mathbf{E}_y$ plane. A vertical gravitational force acts on the particle in the *negative* $\mathbf{E}_y$-direction.




**(a)** Starting from the standard representations for the position vector

$$\mathbf{r} = x\mathbf{E}_x + y\mathbf{E}_y = r\mathbf{e}_r \tag{1}$$

for the particle, derive expressions for the velocity $\mathbf{v}$ and acceleration $\mathbf{a}$ vectors in the given Cartesian *and* the standard polar coordinate bases.
Hint (no need to derive): $\dot{\mathbf{e}}_r = \dot{\theta}\mathbf{e}_\theta$ and $\dot{\mathbf{e}}_\theta = -\dot{\theta}\mathbf{e}_r$.

**(b)** Draw a free-body diagram of the particle and write what is known about each force acting on the particle.

**(c)** Show that the scalar differential equations governing the motion of the particle are, in polar coordinates,

$$-K(r-L) - mg\sin\theta = m\left(\ddot{r} - r\dot{\theta}^2\right) \tag{2a}$$
$$-mg\cos\theta = m\left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right). \tag{2b}$$

**(d)** What law of motion did you use to find (2a) and (2b)?

**(e)** What must $r$ and $\theta$ be if the particle is at rest?

**Problem 2** (20 points)**.** Give short descriptions (one-to-two sentences) of the following.

**(a)** particles

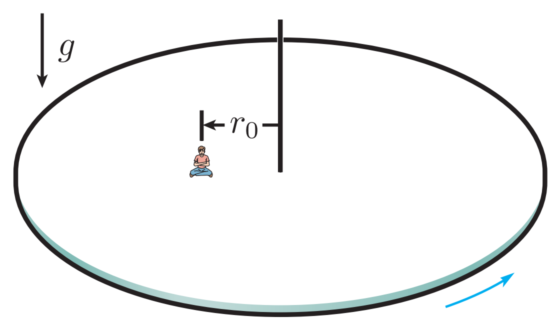
**(b)** force

**(c)** (coordinate) bases

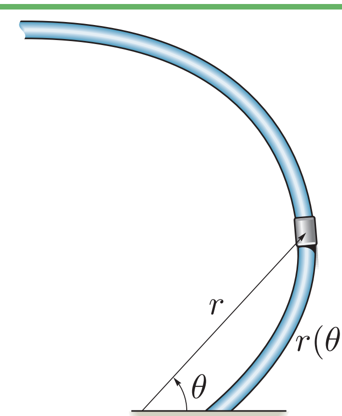
**(d)** the Sennet-Frenet basis

**(e)** friction forces

**Problem 3** (30 points)**.** The figure shows a boy of mass $m$ sitting on a disk, a distance $r_0$ from its center. If the disk, starting from rest, begins spinning such that the boy's speed $v$ increases at a constant rate $\dot{v} = a_0$, how long until the boy slips? The coefficient of static friction between the boy and the disc is $\mu_s$. Write your answer in terms of $m$, $r_0$, $a_0$, $\mu_s$, and the gravitational constant $g$. Does $a_0$ have a maximum above which the boy slips immediately?
If not, explain; if so, what is it?




**Problem 4** (20 points)**.** The figure shows a collar with mass, sliding on a rail with known shape described by $r(\theta)$. Explain in detail *how* you would find the equation of motion of the particle. Describe *why* you would take each step. Do not actually perform the work, just describe what you *would* do.



## Solutions

### Solution to Problem 1.

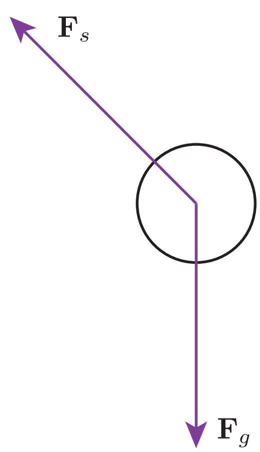
**(a)** In the Cartesian basis:

$$\mathbf{r} = x\mathbf{E}_x + y\mathbf{E}_y$$
$$\mathbf{v} = \dot{\mathbf{r}} = \dot{x}\mathbf{E}_x + \dot{y}\mathbf{E}_y$$
$$\mathbf{a} = \dot{\mathbf{v}} = \ddot{x}\mathbf{E}_x + \ddot{y}\mathbf{E}_y.$$

In the polar basis:

$$\begin{aligned}
\mathbf{r} &= r\mathbf{e}_r \\
\mathbf{v} &= \dot{\mathbf{r}} \\
&= \dot{r}\mathbf{e}_r + r\dot{\mathbf{e}}_r \\
&= \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta \\
\mathbf{a} &= \dot{\mathbf{v}} \\
&= \ddot{r}\mathbf{e}_r + \dot{r}\dot{\theta}\mathbf{e}_\theta + \dot{r}\dot{\theta}\mathbf{e}_\theta + r\ddot{\theta}\mathbf{e}_\theta + r\dot{\theta}\dot{\mathbf{e}}_\theta \\
&= \left(\ddot{r} - r\dot{\theta}^2\right)\mathbf{e}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\mathbf{e}_\theta.
\end{aligned}$$

**(b)** The free-body diagram is:




The gravity force is

$$\begin{aligned}
\mathbf{F}_g &= -mg\mathbf{E}_y && (3)\\
&= -mg(\sin(\theta)\mathbf{e}_r + \cos(\theta)\mathbf{e}_\theta). && (4)
\end{aligned}$$

The spring force is

$$\begin{aligned}
\mathbf{F}_s &= -K(||\mathbf{r}|| - L)\mathbf{r}/||\mathbf{r}|| && (5a)\\
&= -K(r - L)\mathbf{e}_r. && (5b)
\end{aligned}$$

**(c)** Using Newton's Second Law in polar coordinates:

$$\mathbf{F} = m\mathbf{a} \qquad (6)$$

$$\begin{bmatrix} -K(r-L) - mg\sin(\theta) \\ -mg\cos(\theta) \end{bmatrix} = m \begin{bmatrix} \ddot{r} - r\dot{\theta}^2 \\ r\ddot{\theta} + 2\dot{r}\dot{\theta} \end{bmatrix}. \qquad (7)$$

**(d)** We used Newton's Second Law (also known as Euler's First Law). Note that "gravitation" is incorrect because (1) it is not a law of motion but one of the fundamental forces of nature and (2) alone, it cannot give the equations of motion.

**(e)** All the time-derivatives must be zero. From (2b), $\theta = 90\,\text{deg} \pm n \times 180\,\text{deg}$, where $n \in \mathbb{Z}$. The solution only makes sense *physically* if $\theta = 270\,\text{deg} \pm n \times 360\,\text{deg}$. From (2a), $r = L + mg/K$.

□

### Solution to Problem 2.

**(a)** A particle is an *abstraction* that *models* for us a body in spacetime, the internal structure of which we are ignoring.

**(b)** A force is an efficient *cause of mechanical motion*.

**(c)** A coordinate basis is a collection of vectors that span a vector space. Any vector in the space may be constructed as a linear combination of basis vectors. We have primarily used three bases for $\mathbb{R}^3$: Cartesian, cylindrical polar, and Sennet-Frenet.

**(d)** The Sennet-Frenet basis associates points on a spacecurve to a set of three basis vectors $\mathbf{e}_t$ (tangential to the path), $\mathbf{e}_n$ (normal to the path), and $\mathbf{e}_b$ (binormal to the path). This basis is often useful for problems along arbitrary paths, and those for which a convenient quantity is available or desired, like the tangential force or speed.

**(e)** Friction forces always oppose *relative motion* between a particle and a surface or spacecurve. We often model it as being proportional to the normal force. When static friction is overcome by a force, dynamic friction opposes the motion, typically with a different constant of proportionality.

□

### Solution to Problem 3.

**Kinematics** Use the cylindrical polar basis. Set the origin at the center of the disk, with the $\mathbf{e}_r$–$\mathbf{e}_\theta$ plane coincident with its surface. The pertinent kinematics are:

$$\begin{aligned}
\mathbf{r} &= r\mathbf{e}_r = r_0\mathbf{e}_r && (8a)\\
\mathbf{v} &= \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta = r_0\dot{\theta}\mathbf{e}_\theta && (8b)\\
\mathbf{a} &= \left(\ddot{r} - r\dot{\theta}^2\right)\mathbf{e}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\mathbf{e}_\theta \\
&= -r_0\dot{\theta}^2\mathbf{e}_r + r_0\ddot{\theta}\mathbf{e}_\theta. && (8c)
\end{aligned}$$

**Forces** Gravitational $\mathbf{F}_g = -mg\mathbf{E}_z$, normal $\mathbf{N} = N\mathbf{E}_z$, and friction $\mathbf{F}_f = F_r\mathbf{e}_r + F_\theta\mathbf{e}_\theta$ forces act on the particle.

We know that the threshold above which the particle slips is

$$||\mathbf{F}_f|| = \mu_s ||\mathbf{N}|| = \mu_s |N|. \tag{9}$$

**Newton's Second Law** In the cylindrical polar basis, $\mathbf{F} = m\mathbf{a}$ can be written:

$$\begin{bmatrix} F_r \\ F_\theta \\ N - mg \end{bmatrix} = m \begin{bmatrix} -r_0\dot{\theta}^2 \\ r_0\ddot{\theta} \\ 0 \end{bmatrix} \tag{10}$$

**Analysis** We are given that $\dot{v} = r_0\ddot{\theta} = a_0$, which, if we integrate, gives $\dot{\theta} = a_0 t/r_0$. This is always true, so if we know $\dot{\theta}_{\text{slip}}$, we know at what time $t_{\text{slip}}$ this takes place. From the $\mathbf{E}_z$ scalar equation, $N = mg$. At the moment the particle begins to slip,

$$||\mathbf{F}_f|| = \mu_s |N| \tag{11a}$$

$$\sqrt{F_r^2 + F_\theta^2} = \mu_s m g \tag{11b}$$

$$m\sqrt{r_0^2\dot{\theta}_{\text{slip}}^4 + a_0^2} = \mu_s m g \tag{11c}$$

$$\dot{\theta}_{\text{slip}} = \frac{\left((\mu_s g)^2 - a_0^2\right)^{1/4}}{r_0^{1/2}} \tag{11d}$$

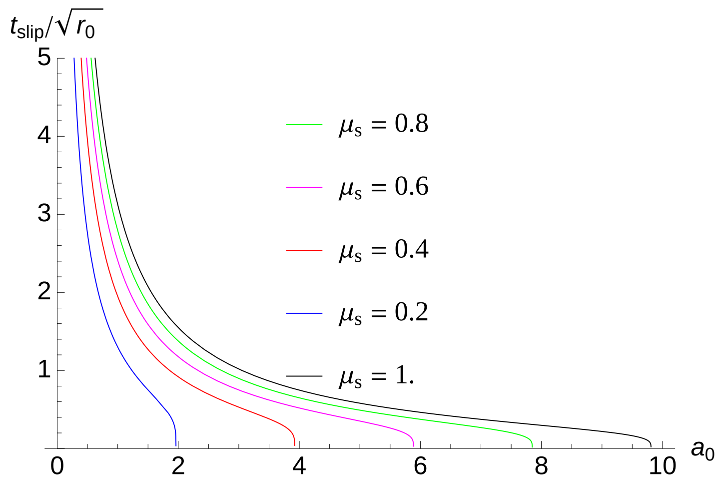
Therefore,

$$\boxed{t_{\text{slip}} = \left(\frac{r_0}{a_0}\right)^{1/2}\left(\left(\frac{\mu_s g}{a_0}\right)^2 - 1\right)^{1/4}}. \tag{12}$$

The solution is only valid for

$$\boxed{a_0 \le \mu_s g} \tag{13}$$

because (11d) must be real. The physical interpretation is that the particle will slip immediately ($t_{\text{slip}} = 0$) if $a_0 > \mu_s g$. The plot below (with $g = 9.81\,\text{m/sec}^2$) shows the relationships between the variables. Notice that with larger friction coefficients, the acceleration can be greater for a longer time without slipping. And if we increase the radius, the particle slips *later*.

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**Solution to Problem 4.** I want to see a discussion of *coordinate system* (polar or Sennet-Frenet), *origin* (origin of $r$), *kinematics*, *forces* (e.g. free-body diagram and specific forces like gravity, friction, and normal forces from the rail), *Newton's Second Law*, and *analysis* leading to the equations of motions from $\mathbf{F} = m\mathbf{a}$. Specific mention that the equations of motion are, in general, differential equations is appreciated. For each step, *justification* should be given; e.g. "Newton's Second Law will yield the equation of motion." □