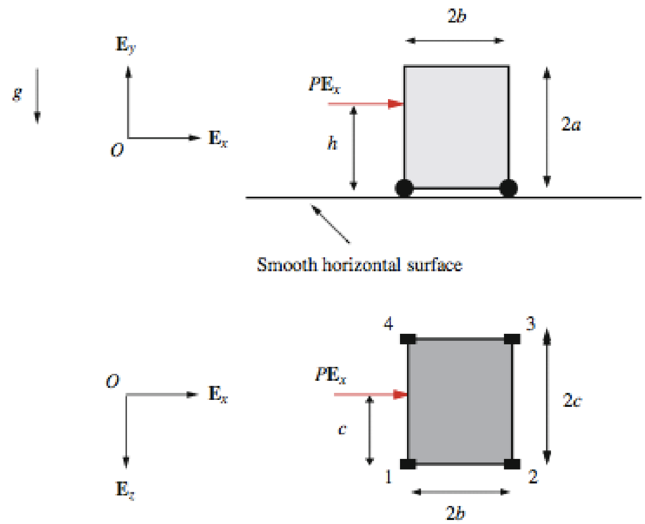
Section AA
ME 230
Finals review

ME230 Kinematics and Dynamics
Rigid Body Dynamics

1) O'Reilly 9.3 Consider the cart shown below,



Suppose that the applied force $\mathbf{P} = 0$, but the front wheels are driven. The driving force on the respective front wheels is assumed to be

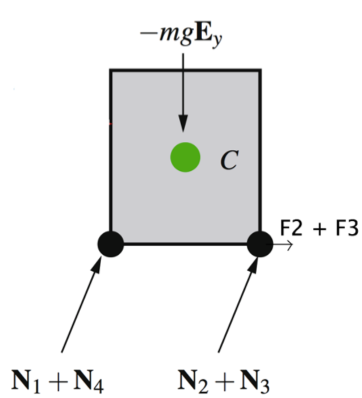
$$\mathbf{F}_2 = \mu N_{2,y}\mathbf{E}_x, \qquad \mathbf{F}_3 = \mu N_{3,y}\mathbf{E}_x$$

where μ is a constant. Calculate the resulting acceleration vector of the center of mass of the cart.

As for the example shown in the book, we first start by with the kinematics and choose a Cartesian coordinate system to describe the position of the center of mass, $\mathbf{x}_C$,

$$\mathbf{x}_C = x\hat{E}_x + y_0\hat{E}_y + z_0\hat{E}_z,$$

where y_0 and z_0 are constants because we consider the case where all 4 wheels of the cart never leave the ground.



The resultant force acting on the system is given by,

$$\begin{aligned}\mathbf{F} = {} & \mu N_{2,y}\hat{E}_x + \mu N_{3,y}\hat{E}_x - mg\hat{E}_y \\ & + (N_{1,y} + N_{2,y} + N_{3,y} + N_{4,y})\hat{E}_y \\ & + (N_{1,z} + N_{2,z} + N_{3,z} + N_{4,z})\hat{E}_z\end{aligned}$$

Resultant moment about center of mass is given by,

$$\begin{aligned}\mathbf{M} = {} & \mu N_{2,y}a\hat{E}_z + \mu N_{3,y}a\hat{E}_z \\ & + c(-N_{1,y} - N_{2,y} + N_{3,y} + N_{4,y})\hat{E}_x \\ & - a(N_{1,z} + N_{2,z} + N_{3,z} + N_{4,z})\hat{E}_x \\ & + b(N_{1,z} - N_{2,z} - N_{3,z} + N_{4,z}))\hat{E}_y \\ & + b(-N_{1,y} + N_{2,y} + N_{3,y} - N_{4,y})\hat{E}_z\end{aligned}$$

Use force and moment balance for the cart rigid body we have the following six equations

$$\mu(N_{2,y} + N_{3,y}) = m\ddot{x}_C \tag{1}$$
$$N_{1,y} + N_{2,y} + N_{3,y} + N_{4,y} - mg = 0 \tag{2}$$
$$N_{1,z} + N_{2,z} + N_{3,z} + N_{4,z} = 0 \tag{3}$$
$$\mu(N_{2,y} + N_{3,y})a + b(-N_{1,y} + N_{2,y} + N_{3,y} - N_{4,y}) = 0 \tag{4}$$
$$c(-N_{1,y} - N_{2,y} + N_{3,y} + N_{4,y}) - a(N_{1,z} + N_{2,z} + N_{3,z} + N_{4,z}) = 0 \tag{5}$$
$$b(N_{1,z} - N_{2,z} - N_{3,z} + N_{4,z})) = 0 \tag{6}$$

Since we have 9 unknowns and only 6 equations,, the system is indeterminate and we need to make some assumptions to come up with extra equations or eliminate some unknowns. We make the following assumption

$$N_{1,y} = N_{4,y} = N_{front}, \qquad N_{2,y} = N_{3,y} = N_{rear}, \qquad N_{1,z} = N_{2,z} = N_{3,z} = N_{4,z} = 0$$

This gives us the following equations

$$2\mu N_{front} = m\ddot{x}_C \tag{7}$$
$$2N_{front} + 2N_{rear} - mg = 0 \tag{8}$$
$$\mu(N_{front})a + b(N_{front} - N_{rear}) = 0 \tag{9}$$

where we have 3 equations and 3 unknowns which we can solve. Thus, from (9) we have,

$$N_{front} = \left(\frac{b}{\mu a + b}\right) N_{rear}$$

Using above in (8) we have,

$$\boxed{N_{rear} = \left(\frac{\mu a + b}{\mu a + 2b}\right)\frac{mg}{2}}, \qquad \Rightarrow \boxed{N_{front} = \left(\frac{b}{\mu a + 2b}\right)\frac{mg}{2}}$$

Using this in (7) we have,

$$\ddot{x}_C = \left(\frac{\mu b g}{\mu a + 2b}\right) = K \tag{10}$$

which is the acceleration of the center of mass of the cart. ■

2) Euler's Disk (spinning/rolling)

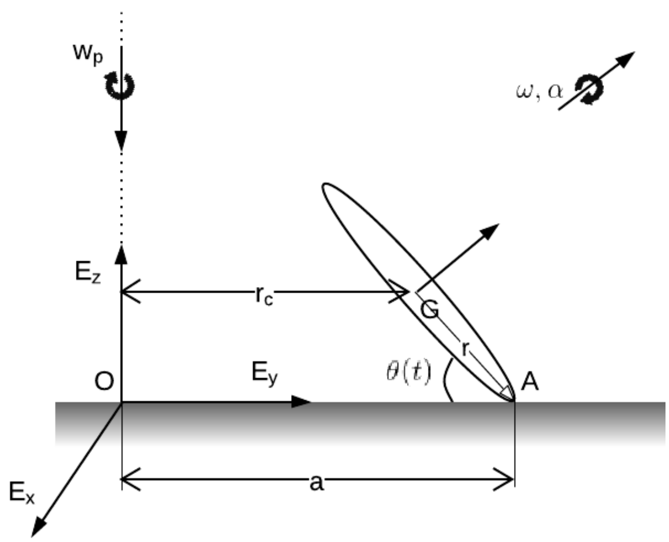
Link to a video depicting the motion of the Euler's disk:

`https://www.youtube.com/watch?v=qdYS8PyOZ7w`

As can be seen from the video, the spinning/rolling action of the Euler's disk is similar to what happens when you spin a coin on a flat surface. The disk spins faster and faster as time goes on before eventually hitting the ground and stopping.
How is it that the angular momentum of the disk keeps increasing with time, seemingly going against our ideas of conservation of angular momentum and energy?

To analyze the dynamics behind the motion of the Euler's disk, we first draw a schematic of the disk as shown on the right, making the following simplifying assumptions:

1. ω_p, the rate of precession, ω_s, the rate of spin, θ, the angle the disk makes with the horizontal, are assumed constant (valid since the rate at which these quantities change with time are small).
2. The disk rolls on the flat surface without slipping.
3. The disk is thin relative to its radius.
4. The disk rolls such that the diameter of the circular path traversed by point A approaches the diameter of the disk itself, as θ becomes small. This implies, we can assume the distance of G from the $\hat{E}_z$ axis, $r_C = a - r\cos\theta = 0$ in the figure to the right.



Now, the angluar velocity of the disk with respect to the ground frame of reference is given by,

$$\vec{\omega} = \omega_s \sin\theta \hat{e}_y + (\omega_s \cos\theta - \omega_p)\hat{E}_z \tag{11}$$

where, $[\hat{e}_x, \hat{e}_y, \hat{e}_z]$ is the rotating frame of reference while $[\hat{E}_x, \hat{E}_y, \hat{E}_z]$ is the fixed global frame of reference. In the above, $\hat{e}_z = \hat{E}_z$. We have the relation between the two frames of reference as,

$$\begin{bmatrix} \hat{e}_x \\ \hat{e}_y \\ \hat{e}_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{E}_x \\ \hat{E}_y \\ \hat{E}_z \end{bmatrix}$$

and also,

$$\frac{d\hat{e}_x}{dt} = \vec{\omega}_p \times \hat{e}_x = \omega_p(-\hat{e}_z \times \hat{e}_x) = -\omega_p \hat{e}_y, \tag{12}$$

$$\frac{d\hat{e}_y}{dt} = \vec{\omega}_p \times \hat{e}_y = \omega_p(-\hat{e}_z \times \hat{e}_y) = \omega_p \hat{e}_x, \tag{13}$$

and $\frac{d\hat{e}_z}{dt} = 0$.

Now, the angular acceleration of the disk with respect to the ground is given by,

$$\begin{aligned} \vec{\alpha} = \frac{d\vec{\omega}}{dt} &= \frac{d[\omega_s \sin\theta \hat{e}_y]}{dt} + \frac{d[(\omega_s \cos\theta - \omega_p)\hat{e}_z]}{dt} \\ &= \omega_s \sin\theta \frac{d\hat{e}_y}{dt} + 0 \end{aligned} \tag{14}$$

since, $\omega_s, \theta, \omega_p$ and $\hat{e}_z = \hat{E}_z$ are assumed to be constant with respect to time. Now, using (3) in (4) we have,

$$\vec{\alpha} = \omega_s \omega_p \sin\theta \hat{e}_x \tag{15}$$

Now consider the point A on the disk. The linear velocity of A is given by,

$$\vec{v}_A = \vec{v}_G + \omega_s(\sin\theta \hat{e}_y + \cos\theta \hat{e}_z) \times r(\cos\theta \hat{e}_y - \sin\theta \hat{e}_z)$$

and,

$$\vec{v}_G = -\omega_p \hat{e}_z \times a\hat{e}_y = a\omega_p \hat{e}_x$$

Thus, we have

$$\vec{v}_A = (a\omega_p - r\omega_s)\hat{e}_x$$

For pure rolling, the point A has zero velocity. This implies,

$$a\omega_p = r\omega_s$$

Also, $a = r\cos\theta$ (from assumption 4) which gives us the relation between the rate of precession and the rate of spin as,

$$\omega_p = \frac{\omega_s}{\cos\theta} \tag{16}$$

Now, for the condition that $r_c = 0$, the point G (the center of mass of the disk) is stationary, implying that the acceleration of the point G is zero. Thus, the normal force acting at point A, $\vec{N} = N\hat{e}_z$ balances the gravitational force, $-mg\hat{e}_z$ giving,

$$N = mg$$

Also, the torque due to this normal force results in ,

$$\tau = r(\cos\theta \hat{e}_y - \sin\theta \hat{e}_z) \times mg(\hat{e}_z) = I_{Gx}\vec{\alpha}$$

where, $I_{Gx} = mr^2/4$ for the thin disk and $\vec{\alpha}$ is given by (5). Thus, we have,

$$mgr\cos\theta\hat{e}_x = \frac{mr^2}{4}\omega_s\omega_p\sin\theta\hat{e}_x \tag{17}$$

Using (6) in (7) we have,

$$\omega_p^2 = \frac{4g}{r\sin\theta}$$

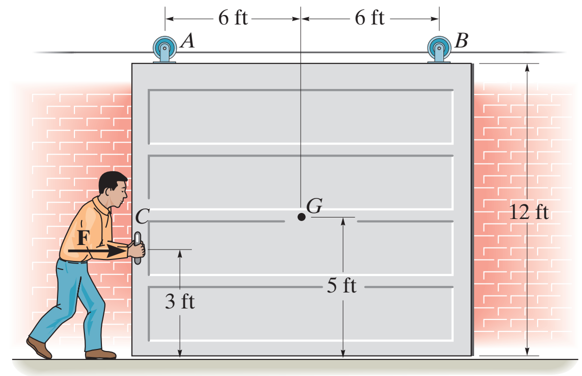
$$\Rightarrow \boxed{\omega_p = \sqrt{\frac{4g}{r\sin\theta}}}$$

This implies that as $\theta \to 0$ we have that $\omega_p \to \infty$ due to the $\sin\theta$ term in the denominator. This is consistent with the rapid increase in the angular velocity we see in a real Euler's disk as θ becomes small. In reality, there is a bit of slipping (friction) at point A (along with rolling friction) which results in the disk losing energy. This results in an equal loss in potential energy (conservation of energy) leading to a loss in height of the center of mass, G (leading to a decrease in θ). Thus, ω_p keeps increasing as θ keeps decreasing upto the point where the disk hits the ground and is suddenly brought to a stop.

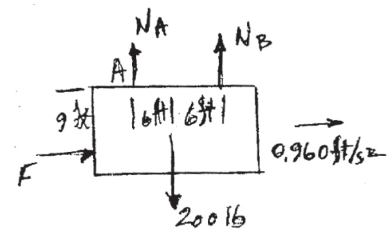
Although we have neglected friction, angular momentum is still **not** conserved about the center of mass G due to the net external torque acting about G due to the normal reaction force from the ground at point A. ∎

17–25.

The door has a weight of 200 lb and a center of gravity at G. Determine the constant force F that must be applied to the door to push it open 12 ft to the right in 5 s, starting from rest. Also, find the vertical reactions at the rollers A and B.



SOLUTION



$(\overset{+}{\rightarrow})\ s = s_0 + v_0 t + \frac{1}{2} a_G t^2$

$$12 = 0 + 0 + \frac{1}{2} a_G (5)^2$$

$a_c = 0.960\ \text{ft/s}^2$

$\overset{+}{\rightarrow} \Sigma F_x = m(a_G)_x; \qquad F = \frac{200}{32.2}(0.960)$

$F = 5.9627\ \text{lb} = 5.96\ \text{lb}$ **Ans.**

$\zeta + \Sigma M_A = \Sigma (M_k)_A; \qquad N_B(12) - 200(6) + 5.9627(9) = \frac{200}{32.2}(0.960)(7)$

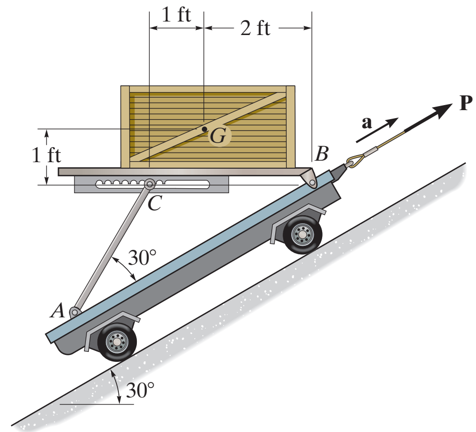
$N_B = 99.0\ \text{lb}$ **Ans.**

$+\uparrow \Sigma F_y = m(a_G)_y; \qquad N_A + 99.0 - 200 = 0$

$N_A = 101\ \text{lb}$ **Ans.**

17–29.

If the strut AC can withstand a maximum compression force of 150 lb before it fails, determine the cart's maximum permissible acceleration. The crate has a weight of 150 lb with center of gravity at G, and it is secured on the platform, so that it does not slide. Neglect the platform's weight.



SOLUTION

Equations of Motion: $\mathbf{F}_{AC}$ in terms of **a** can be obtained directly by writing the moment equation of motion about B.

$+\Sigma M_B = \Sigma(M_k)_B;$

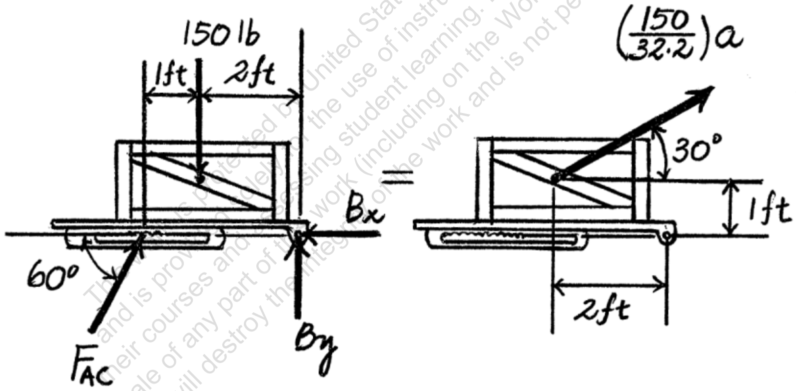
$$150(2) - F_{AC}\sin 60^\circ(3) = -\left(\frac{150}{32.2}\right)a\cos 30^\circ(1) - \left(\frac{150}{32.2}\right)a\sin 30^\circ(2)$$

$$F_{AC} = (3.346a + 115.47)\text{ lb}$$

Assuming AC is about to fail,

$$F_{AC} = 150 = 3.346a + 115.47$$

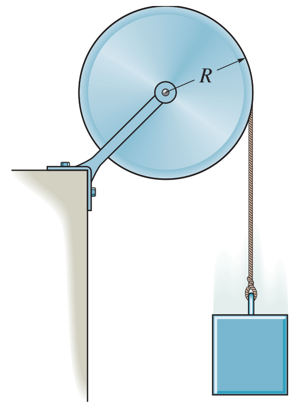
$$a = 10.3\text{ ft/s}^2 \qquad \textbf{Ans.}$$



(a)

***17–68.**

The disk has a mass M and a radius R. If a block of mass m is attached to the cord, determine the angular acceleration of the disk when the block is released from rest. Also,what is the velocity of the block after it falls a distance $2R$ starting from rest?



SOLUTION

$\circlearrowleft +\Sigma M_O = \Sigma(M_k)_O; \qquad mgR = \frac{1}{2}MR^2(\alpha) + m(\alpha R)R$

$$\alpha = \frac{2mg}{R(M + 2m)} \qquad \textbf{Ans.}$$

$a = \alpha R$

$v^2 = v_0^2 + 2\,a(s - s_0)$

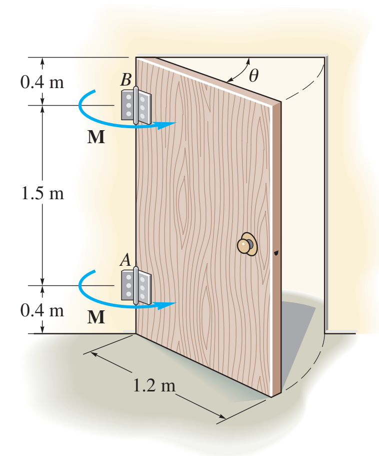
$v^2 = 0 + 2\left(\frac{2mgR}{R(M + 2m)}\right)(2R - 0)$

FBD & KB

$$v = \sqrt{\frac{8mgR}{(M + 2m)}} \qquad \textbf{Ans.}$$

17–70.

The door will close automatically using torsional springs mounted on the hinges. If the torque on each hinge is $M = k\theta$, where θ is measured in radians, determine the required torsional stiffness k so that the door will close ($\theta = 0°$) with an angular velocity $\omega = 2$ rad/s when it is released from rest at $\theta = 90°$. For the calculation, treat the door as a thin plate having a mass of 70 kg.



SOLUTION

$$\Sigma M_A = I_A\alpha; \qquad 2M = -\left[\frac{1}{12}(70)(1.2)^2 + 70(0.6)^2\right](\alpha)$$

$$M = -16.8\alpha$$

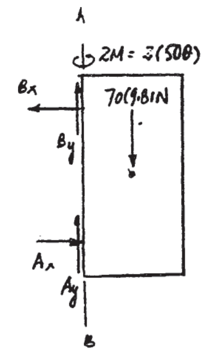
$$k\theta = -16.8\alpha$$

$$\alpha\, d\theta = \omega\, d\omega$$

$$-k\int_{\frac{\pi}{2}}^{0}\theta d\theta = 16.8\int_{0}^{2}\omega\, d\omega$$

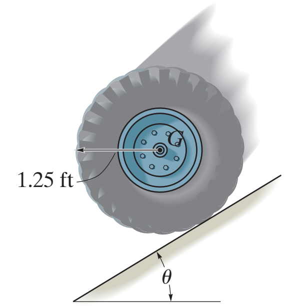
$$\frac{k}{2}\left(\frac{\pi}{2}\right)^2 = \frac{16.8}{2}(2)^2$$

$$k = 27.2\ \text{N}\cdot\text{m/rad}$$ **Ans.**



17–95.

The wheel has a weight of 30 lb and a radius of gyration of $k_G = 0.6$ ft. If the coefficients of static and kinetic friction between the wheel and the plane are $\mu_s = 0.2$ and $\mu_k = 0.15$, determine the maximum angle θ of the inclined plane so that the wheel rolls without slipping.



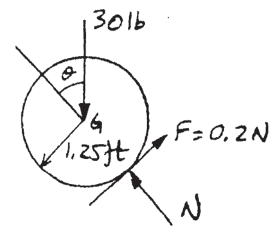
SOLUTION

Since wheel is on the verge of slipping:

$$+\swarrow \Sigma F_x = m(a_G)_x; \qquad 30 \sin\theta - 0.2N = \left(\frac{30}{32.2}\right)(1.25\alpha) \qquad \textbf{(1)}$$

$$+\nwarrow \Sigma F_y = m(a_G)_y; \qquad N - 30\cos\theta = 0 \qquad \textbf{(2)}$$

$$\circlearrowleft + \Sigma M_C = I_G\,\alpha; \qquad 0.2N(1.25) = \left[\left(\frac{30}{32.2}\right)(0.6)^2\right]\alpha \qquad \textbf{(3)}$$



Substituting Eqs.(2) and (3) into Eq. (1),

$$30\sin\theta - 6\cos\theta = 26.042\cos\theta$$

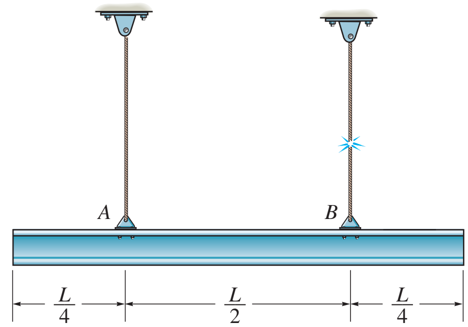
$$30\sin\theta = 32.042\cos\theta$$

$$\tan\theta = 1.068$$

$$\theta = 46.9^\circ \qquad \textbf{Ans.}$$

***17–116.**

The uniform beam has a weight W. If it is originally at rest while being supported at A and B by cables, determine the tension in cable A if cable B suddenly fails. Assume the beam is a slender rod.

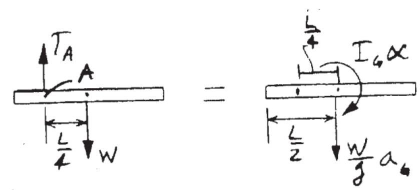


SOLUTION

$$+\uparrow \Sigma F_y = m(a_G)_y; \qquad T_A - W = -\frac{W}{g}a_G$$

$$\circlearrowleft + \Sigma M_A = I_A\alpha; \qquad W\left(\frac{L}{4}\right) = \left[\frac{1}{12}\left(\frac{W}{g}\right)L^2\right]\alpha + \frac{W}{g}\left(\frac{L}{4}\right)\alpha\left(\frac{L}{4}\right)$$

$$1 = \frac{1}{g}\left(\frac{L}{4} + \frac{L}{3}\right)\alpha$$



Since $a_G = \alpha\left(\frac{L}{4}\right)$.

$$\alpha = \frac{12}{7}\left(\frac{g}{L}\right)$$

$$T_A = W - \frac{W}{g}(\alpha)\left(\frac{L}{4}\right) = W - \frac{W}{g}\left(\frac{12}{7}\right)\left(\frac{g}{L}\right)\left(\frac{L}{4}\right)$$

$$T_A = \frac{4}{7}W$$ **Ans.**

Also,

$$+\uparrow \Sigma F_y = m(a_G)_y; \qquad T_A - W = -\frac{W}{g}a_G$$

$$\circlearrowleft + \Sigma M_G = I_G\,\alpha; \qquad T_A\left(\frac{L}{4}\right) = \left[\frac{1}{12}\left(\frac{W}{g}\right)L^2\right]\alpha$$

Since $a_G = \frac{L}{4}\alpha$

$$T_A = \frac{1}{3}\left(\frac{W}{g}\right)L\alpha$$

$$\frac{1}{3}\left(\frac{W}{g}\right)L\alpha - W = -\frac{W}{g}\left(\frac{L}{4}\right)\alpha$$

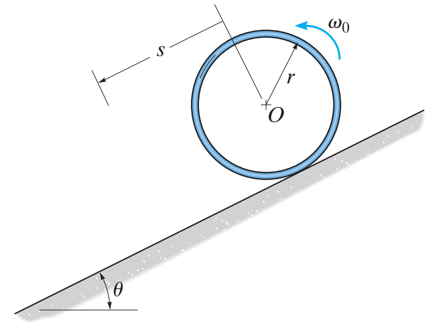
$$\alpha = \frac{12}{7}\left(\frac{g}{L}\right)$$

$$T_A = \frac{1}{3}\left(\frac{W}{g}\right)L\left(\frac{12}{7}\right)\left(\frac{g}{L}\right)$$

$$T_A = \frac{4}{7}W$$ **Ans.**

18–17.

The center O of the thin ring of mass m is given an angular velocity of ω_0. If the ring rolls without slipping, determine its angular velocity after it has traveled a distance of s down the plane. Neglect its thickness.



SOLUTION

$$T_1 + \Sigma U_{1-2} = T_2$$

$$\frac{1}{2}(mr^2 + mr^2)\omega_0^{\,2} + mg(s \sin\theta) = \frac{1}{2}(mr^2 + mr^2)\omega^2$$

$$\omega = \sqrt{\omega_0^{\,2} + \frac{g}{r^2}s\sin\theta}$$ **Ans.**