Part 0: Experimental Facts

What is dynamics?
Dynamics is the study of mechanical motion.

Where can we begin with dynamics?
We must be attentive to the data of our experience.
We must begin with experimental facts.

Chapter 0: Space, Time, + Motion

Space:
- is 3-dimensional
- has no origin

Time:
- is 1-dimensional

0.1 Galileo's Principle of Relativity (Arnold)

There exist coordinate systems (called inertial) possessing the following two properties:

1. All the laws of nature at all moments of time are the same in all inertial coordinate sysems.
2. All coordinate systems in uniform rectilinear motion with respect to an inertial one are themselves inertial.

0.2 Newton's Principle of Determinacy (Arnold)

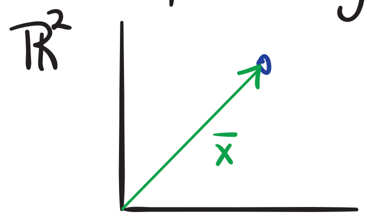
The initial state of a mechanical system
(the totality of positions and velocities of its points at some moment of time)
uniquely determines all of its [future] motion.

0.3 Spacetime is affine (Euclidean Space) (Arnold)

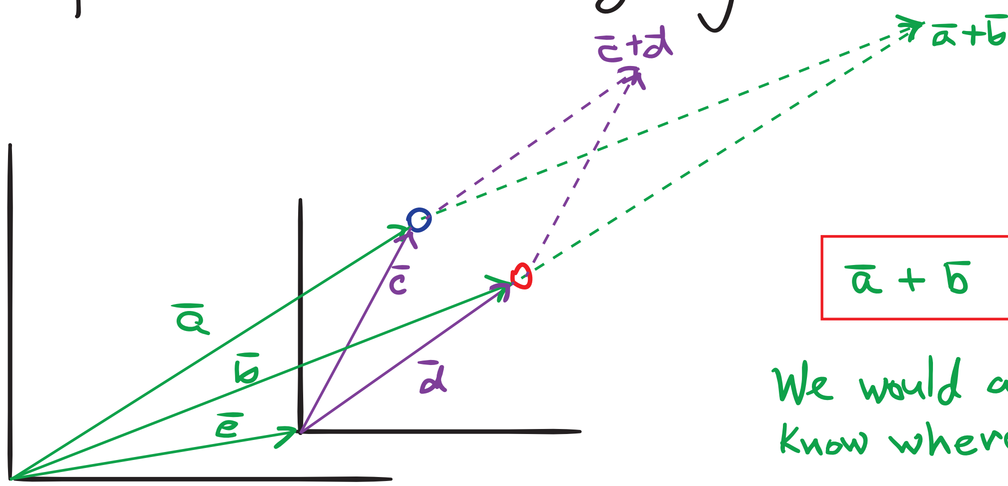
We often represent the locations of objects in spacetime with n-tuples of real numbers, e.g. (1 meter up, 2.3 meters to the right).
2-tuple, i.e. double

The set of all n-tuples of real numbers $\mathbb{R}^n$ is a vector space.
We are used to representing points in space as vectors.

E.g.



But spacetime doesn't behave globally like a vector space.

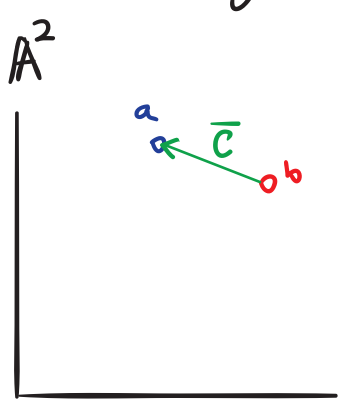


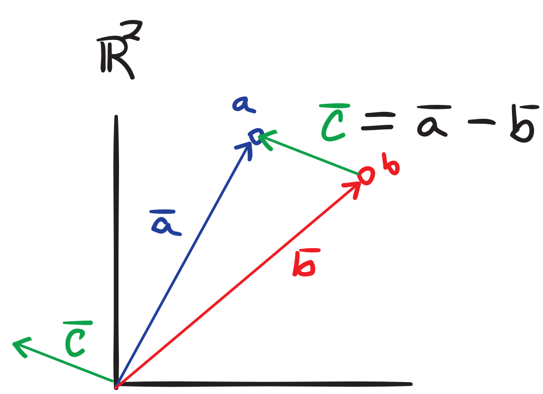
$$\boxed{\bar{a}+\bar{b} \neq \bar{c}+\bar{d}}$$

We would all have to know where the origin is!

Spacetime is actually affine. Affine spaces are like vector spaces, but they have no origin.

We will remember that spacetime is affine, but when we solve problems we will assign an origin and work with a vector space.





Finally, we still need to define distances. The Euclidean norm defines the length of a vector $\bar{x}$: (in vector-not affine-space)

$$\boxed{\|\bar{x}\| = \sqrt{\bar{x}\cdot\bar{x}}} \, .$$

The Euclidean metric or "distance function" is (in vector space)

$$\boxed{d(\bar{x},\bar{y}) = \|\bar{x}-\bar{y}\|} \, .$$

An affine space with a Euclidean metric is a Euclidean Space $\mathbb{E}^n$.

Our model of spacetime is a Eulidean Space $\mathbb{E}^4$ (3 spatial, 1 time).

We will often consider only subspaces of $\mathbb{E}^4$!

0.4 Newton's equation of Motion

All future motions of a system are uniquely determined by their initial positions $\bar{r}(t_0)\in\mathbb{R}^n$ and velocities $\bar{v}(t_0)=\dot{\bar{r}}(t_0)\in\mathbb{R}^n$.

In particular, the acceleration $\bar{a}=\dot{\bar{v}}=\ddot{\bar{r}}$ is determined:

$$\boxed{\bar{a} = \bar{D}(\bar{r},\bar{v},t)} \, ,$$

Assuming that the laws of physics remain constant (Galilean relativity), for a closed system,

$$\boxed{\bar{a} = \bar{D}(\bar{r},\bar{v})} \, .$$

Determining $\bar{D}$ for any particular system is an experimental endeavor.

Certain common characteristics will arise, however. For instance, a spring often behaves in a certain way (influences $\bar{a}$ in a certain recognizable manne that depends of $\bar{r}$ + $\bar{v}$).

Part 1: Dynamics of a single particle

What is a particle? A particle is an abstraction that models for us a body in spacetime thats internal structure we are ignoring.

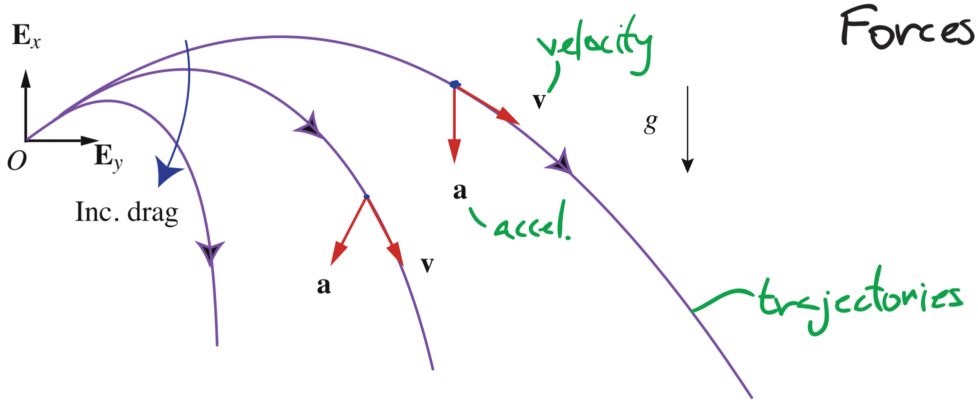
Chapter 1: Elementary particle dynamics

Topics:
- particle kinematics
- particle dynamics
- Euler's first law (Newton's second law)

1.1 An example

A projectile particle:

Forces: drag and gravity

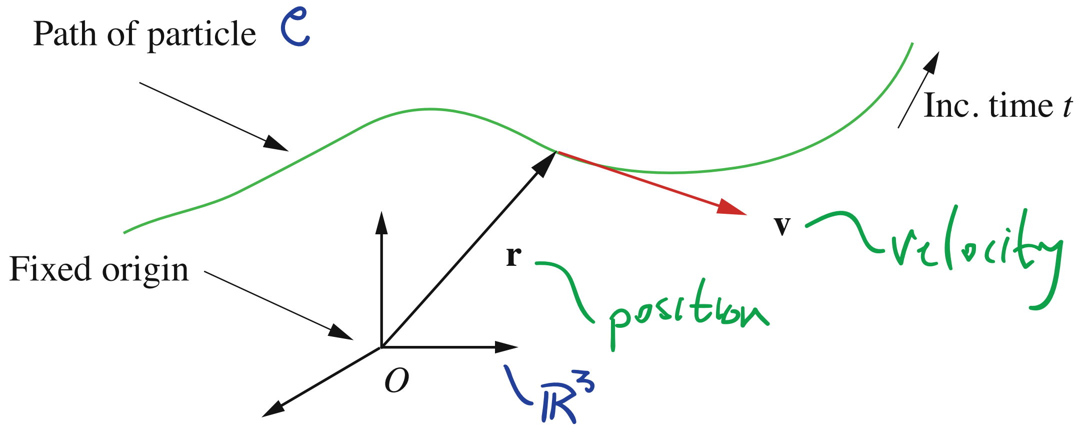


1.2 Kinematics of a particle

What is kinematics? Kinematics is the study of the motion of bodies without concern for the efficient causes* of the motion.

*See Aristotle or Heidegger's "The Question Concerning Tech."

We will consider motion in a four-dimensional Euclidean Space $\mathbb{E}^4$ (3 spatial, 1 time).



Definitions

Position vector $\bar{r} \in \mathbb{R}^n$: vector that describes the location of a particle at some time.

Absolute velocity vector-valued function $\bar{v}$: the time-rate of change of the location of a particle. It can be computed from $\bar{r}$ by:

$$\boxed{\bar{v} = \frac{d\bar{r}}{dt} = \dot{\bar{r}}}$$

Speed v: the magnitude of the velocity. It can be computed by:

$$\boxed{v = \|\bar{v}\| = \sqrt{\langle \bar{v}, \bar{v} \rangle} = \sqrt{\bar{v} \cdot \bar{v}}}$$

Absolute acceleration vector-valued function $\bar{a}$: the time-rate of change of the velocity. It can be computed by:

$$\boxed{\bar{a} = \frac{d\bar{v}}{dt} = \dot{\bar{v}}}$$

Distance along a path (arclength) s: the time derivative is the speed:

$$\boxed{\frac{ds}{dt} = \|\bar{v}\|}$$

This can be integrated to find the distance travelled by the particle along its path $\mathcal{C}$ during the time interval $(t-t_0)$:

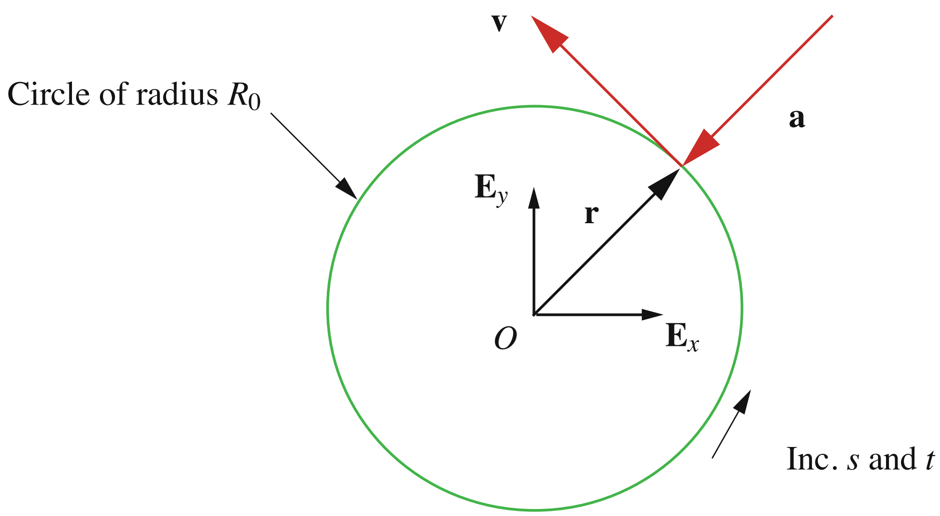
$$\boxed{s(t)-s_0=\int_{t_0}^{t}\frac{ds(\tau)}{dt}\,d\tau=\int_{t_0}^{t}\sqrt{\bar{v}(\tau)\cdot\bar{v}(\tau)}\;d\tau}\;.$$

Thus far, we haven't restricted ourselves to any specific coordinate system. All the above applies for all coordinates.

1.3 A Circular Motion (an example)

$$\bar{r}=\bar{r}(t)=R_0\left(\cos(\omega t)\bar{E}_x+\sin(\omega t)\bar{E}_y\right)$$

constant angular velocity



$\bar{E}_x, \bar{E}_y$: Cartesian basis vectors

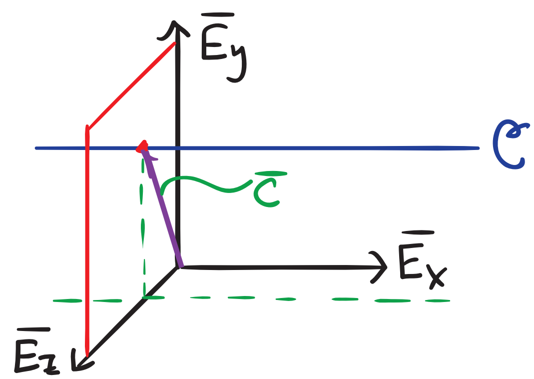
Compute $\bar{v}(t)$: $\quad \bar{v}(t)=\dot{\bar{r}}(t)=R_0\,\omega\left(-\sin(\omega t)\bar{E}_x+\cos(\omega t)\bar{E}_y\right)$

Compute arclength:

$$s(t)-s_0=\int_{t_0}^{t}\frac{ds(\tau)}{dt}\,d\tau=\int_{t_0}^{t}\sqrt{\bar{v}(\tau)\cdot\bar{v}(\tau)}\;d\tau=\int_{t_0}^{t}\sqrt{\omega^2R_0}\,d\tau=R_0\,\omega(t-t_0).$$

1.4 Rectilinear Motions

$$\bar{r}=\bar{r}(t)=x(t)\bar{E}_x+\bar{c}$$
$$\bar{v}=\bar{v}(t)=\dot{x}(t)\bar{E}_x=v(t)\bar{E}_x$$
$$\bar{a}=\bar{a}(t)=\ddot{x}(t)\bar{E}_x=a(t)\bar{E}_x$$



1.4.1 Given $a = a(t)$

$$v(t) = v(t_0) + \int_{t_0}^{t} a(u)\,du$$

$$x(t) = x(t_0) + \int_{t_0}^{t} v(\tau)\,d\tau$$

1.4.2 Given $a = \hat{a}(v)$

Find $\hat{x}(v)$

We use the useful identity $\boxed{a = \frac{dv}{dt} \overset{\text{chain rule}}{=} \frac{dv}{dx}\underbrace{\frac{dx}{dt}}_{v} = v\frac{dv}{dx}}$:

$$\hat{a}(v) = v\frac{dv}{d\hat{x}} \Rightarrow d\hat{x} = \frac{v}{\hat{a}(v)}dv \Rightarrow \hat{x}(v) = \hat{x}(v_0) + \int_{v_0}^{v} \frac{u}{a(u)}\,du.$$

Find $\hat{t}(v)$

In order to find $\hat{t}(v)$, reason as follows:

$$\hat{a}(v) = \frac{dv}{d\hat{t}} \Longrightarrow d\hat{t} = \frac{dv}{\hat{a}(v)} \Longrightarrow \hat{t}(v) = \hat{t}(v_0) + \int_{v_0}^{v} \frac{1}{a(u)}\,du.$$

1.4.3 Given $a = \hat{a}(x)$

Find $\hat{v}(x)$

$$a = \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx} \Longrightarrow a\,dx = v\,dv$$

$$\Longrightarrow \int_{x_0}^{x} \hat{a}(u)\,du = \int_{v_0}^{v} \tau\,d\tau$$

$$= \frac{1}{2}(\hat{v}^2 - \hat{v}_0^2) \Longrightarrow \hat{v}^2(x) = \hat{v}^2(x_0) + 2\int_{x_0}^{x} \hat{a}(u)\,du$$

Find $\hat{t}(x)$

$$\hat{v}(x) = \frac{d\hat{x}}{d\hat{t}} \Rightarrow d\hat{t} = \frac{d\hat{x}}{\hat{v}(x)} \Longrightarrow \hat{t}(x) = \hat{t}(x_0) + \int_{x_0}^{x} \frac{1}{\hat{v}(u)}\,du$$

1.5 Kinetics of a particle

What is Kinetics? Kinetics is the study of the efficient causes of mechanical motion.

We call the efficient causes of mechanical motion forces.

What is mass? Mass is a body's resistance to being accelerated by a force and a body's strength of gravitational attraction.

Newton's second law / Euler's first law

Earlier we said that $\bar{a} = \bar{D}(\bar{r},\bar{v})$. Newton and Euler specified $\bar{D}(\bar{r},\bar{v})$ such that the resultant external force acting on a particle $\bar{F}$ can be written

$$\boxed{\bar{F} = \frac{d\bar{G}}{dt} = m\bar{a}}.$$

$\bar{G}$ is the linear momentum: $\bar{G} = m\cdot\bar{v}$. So the resultant force is the time rate change of linear momentum.

It is important to remember that $\bar{a}$ is the absolute accel.

In the Cartesian basis $(\bar{E}_x, \bar{E}_y, \bar{E}_z)$, the eq can be written:

$$F_x\bar{E}_x + F_y\bar{E}_y + F_z\bar{E}_z = m(a_x\bar{E}_x + a_y\bar{E}_y + a_z\bar{E}_z)$$

$$\begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix} = m \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}.$$

1.5.1 Action + Reaction (Newton's Third Law) (Spivak)

If an object A exerts a force $\bar{F}$ on object B, then B exerts $-\bar{F}$ on A.

1.5.2 The Four Steps

$\bar{F}=m\bar{a}$ can be used to analyze mechanical systems. We suggest these four steps.

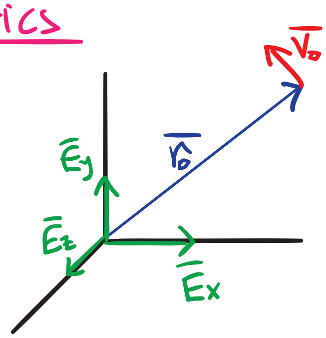
1. Pick an origin and a coordinate basis, then establish expressions for $\bar{r}, \bar{v}$, and $\bar{a}$ (Kinematics).
2. Draw a free-body diagram.
3. Write $\bar{F}=m\bar{a}$.
4. Perform the analysis.

1.6 A particle under the influence of gravity (an example)

Consider a particle of mass m launched from $\bar{r}_0$ with velocity $\bar{v}_0$ at time $t=0$. Include the gravitational force but not the drag force. Find $\bar{r}(t)$.

1.6.1 Kinematics

Cartesian basis:



$$\bar{r} = x\bar{E}_x + y\bar{E}_y + z\bar{E}_z$$

$$\bar{a} = \ddot{\bar{r}} = \ddot{x}\bar{E}_x + \ddot{y}\bar{E}_y + \ddot{z}\bar{E}_z$$

1.6.2 Free-body diagram

Does the direction of the arrow in the FBD matter? No! We draw it only to help visualize the forces. Signs correspond to our coordinates.

1.6.3 $\bar{F}=m\bar{a}$

First, write the gravitational force eq.: $\bar{F}_g = -mg\bar{E}_y$. (negative!)

$\bar{F}$ is the resultant force, so it is the sum of forces.
In this case, there's only one force, so:

$$\bar{F} = \bar{F}_g = -mg\bar{E}_y.$$

Now we can relate $\bar{F}$ and $\bar{a}$ with

$$\bar{F} = m\bar{a}$$
$$-mg\bar{E}_y = m(a_x\bar{E}_x + a_y\bar{E}_y + a_z\bar{E}_z)$$

$$\Rightarrow \quad \bar{a} = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} = \begin{bmatrix} 0 \\ -g \\ 0 \end{bmatrix}$$

1.6.4 Analysis (find $\bar{r}(t)$)

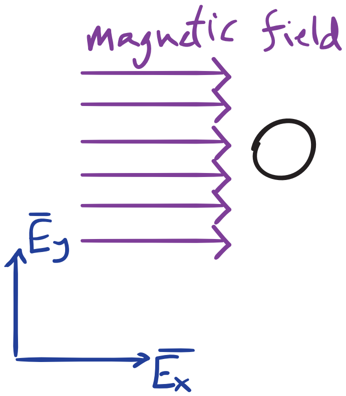
We have $\bar{a}$. Given our initial state $\bar{r}_0$ and $\bar{v}_0$, we want to know $\bar{r}(t)$.

$$\bar{v}(t) = \bar{v}_0 + \int_0^t \bar{a}(\tau)\,d\tau = \bar{v}_0 + \begin{bmatrix} 0 \\ \int_0^t a_y\,d\tau \\ 0 \end{bmatrix} = \bar{v}_0 - gt\bar{E}_y \, .$$

$$\bar{r}(t) = \bar{r}_0 + \int_0^t \bar{v}(\tau)\,d\tau = \bar{r}_0 + \int_0^t \bar{v}_0\,d\tau + \begin{bmatrix} 0 \\ \int_0^t -g\tau\,d\tau \\ 0 \end{bmatrix}$$

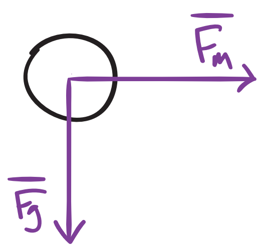
$$\Rightarrow \quad \boxed{\bar{r}(t) = \bar{r}_0 + \bar{v}_0 t - \frac{1}{2}gt^2\bar{E}_y}$$

1.7 A particle in magnetic + gravitational fields (example)



The ball is released from rest in a horizontal mag. field + vertical gravitational field.
If we measured the velocity as a function of time to be approx.
$\bar{V}(t) = \left(t^2\,\bar{E}_x - 9.8t\,\bar{E}_y\right)\frac{m}{sec}$ what is $\bar{F}_m(t)$, the magnetic field force?

1.7.1 FBD



1.7.2 $\bar{F} = m\bar{a}$

Known force: $\bar{F}_g = -mg\,\bar{E}_y$

Unknown force: $\bar{F}_m$

Resultant force:

$$\bar{F} = \begin{matrix} & F_m & \bar{E}_x \\ - & mg & \bar{E}_y \end{matrix}$$

$\bar{F} = m\bar{a}$:

$$F_m\,\bar{E}_x - mg\,\bar{E}_y = m\left(a_x\bar{E}_x + a_y\bar{E}_y\right)$$

1.7.3 Analysis (find $F_m(t)$)

Solve for $\bar{a}$: $\bar{a} = \begin{bmatrix} \frac{1}{m}F_m \\ -g \end{bmatrix}$

From the measurement, we know:

$$\bar{a}(t) = \dot{\bar{v}}(t) = \begin{bmatrix} 2t \\ -9.8 \end{bmatrix} \qquad \therefore$$

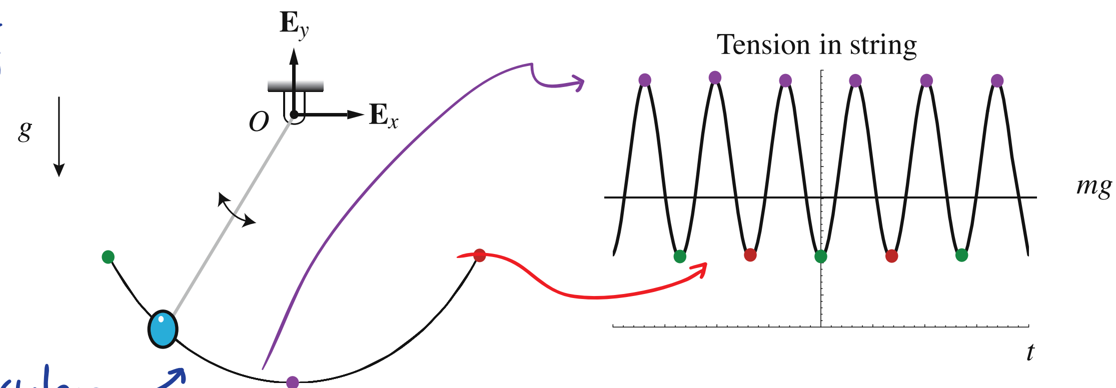
$\boxed{F_m(t) = 2mt}$ and $g = -9.8\ m/sec^2$ (Earth)

Chapter 2: Cylindrical Polar Coordinates

Topics:
- cylindrical polar coordinates
- basis $(\bar{e}_r, \bar{e}_\theta, \bar{E}_z)$
- kinematics + kinetics of particles w/ $(\bar{e}_r, \bar{e}_\theta, \bar{E}_z)$

2.1 The Cylindrical Polar Coordinate System

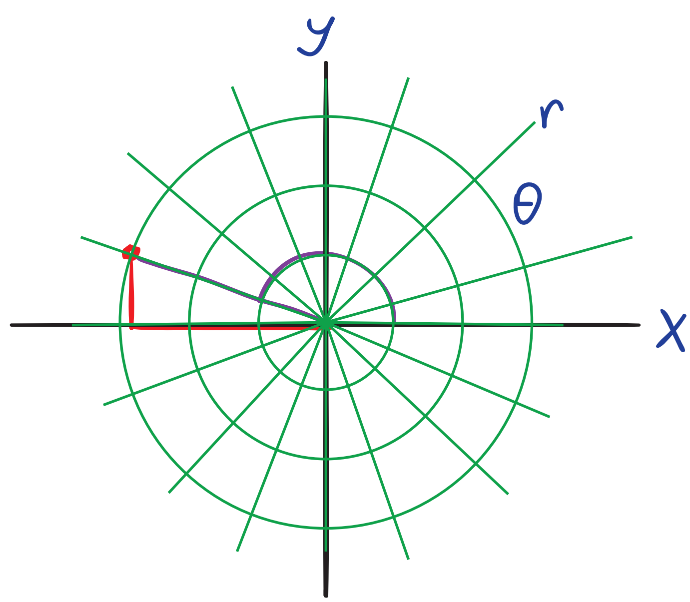
An example of a good system for polar coordinates:



circular path

We will now define the cylindrical polar coordinate system $\{r, \theta, z\}$ in terms of the Cartesian system $\{x, y, z\}$.

$$r = \sqrt{x^2 + y^2}$$
$$\theta = \arctan(y/x)$$
$$z = z$$



Going the other way (assuming $\neg(x=0 \wedge y=0)$):

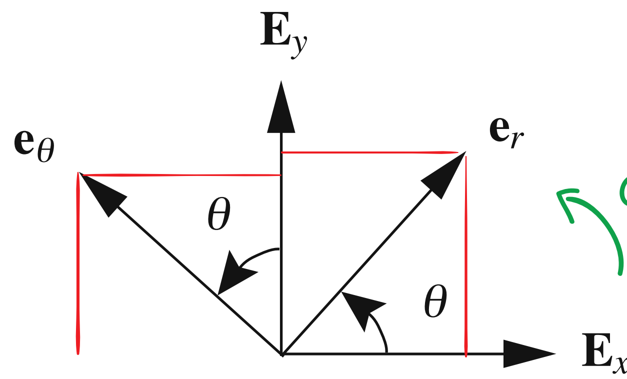
$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = z$$

Now we can write our position vector $\bar{r}$ as

$$\bar{r} = x\bar{E}_x + y\bar{E}_y + z\bar{E}_z$$
$$= r\cos\theta\,\bar{E}_x + r\sin\theta\,\bar{E}_y + z\bar{E}_z$$

Define the basis vectors as:

$$(*)\quad \begin{bmatrix} \bar{e}_r \\ \bar{e}_\theta \\ \bar{E}_z \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \bar{E}_x \\ \bar{E}_y \\ \bar{E}_z \end{bmatrix}$$



CCW!

Rewriting our position vector:

$$\bar{r} = \underbrace{r\cos\theta\,\bar{E}_x + r\sin\theta\,\bar{E}_y}_{=r\bar{e}_r} + z\bar{E}_z$$
$$= r\bar{e}_r + z\bar{E}_z \longleftarrow \text{note: no } \theta\bar{e}_\theta\,!$$

2.2 Velocity + Acceleration Vectors

Velocity

$$\bar{v} = \dot{\bar{r}} = \dot{r}\bar{e}_r + r\dot{\bar{e}}_r + \dot{z}\bar{E}_z$$

From $(*)$ we know that $\boxed{\dfrac{d\bar{e}_\theta}{d\theta} = -\bar{e}_r}$ + $\boxed{\dfrac{d\bar{e}_r}{d\theta} = \bar{e}_\theta}$.

Combining this with the chain rule $(\dot{\bar{e}}_r = \dot{\theta}\, d\bar{e}_r/d\theta)$,

$$\boxed{\bar{v} = \dot{r}\bar{e}_r + r\dot{\theta}\bar{e}_\theta + \dot{z}\bar{E}_z}$$

Acceleration

$$\bar{a} = \dot{\bar{v}} \qquad \therefore$$

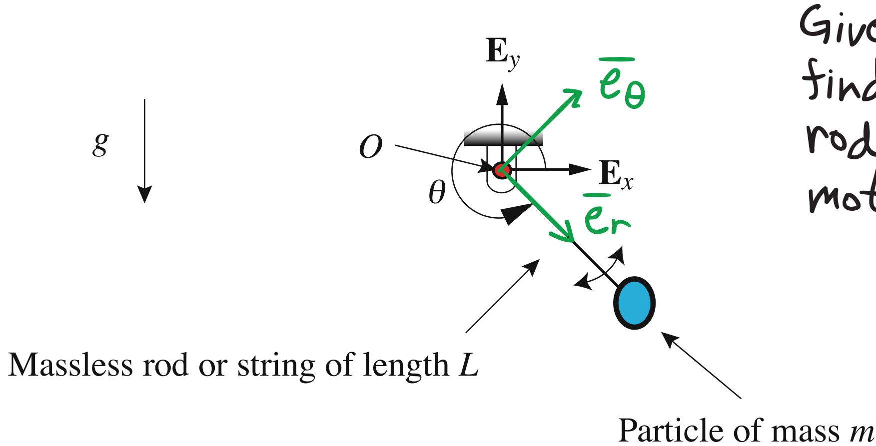
$$\boxed{\bar{a} = (\ddot{r} - r\dot{\theta}^2)\bar{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\bar{e}_\theta + \ddot{z}\bar{E}_z}$$

2.3 Kinetics of a Particle

Writing $\bar{F} = m\bar{a}$ in cylindrical polar coordinates,

$$\boxed{\begin{aligned} \bar{F} = &\ m(\ddot{r} - r\dot{\theta}^2) && \bar{e}_r \\ &+ m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) && \bar{e}_\theta \\ &+ m\ddot{z} && \bar{E}_z \end{aligned}}\ .$$

2.4 Planar Pendulum (example)



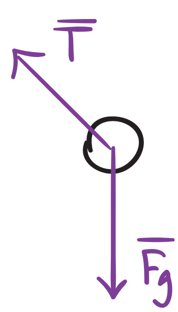
Given an initial state $\bar{r}_0, \bar{v}_0$, find the tension in the string/rod and the equations of motion.

2.4.1 Kinematics

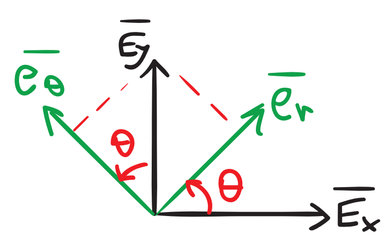
Position: $\bar{r} = L\bar{e}_r$ Velocity: $\bar{v} = \dot{\bar{r}} = L\dot{\bar{e}}_r = L\dot{\theta}\bar{e}_\theta$

Acceleration:
$$\begin{aligned} \bar{a} = \dot{\bar{v}} &= L\ddot{\theta}\bar{e}_\theta + L\dot{\theta}\dot{\bar{e}}_\theta \\ &= L\ddot{\theta}\bar{e}_\theta + L\dot{\theta}\frac{d\bar{e}_\theta}{d\theta}\frac{d\theta}{dt} \\ &= L\ddot{\theta}\bar{e}_\theta - L\dot{\theta}^2\bar{e}_r \qquad (d\bar{e}_\theta/d\theta = -\bar{e}_r, \text{ see } 2.2) \end{aligned}$$

2.4.2 FBD + Forces



Unknown: $\bar{T}$
Known: $\bar{F}_g$



$$\bar{F}_g = -mg\bar{E}_y = -mg(\sin\theta\, \bar{e}_r + \cos\theta\, \bar{e}_\theta)$$

But we do know something about $\overline{T}$: $\overline{T} = T\overline{e}_r$.

So $\overline{F} = \overline{F}_g + \overline{T} = -mg\cos\theta\,\overline{e}_\theta - mg\sin\theta\,\overline{e}_r + T\overline{e}_r$
$= -mg\cos\theta\,\overline{e}_\theta$
$+ (T - mg\sin\theta)\,\overline{e}_r$

2.4.3 $\overline{F} = m\overline{a}$

$\overline{F} = m\overline{a}$ in the (r, θ, z)-basis:

$$\begin{bmatrix} T - mg\sin\theta \\ -mg\cos\theta \end{bmatrix} = m\begin{bmatrix} -L\dot{\theta}^2 \\ L\ddot{\theta} \end{bmatrix}$$

2.4.4 Analysis

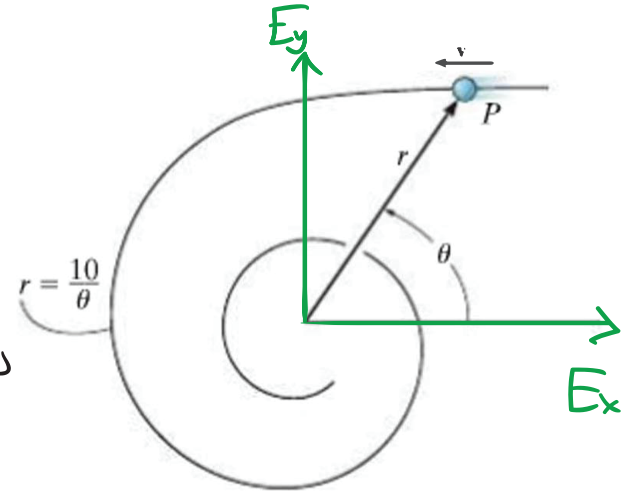
The $\overline{e}_\theta$ equation is an ODE from which we can find $\theta(t)$. Once we have $\theta(t)$, we have $\dot{\theta}(t)$ + $\ddot{\theta}(t)$, and we can solve the $\overline{e}_r$ equation for

$$T(t) = mg\sin\theta - mL\dot{\theta}^2 .$$

However, the $\overline{e}_\theta$ equation is hard to solve. We can linearize it about some $\theta = \hat{\theta}$ to get a good approximation nearby.

Example (Hibbeler 12-175)

A particle P moves along the spiral path $r = 10/\theta$ ft, where θ is in radians. If it maintains a constant speed of $v = v_0$, determine v_r and v_θ as functions of θ.



$$\bar{v} = v_r \bar{e}_r + v_\theta \bar{e}_\theta = \dot{r}\bar{e}_r + r\dot{\theta}\bar{e}_\theta$$

$$v_r = \dot{r} \qquad \text{and} \qquad v_\theta = r\dot{\theta}$$

Compute $\dot{r}$: $\quad \dot{r} = -\frac{10}{\theta^2}\dot{\theta}$

How can we find $\dot{\theta}$? We know $v = v_0$, and

$$\|\bar{v}\| = \sqrt{v_r^2 + v_\theta^2} = \sqrt{\left(-\frac{10}{\theta^2}\dot{\theta}\right)^2 + \left(\frac{10}{\theta}\dot{\theta}\right)^2} = v_0.$$ Solving for $\dot{\theta}$,

$$\frac{100\,\dot{\theta}^2}{\theta^4} + \frac{100\,\dot{\theta}^2}{\theta^2} = v_0^2$$

$$\dot{\theta} = \pm\left(\frac{v_0^2}{100\left(1/\theta^4 + 1/\theta^2\right)}\right)^{1/2} = \pm\left(\frac{v_0^2}{100}\,\frac{\theta^4}{1+\theta^2}\right)^{1/2}$$

we can take the + because $\theta \geq 0$.

So,

$$v_r = -\frac{10}{\theta^2}\dot{\theta} = -\frac{v_0}{\sqrt{1+\theta^2}}$$

$$v_\theta = r\dot{\theta} = \frac{10}{\theta}\,\frac{v_0}{10}\,\frac{\theta^2}{\sqrt{1+\theta^2}} = \frac{v_0\,\theta}{\sqrt{1+\theta^2}}.$$

Was this a kinematics or kinetics problem? Kinematics.

Why did we use cylindrical polar coordinates? $r(\theta)$.

Chapter 3: Particles + Space Curves

Topics:
- Differential geometry of space curves
- Serret-Frenet basis vectors $\{\bar{e}_t, \bar{e}_n, \bar{e}_b\}$
- Rate-of-change of $\{\bar{e}_t, \bar{e}_n, \bar{e}_b\}$
- Examples of spacecurves
- Application to mechanics

} Kinematics (first four topics)

} Kinetics (last topic)

3.1 Space Curves

A space curve is a curved path in space. Rectilinear and circular paths are special types of space curves.

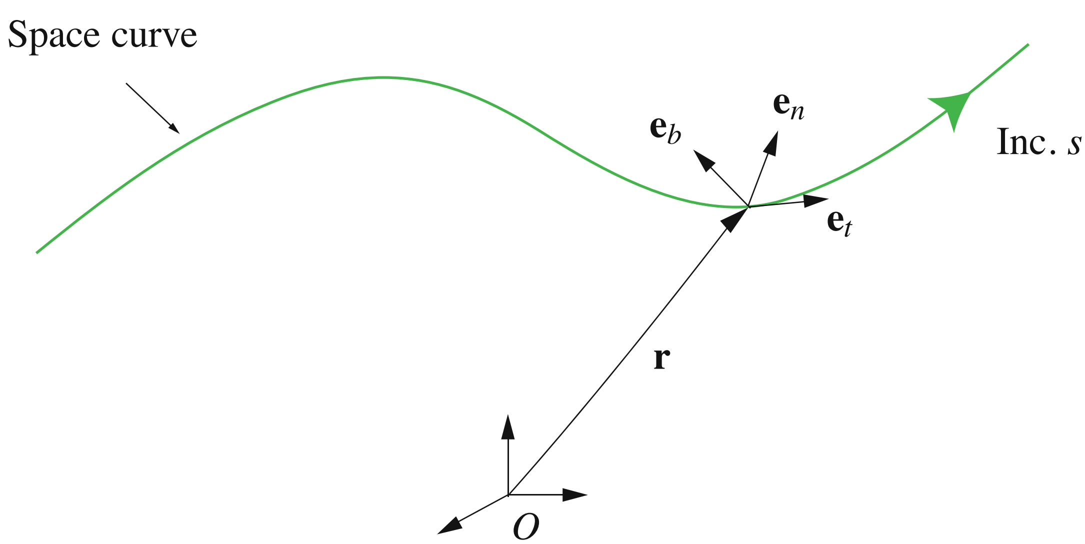
3.1.1 The Arc-Length Parameter

Position vector: $\bar{r} = x\,\bar{E}_x + y\,\bar{E}_y + z\,\bar{E}_z$

Arc-length s: $\dfrac{ds}{dt} = \left\| \dfrac{d\bar{r}}{dt} \right\| = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}$

So we can parameterize the space curve $\mathcal{C}$ by s:

$$\bar{r} = \hat{\bar{r}}(s).$$



We define the Serret-Frenet basis vectors $\{\bar{e}_t, \bar{e}_n, \bar{e}_b\}$ for a point $P \in \mathcal{C}$.

unit tangent vector: $\boxed{\bar{e}_t = \hat{e}_t(s) \triangleq \dfrac{d\bar{r}}{ds}}$

To define $\bar{e}_n$, let's consider that (from):

$$\frac{d^2\bar{r}}{ds^2} = \frac{d\bar{e}_t}{ds} .$$

We want $\bar{e}_n$ to be perpendicular to $\bar{e}_t$ and we want it to have unit length (i.e. $\|\bar{e}_n\| = 1$). From the fact that $\|\bar{e}_t\| = 1$, we have:

$$\bar{e}_t \cdot \bar{e}_t = 1 \quad \xRightarrow{d/ds} \quad \frac{d\bar{e}_t}{ds} \cdot \bar{e}_t + \frac{d\bar{e}_t}{ds} \cdot \bar{e}_t = 0$$

$$\frac{d\bar{e}_t}{ds} \cdot \bar{e}_t = 0$$

Therefore, $d\bar{e}_t/ds$ is perpendicular to $\bar{e}_t$! We use this fact to define $\bar{e}_n$ as follows:

curvature of $\mathcal{C}$

$$\boxed{\kappa \bar{e}_n = \frac{d\bar{e}_t}{ds}}$$

unit principle normal vector

where $\kappa = \hat{\kappa}(s)$ is the curvature of $\mathcal{C}$ at some point P.
We also define the radius of curvature:

$$\rho = \hat{\rho}(s) = \frac{1}{\kappa} .$$

When $d\bar{e}_t/ds = \bar{0}$ for some s, $\kappa = 0$, $\rho \to \infty$, and $\bar{e}_n$ is not uniquely defined. The final unit vector is defined as:

$$\boxed{\bar{e}_b = \hat{e}_b(s) = \bar{e}_t \times \bar{e}_n}$$

We call the $\bar{e}_t$-$\bar{e}_n$ plane the osculating plane and the $\bar{e}_t$-$\bar{e}_b$ the rectifying plane.

A vector in $\bar{b} \in \mathbb{E}^3$ can be written:

$$\bar{b} = b_x \bar{E}_x + b_y \bar{E}_y + b_z \bar{E}_z = b_t \bar{e}_t + b_n \bar{e}_n + b_b \bar{e}_b \,.$$

3.2 The Serret-Frenet formulae

These formulas describe the rate-of-change of $\{\bar{e}_t, \bar{e}_n, \bar{e}_b\}$ in terms of $\{\bar{e}_t, \bar{e}_n, \bar{e}_b\}$. The first one we saw above:

$$\boxed{\frac{d\bar{e}_t}{ds} = \kappa \bar{e}_n} \,.$$

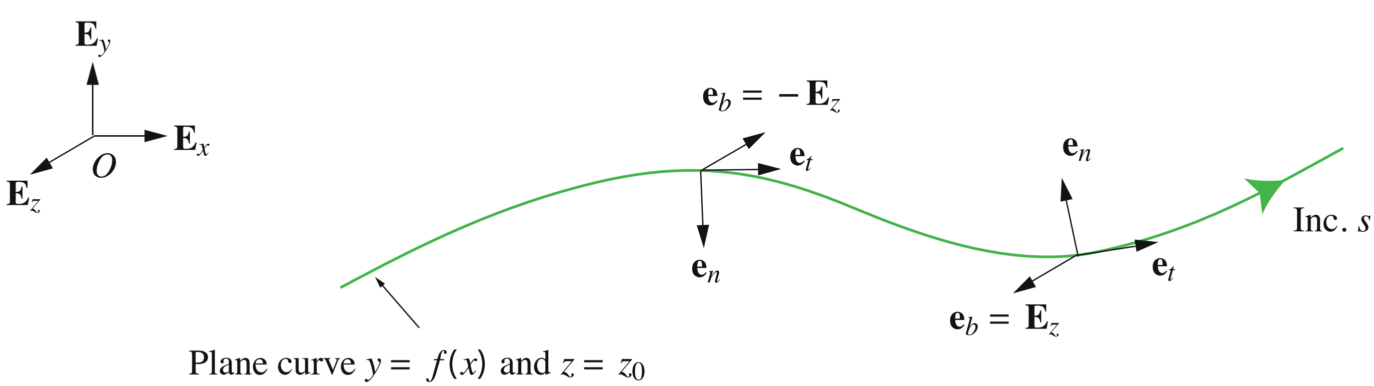
The others are derived in O'Reilly.

$$\boxed{\frac{d\bar{e}_b}{ds} = -\tau \bar{e}_n} \qquad \boxed{\frac{d\bar{e}_n}{ds} = -\kappa \bar{e}_t + \tau \bar{e}_b}$$

where $\tau = \hat{\tau}(s)$ is called the torsion.

3.3 Examples of Space Curves

3.3.1 A Curve on a Plane



Position: $\bar{r} = x\bar{E}_x + f(x)\bar{E}_y + z_0 \bar{E}_z$

Arclength (assuming s + x increase in the same direction):

$$s = s(x) = \int_{x_0}^{x} \sqrt{1 + \left(\frac{df}{dx}\right)^2}\, du + s(x_0)$$

Serret-Frenet basis vectors:

$$\bar{e}_t = \bar{e}_t(x) = \frac{d\bar{r}}{ds} = \frac{d\bar{r}}{dx}\frac{dx}{ds} = \frac{1}{\sqrt{1+\left(df/dx\right)^2}}\left(\bar{E}_x + \frac{df}{dx}\bar{E}_y\right)$$

$$\bar{e}_n = \bar{e}_n(x) = \frac{\operatorname{sgn}(d^2f/dx^2)}{\sqrt{1+\left(df/dx\right)^2}}\left(\bar{E}_y - \frac{df}{dx}\bar{E}_x\right)$$

$$\bar{e}_b = \operatorname{sgn}(d^2f/dx^2)\,\bar{E}_z$$

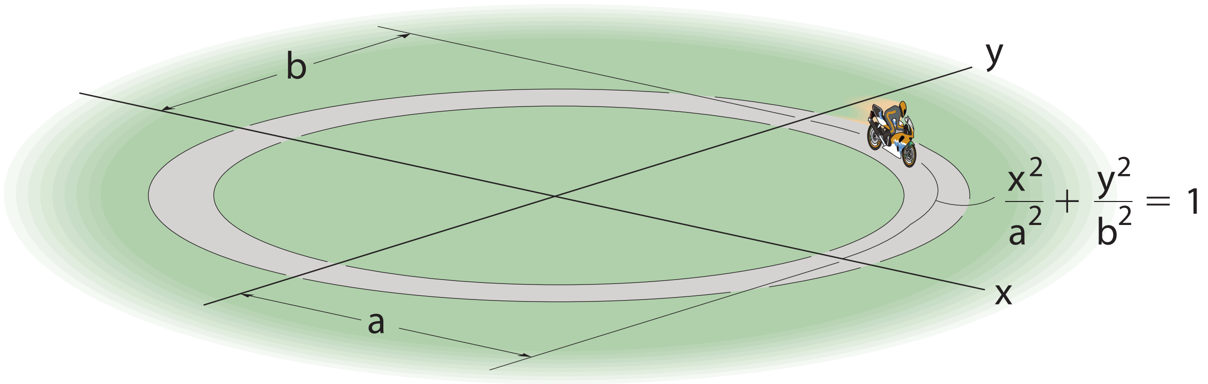
Curvature and torsion:

$$\kappa = \kappa(x) = |d^2f/dx^2| \Big/ \left(1+\left(df/dx\right)^2\right)^{3/2}$$

$$\tau = 0$$

Here κ can be interpreted as a rate of rotation of $\bar{e}_t$ and $\bar{e}_n$ about $\bar{e}_b = \pm\bar{E}_z$.

Example (Hibbeler 12-158)



The motorcycle travels along the elliptical track at a constant speed v. Determine the greatest magnitude of the acceleration if $a > b$.

We have a curve on a plane: $\bar{r} = x\bar{E}_x + y\bar{E}_y$.

The acceleration is $\bar{a} = 0\,\bar{e}_t + a_n\bar{e}_n = a_n\bar{e}_n$.

More specifically, $\bar{a} = \ddot{\bar{r}} = \frac{d\dot{\bar{r}}}{dt} = \frac{d}{dt}\left(\frac{d\bar{r}}{ds}\frac{ds}{dt}\right) = \frac{d}{dt}\left(v\,\bar{e}_t\right)$

$$= \cancelto{0}{\frac{dv}{dt}}\,\bar{e}_t + v\frac{d\bar{e}_t}{dt} = v\frac{d\bar{e}_t}{ds}\frac{ds}{dt} = \kappa v^2\bar{e}_n$$

v is constant, so maximizing the curvature κ is tantamount to maximizing $\|\bar{a}\|$. Since the curve is symmetric about the x-axis, we can take one half of it for analysis: $\bar{r} = x\bar{E}_x + f(x)\bar{E}_y$. The results of O'Reilly, Section 3.3.1 apply:

$$\kappa = \kappa(x) = \frac{\left|\frac{d^2f}{dx^2}\right|}{\left(1 + \left(\frac{df}{dx}\right)^2\right)^{3/2}}.$$

Let's compute d^2f/dx^2 separately:

$$f(x) = b\left(1 - \frac{x^2}{a^2}\right)^{1/2}$$

$$\frac{df}{dx} = \frac{b}{a^2} x \left(1 - \frac{x^2}{a^2}\right)^{-1/2}$$

$$\frac{d^2f}{dx^2} = \frac{b}{a^2}\left(1 - \frac{x^2}{a^2}\right)^{-1/2} + \frac{b}{a^2} x \left(1 - \frac{x^2}{a^2}\right)^{-3/2}\left(-\frac{1}{2}\right)\left(-\frac{2x}{a^2}\right)$$

$$= \frac{b}{a^2}\left(1 - \frac{x^2}{a^2}\right)^{-1/2} + \frac{b}{a^4} x^2 \left(1 - \frac{x^2}{a^2}\right)^{-3/2}$$

$$K = \frac{ab}{\left(1 + \frac{b^2}{a^2}\frac{x^2}{a^2 - x^2}\right)^{3/2} |a^2 - x^2|^{3/2}} = \frac{ab}{\left(a^2 - x^2 + \frac{b^2}{a^2} x^2\right)^{3/2}}$$

$$K_{max} = \lim_{x \to a} K = \frac{ab}{\left(a^2 - a^2 + \frac{b^2}{a^2} a^2\right)^{3/2}} = \frac{ab}{b^3} = \frac{a}{b^2}$$

Therefore,

$$\bar{a}_{max} = K_{max} v^2 \bar{e}_n = \frac{a v^2}{b^2} \bar{e}_n$$

and

$$\|\bar{a}_{max}\| = \frac{a v^2}{b^2} \quad .$$

Check units: $\frac{L}{T^2} \stackrel{?}{=} \frac{L(L/T)^2}{L^2}$ OK

3.4 Application to Particle Mechanics

We make two identifications:

1. The space curve $\mathscr{C}$ is identified as the path of a particle.
2. The arclength parameter s is considered to be a function of time t.

Kinematics

functional

Position: $\bar{r} = x\bar{E}_x + y\bar{E}_y + z\bar{E}_z = \bar{r}(t) = \hat{r}(s(t))$.

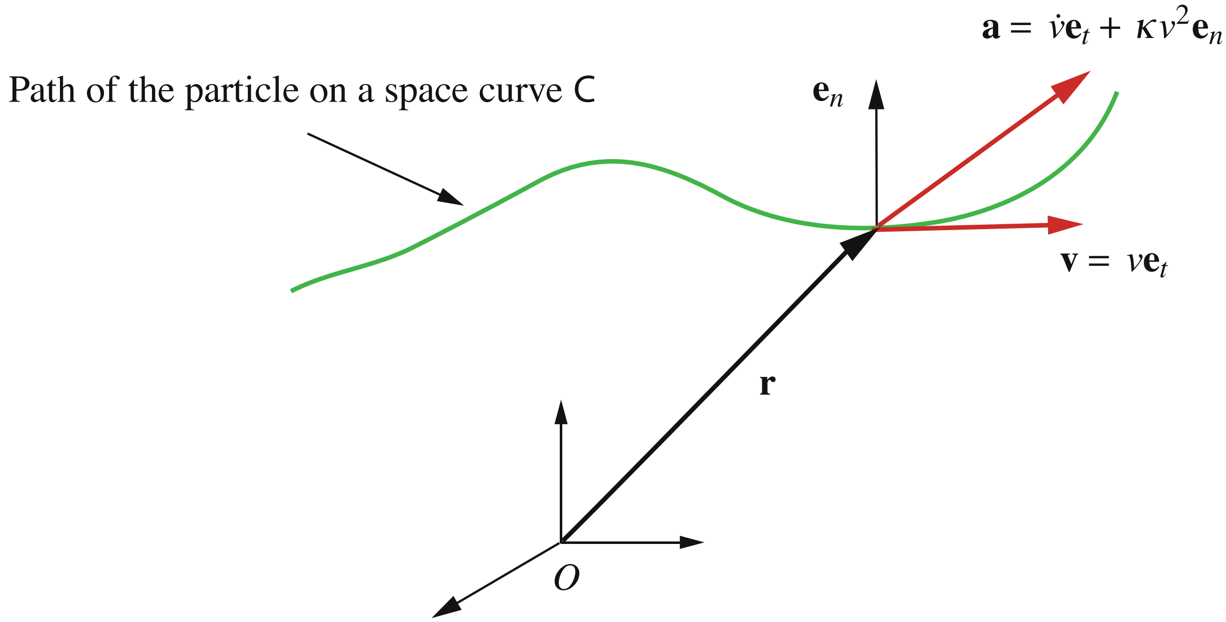
Velocity:

$$\bar{v} = \dot{x}\bar{E}_x + \dot{y}\bar{E}_y + \dot{z}\bar{E}_z = \dot{\bar{r}}(t) = \frac{d\bar{r}}{ds}\frac{ds}{dt} = \frac{ds}{dt}\bar{e}_t = v\bar{e}_t$$

Acceleration:

$$\bar{a} = \dot{\bar{v}} = \frac{d^2s}{dt^2}\bar{e}_t + K\left(\frac{ds}{dt}\right)^2\bar{e}_n = \dot{v}\bar{e}_t + Kv^2\bar{e}_n.$$

Remarkably, $\bar{a}$ lies entirely in the osculating plane.



Kinetics

For a particle of mass m, $\bar{F} = m\bar{a}$.

The resultant force $\bar{F}$ can be written as

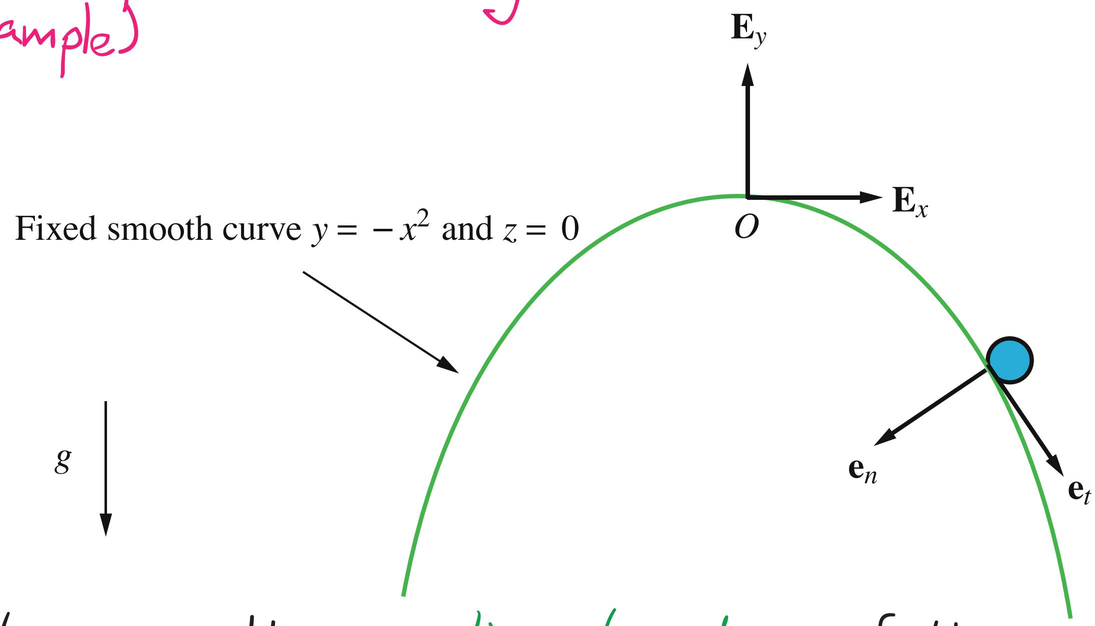
$$\bar{F} = F_t \bar{e_t} + F_n \bar{e_n} + F_b \bar{e_b}.$$

So, in the Sennet-Frenet basis,

$$\boxed{\begin{bmatrix} F_t \\ F_n \\ F_b \end{bmatrix} = m \begin{bmatrix} \dot{v} \\ \kappa v^2 \\ 0 \end{bmatrix}}.$$

Note that $F_b = 0$, so $\bar{F}$ is also entirely in the osculating plane.

3.5 A particle moving on a fixed curve under gravity (Example)



Determine the equation of motion of the particle and the force exerted "by the curve" to keep the particle on the curve.

3.5.1 Kinematics

From 3.3.1 we know that the arclength is (with $x_0 = 0$)

$$s = s(x) = \int_0^x \sqrt{1 + \left(\frac{df}{dx}\right)^2}\, du + s(0)$$

We have $df/dx = d(-x^2)/dx = -2x$, so

$$s(x) = \int_0^x \sqrt{1 + (-2u)^2}\, du + s(0) = \frac{x}{2}\sqrt{1+4x^2} + \frac{1}{4}\sinh^{-1}(2x) + s(0)$$

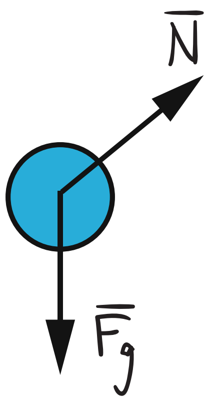
We can also compute the curvature:

$$\kappa = \kappa(x) = |d^2f/dx^2| \Big/ \left(1 + (df/dx)^2\right)^{3/2} = 2/(1+4x^2)^{3/2}$$

Acceleration: $\bar{a} = \dot{\bar{v}} = \dot{v}\,\bar{e}_t + \kappa v^2 \bar{e}_n$

$$= \frac{d^2s}{dt^2}\bar{e}_t + \kappa\left(\frac{ds}{dt}\right)^2 \bar{e}_n$$

3.5.2 Forces



Known force: $\bar{F}_g = -mg\,\bar{E}_y$

Unknown force: $\bar{N} = N_n \bar{e}_n + N_b \bar{e}_b$

$$\bar{F} = -mg\,\bar{E}_y + N_n \bar{e}_n + N_b \bar{e}_b$$

We need to write $\bar{E}_y$ in the Sennet-Frenet basis.
Recall:

with $f(x) = -x^2$

$$\bar{e}_t = \frac{1}{\sqrt{1+\left(df/dx\right)^2}}\left(\bar{E}_x + \frac{df}{dx}\bar{E}_y\right) = \frac{1}{\sqrt{1+4x^2}}\left(\bar{E}_x - 2x\bar{E}_y\right)$$

$$\bar{e}_n = \frac{\operatorname{sgn}\left(d^2f/dx^2\right)}{\sqrt{1+\left(df/dx\right)^2}}\left(\bar{E}_y - \frac{df}{dx}\bar{E}_x\right) = \frac{-1}{\sqrt{1+4x^2}}\left(2x\bar{E}_x + \bar{E}_y\right)$$

$$\Rightarrow 2x\,\bar{e}_t + \bar{e}_n = \frac{-4x^2-1}{\sqrt{1+4x^2}}\bar{E}_y \Rightarrow \bar{E}_y = \frac{-2x}{\sqrt{1+4x^2}}\bar{e}_t - \frac{1}{\sqrt{1+4x^2}}\bar{e}_n$$

Now we can write $\bar{F}$ in the Sennet-Frenet basis:

$$\bar{F} = \frac{2xmg}{\sqrt{1+4x^2}}\bar{e}_t + \left(N_n + \frac{mg}{\sqrt{1+4x^2}}\right)\bar{e}_n + N_b\,\bar{e}_b .$$

3.5.3 $\bar{F} = m\bar{a}$

In the Sennet-Frenet basis $\bar{F} = m\bar{a}$ is:

$$\begin{bmatrix} 2xmg/\sqrt{1+4x^2} \\ N_n + mg/\sqrt{1+4x^2} \\ N_b \end{bmatrix} = m \begin{bmatrix} \dot{v} \\ Kv^2 \\ 0 \end{bmatrix}$$

3.5.4 Analysis

First, note that: $v = \frac{ds}{dt} = \frac{ds}{dx}\frac{dx}{dt} = \dot{x}\sqrt{1+4x^2}$
and:

$$\dot{v} = \frac{d^2s}{dt^2} = \frac{4x}{\sqrt{1+4x^2}}\dot{x}^2 + \sqrt{1+4x^2}\,\ddot{x}.$$

The $\bar{e}_t$-equation gives:

$$\frac{2xmg}{\sqrt{1+4x^2}} = m\dot{v} \quad \Rightarrow$$

$$2xg = 4x\dot{x}^2 + (1+4x^2)\ddot{x} \quad \Rightarrow$$

$$\boxed{(1+4x^2)\ddot{x} + 4x\dot{x}^2 - 2xg = 0}$$

which is the equation of motion.

The $\bar{e}_n$-equation gives:

$$N_n = \frac{-mg}{\sqrt{1+4x^2}} + mKv^2 = \frac{-mg}{\sqrt{1+4x^2}} + mK(1+4x^2)\dot{x}^2$$

The $\overline{e_b}$-equation gives: $N_b = 0$.
So the force keeping the particle on the curve is:

$$\boxed{\overline{N} = N_n \overline{e_n}}.$$

So if we solve the equation of motion for $x(t)$, we will also know $\overline{N}(t)$.

Under what conditions will the particle deviate from the given spacecurve?

If $N_n < 0$, because if the force is in the opposite of the normal direction, the path of the particle will be on a new curve with a normal basis vector panted in the opposite direction of the old normal basis vector.

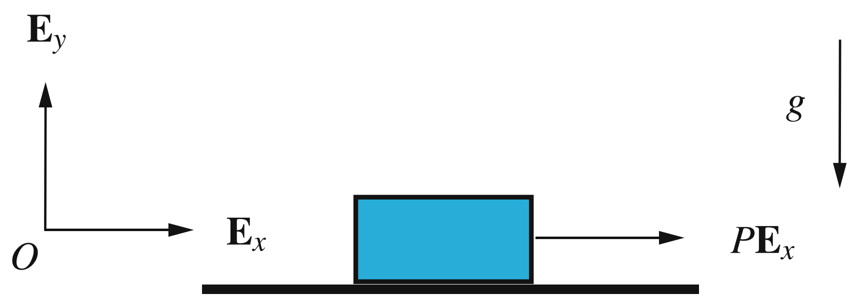
Chapter 4: Friction Forces and Spring Forces

Topics: - friction forces
- spring forces

4.1 An Experiment on Friction

A block of mass m on a surface with a force $P\bar{E}_x$.

What do we observe?



a) small $P \longrightarrow$ block remains at rest

b) beyond some $P = P^* \longrightarrow$ block starts to move

c) once moving $\longrightarrow P = P^{**}$ required to keep it going at a constant speed

d) P^* and P^{**} are proportional to the normal force $\bar{N}$.

Let's model these phenomena.

1) Kinematics

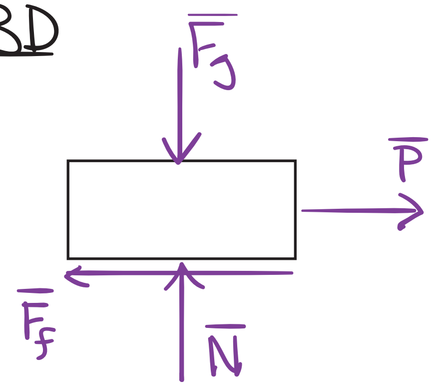
$$\bar{r} = x\bar{E}_x + y_0 E_y + z_0 E_z$$

$$\bar{v} = \dot{\bar{r}} = \dot{x}\bar{E}_x$$

$$\bar{a} = \dot{\bar{v}} = \ddot{x}\bar{E}_x$$

2) Forces

FBD



$\overline{F_g}, \overline{P}, \overline{F_f}, \overline{N}$

$F_g = -mg\overline{E_y}$ $\qquad$ $\overline{P} = P\overline{E_x}$

$P > 0$

What do we know about $\overline{N}$ and $\overline{F_f}$?

$\overline{N} = N\overline{E_y}$

$\overline{F_f}$ is proportional to $\overline{N}$ and opposing motion in the x-direction:

$$\overline{F_f} = F_{fx}\overline{E_x} \qquad (F_{fx} < 0)$$

The two cases give different coefficients:

Static $(P < P^*)$: $\mu = \mu_s$

Dynamic $(P = P^{**})$: $\mu = \mu_d$

3) $\overline{F} = m\overline{a}$ in Cartesian coordinates

$$\begin{bmatrix} P + F_{fx} \\ N - mg \\ 0 \end{bmatrix} = m \begin{bmatrix} \ddot{x} \\ 0 \\ 0 \end{bmatrix}$$

4) Analysis

$N = mg$ $\qquad\qquad F_{fx} = m\ddot{x} - P$

If the block is not moving,

$\ddot{x} = 0 \Rightarrow \overline{F_f} = -P\overline{E_x}$ while $P \leq P^* = \mu_s mg$.

If the block is moving at a constant speed,

$\ddot{x} = 0 \Rightarrow \overline{F_f} = -P\overline{E_x} = -\mu_d |N| E_x$

If the block is accelerating,

$\overline{F_f} = (m\ddot{x} - P)\overline{E_x}$.

The coefficients of static friction μ_s and dynamic friction μ_d depend on the nature of the surface and the block. They are determined experimentally.

4.2 Static and Dynamic Coulomb Friction Forces

The above applies only to flat, stationary surfaces. The following theory applies to:

(a) cases where the surface is curved,
(b) cases where the particle is moving on a space curve,
(c) cases where the spacecurve or surface is moving.

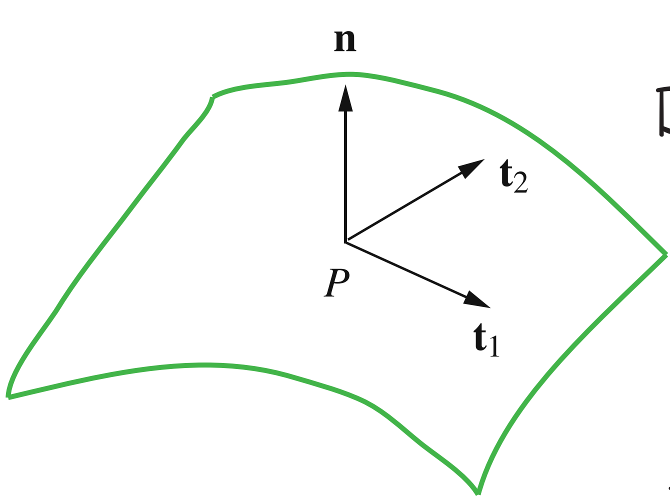
Notation

$\bar{r}, \bar{v}, \bar{a}$ — position, velocity, and acceleration of the particle

$\bar{v}_c$ — velocity of the space curve at the point-of-contact between the curve + particle

$\bar{v}_s$ — velocity of the surface at the point-of-contact between the surface + particle

4.2.1 Particle on a Surface



Define: $\bar{n}$ — normal basis vec.
$\bar{t}_1$ — tangential basis vec.
$\bar{t}_2$ — tangential basis vec.

How can we model friction for a particle on this surface?

Define: $\bar{V}_{rel} = \bar{V} - \bar{V}_s$.

If $\bar{V}_{rel} = \bar{0} \rightarrow$ static friction.
If $\bar{V}_{rel} \neq \bar{0} \rightarrow$ dynamic friction.

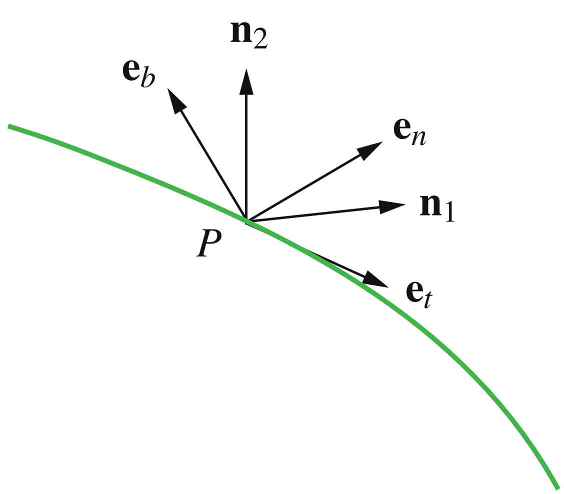
The amount of static friction force is limited by:

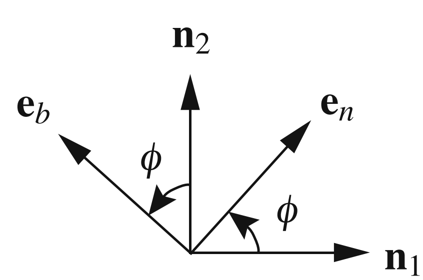
$$\boxed{\|\bar{F}_f\| \leq \mu_s \|\bar{N}\|}.$$

If this is false, $\bar{V}_{rel} \neq \bar{0}$ and

$$\boxed{\bar{F}_f = -\mu_d \|\bar{N}\| \frac{\bar{V}_{rel}}{\|\bar{V}_{rel}\|}}.$$

4.2.2 A Particle on a Space Curve





Define $\bar{n}_1, \bar{n}_2$ — normal basis vectors in the $\bar{e}_n$-$\bar{e}_b$ plane.

Define: $\bar{V}_{rel} = \bar{V} - \bar{V}_c$

If $\bar{v}_{rel} = \vec{0} \rightarrow$ static friction.
If $\bar{v}_{rel} \neq \bar{0} \rightarrow$ dynamic friction.

The amount of static friction force is limited by:

$$\|\bar{F}_f\| \leq \mu_s \|\bar{N}\| .$$

If this is false, $\bar{v}_{rel} \neq \bar{0}$ and

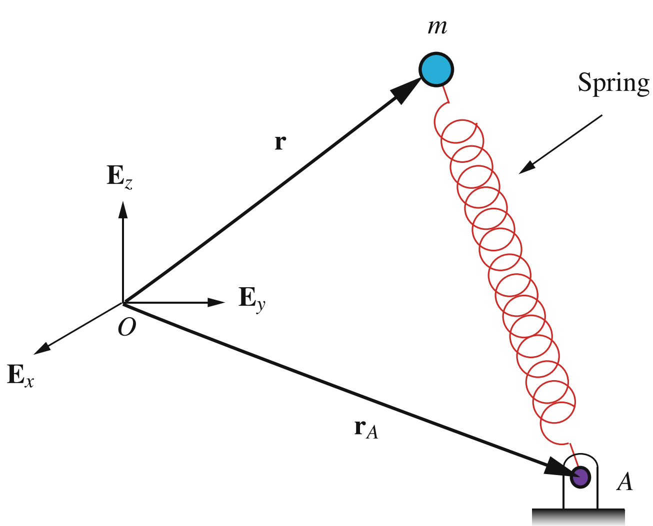
$$\bar{F}_f = -\mu_d \|\bar{N}\| \frac{\bar{v}_{rel}}{\|\bar{v}_{rel}\|} .$$

4.4 Hooke's Law and Linear Springs

Hooke's Law in modern terms:

The force from a spring is proportional to its extension. ("Ut tensio sic vis.")

We call the contant of proportionality K.



Let's explore the dynamics of springs in their "linear" regime.

Assume springs to be massless with unstretched length L.

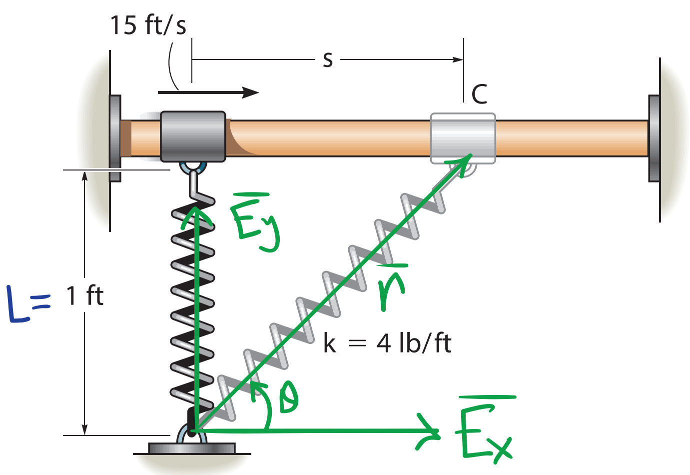
With position vectors as shown above, Hooke's Law can be written:

$$\|\bar{F}_s\| = |k(\|\bar{r} - \bar{r}_A\| - L)| .$$

The force vector is:

$$\bar{F}_s = -k(\|\bar{r} - \bar{r}_A\| - L) \frac{\bar{r} - \bar{r}_A}{\|\bar{r} - \bar{r}_A\|} .$$

13-36 (Hibbeler) Example



Given the 2lb collar on the smooth rod with the spring of unstretched length 1ft, find $\hat{v}(s)$, the velocity as a function of arclength if $\hat{v}(s=0) = v_0$. Also find $\hat{N}(s)$, the force from the rod.

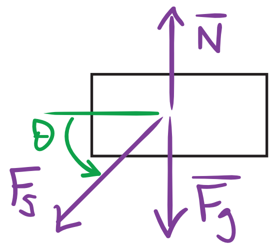
Kinematics

$$\bar{r} = x\bar{E}_x + y\bar{E}_y, \quad \bar{v} = \dot{x}\bar{E}_x + \dot{y}\bar{E}_y, \quad \bar{a} = \ddot{x}\bar{E}_x + \ddot{y}\bar{E}_y$$

$$\hat{r}(s) = s\bar{E}_x + L\bar{E}_y \qquad \|\hat{r}(s)\| = \sqrt{s^2 + L^2}$$

$$\hat{v}(s) = \dot{\hat{r}} = \dot{s}\bar{E}_x \qquad \hat{a}(s) = \dot{\hat{v}} = \ddot{s}E_x$$

Forces



$$\bar{F}_g = -mg\bar{E}_y$$

$$\bar{F}_s = -K(\|\bar{r}\| - L)\frac{\bar{r}}{\|\bar{r}\|} \quad \text{where } L = 1\text{ ft.}$$

$$= -K\frac{\left(\sqrt{s^2+L^2} - L\right)}{\sqrt{s^2+L^2}}(s\bar{E}_x + L\bar{E}_y)$$

$\bar{F} = m\bar{a}$

In the Cartesian basis,

$$\begin{bmatrix} -Ks\left(\sqrt{s^2+L^2} - L\right)/\sqrt{s^2+L^2} \\ N - mg - KL\left(\sqrt{s^2+L^2} - L\right)/\sqrt{s^2+L^2} \end{bmatrix} = m\begin{bmatrix} \ddot{s} \\ 0 \end{bmatrix}$$

Analysis

The $\bar{E}_x$ equation is the equation of motion:

$$-Ks\left(\sqrt{s^2+L^2} - L\right)/\sqrt{s^2+L^2} = m\ddot{s} = m\frac{dv}{dt} = m\frac{dv}{ds}\frac{ds}{dt} = mv\frac{dv}{ds}$$

Separate and solve:

$$-K\int_0^s \frac{\sigma\sqrt{\sigma^2+L^2} - L\sigma}{\sqrt{\sigma^2+L^2}}\,d\sigma = m\int_{v_0}^{v} u\,du$$

$$-K\left(\tfrac{1}{2}s^2 - L\sqrt{s^2+L^2}\right) - L^2K = m(v^2 - v_0^2)$$

$$\boxed{\hat{v}(s) = \pm\left(v_0^2 - \frac{K}{m}\left(\tfrac{1}{2}s^2 - L\sqrt{s^2+L^2}\right) - L^2K/m\right)^{1/2}}$$

The $\overline{E}_y$ equation gives

$$\boxed{\hat{N}(s) = \left(mg + KL\left(\sqrt{s^2+L^2} - L\right)/\sqrt{s^2+L^2}\right)\overline{E}_y}\;.$$

Chapter 5: Power, Work, and Energy

Topics

- The concepts of power, work, and energy
- And their precise definitions
- Work-energy theorem
- Conservative forces and energy conservation

5.1 The Concepts of Work and Power

We will rigorously define work and power momentarily, but we can gain some intuition in the simple case of a constant force acting on a particle in the direction of its motion — work = force × distance moved.

Mechanical power is the rate at which work is performed.

Energy is the ability to perform work.

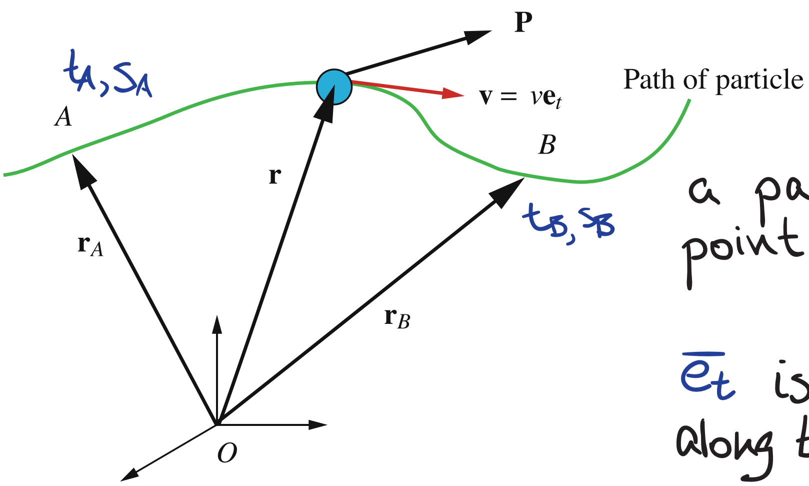
5.2 The Power of a Force

Consider a force $\bar{P}$ acting on a particle of mass m.

Def.: $\boxed{\text{Mechanical Power of } \bar{P} = \bar{P} \cdot \bar{v}}$ ← rate of work by $\bar{P}$

where $\bar{v} = \dot{\bar{r}}$ is, as usual, the absolute velocity.

Therefore, if $\bar{P} \cdot \bar{v} = 0$, $\bar{P}$ does no work.



Consider the work done by a force $\bar{P}$ on a particle from point A to point B.

$\bar{e}_t$ is the unit tangent vector along the particle's path.

The work has several equivalent expressions:

$$W_{AB} = \int_{t_A}^{t_B} \underbrace{\bar{P}\cdot\bar{v}}_{\text{mechanical power}}\, dt$$

$$= \int_{t_A}^{t_B} \bar{P}\cdot\frac{d\bar{r}}{dt}\, dt = \int_{\bar{r}_A}^{\bar{r}_B} \bar{P}\cdot d\bar{r} \qquad (*)$$

$$= \int_{t_A}^{t_B} \bar{P}\cdot\frac{ds}{dt}\bar{e}_t\, dt = \int_{s_A}^{s_B} \bar{P}\cdot\bar{e}_t\, ds$$

$\therefore$ Only the tangential component of $\bar{P}$ does work!

Let's write down $\bar{P}$ and $d\bar{r}$ in different bases:

$$\bar{P} = P_x\bar{E}_x + P_y\bar{E}_y + P_z\bar{E}_z$$
$$= P_r\bar{e}_r + P_\theta\bar{e}_\theta + P_z\bar{E}_z$$
$$= P_t\bar{e}_t + P_n\bar{e}_n + P_b\bar{e}_b$$

$$d\bar{r} = dx\,\bar{E}_x + dy\,\bar{E}_y + dz\,\bar{E}_z$$
$$= dr\,\bar{e}_r + r\,d\theta\,\bar{e}_\theta + dz\,\bar{E}_z$$
$$= ds\,\bar{e}_t$$

From $(*)$, $$W_{AB} = \int_{\bar{r}_A}^{\bar{r}_B} P_x\,dx + P_y\,dy + P_z\,dz$$

$$= \int_{\bar{r}_A}^{\bar{r}_B} P_r\,dr + P_\theta\,r\,d\theta + P_z\,dz$$

$$= \int_{s_A}^{s_B} P_t\,ds$$

5.3 The Work-Energy theorem

Definition: The Kinetic energy of a particle is defined to be

$$T \triangleq \frac{1}{2} m \bar{v} \cdot \bar{v} = \frac{1}{2} m \|\bar{v}\|^2$$

The work-Energy theorem relates the time rate-of-change of the Kinetic energy and the resultant force $\bar{F}$ acting on a particle:

$$\frac{dT}{dt} = \frac{1}{2} m \left(\bar{v} \cdot \dot{\bar{v}} + \dot{\bar{v}} \cdot \bar{v} \right)$$

$$= m \dot{\bar{v}} \cdot \bar{v}$$

because $\bar{F} = m\bar{a}$

$$\Rightarrow \boxed{\dot{T} = \underbrace{\bar{F} \cdot \bar{v}}_{\text{mechanical power of resultant force}}}$$

5.4 Conservative Forces

Let $U = U(\bar{r})$ be the potential energy function. A force $\bar{P}$ is defined to be conservative if

$$\bar{P} = -\text{grad}\, U = \underbrace{-}_{\text{historical convention}} \frac{\partial U}{\partial \bar{r}}.$$

If a force $\bar{P}$ is conservative, then the work done by $\bar{P}$ depends only on the endpoints and not the path!

Let's show this:

$$
\begin{aligned}
W_{AB} &= \int_{\bar{r}_A}^{\bar{r}_B} \bar{P} \cdot d\bar{r} \\
&= -\int_{\bar{r}_A}^{\bar{r}_B} \frac{\partial U}{\partial \bar{r}} \cdot d\bar{r} \\
&= -\int_{\bar{r}_A}^{\bar{r}_B} dU \\
&= U(\bar{r}_A) - U(\bar{r}_B)
\end{aligned}
$$

Therefore, if $\bar{r}_A = \bar{r}_B$ (closed path), $W_{AB} = 0$ (remember: $\bar{P}$ was a conservative force!).

If $\bar{P}$ is conservative, then its mechanical power is:

$$\bar{P} \cdot \bar{v} = -\frac{\partial U}{\partial \bar{r}} \cdot \frac{d\bar{r}}{dt} = -\frac{dU}{dt}.$$

Examples of non-conservative forces:

- tension in inextensible strings/cables
- friction forces
- normal forces

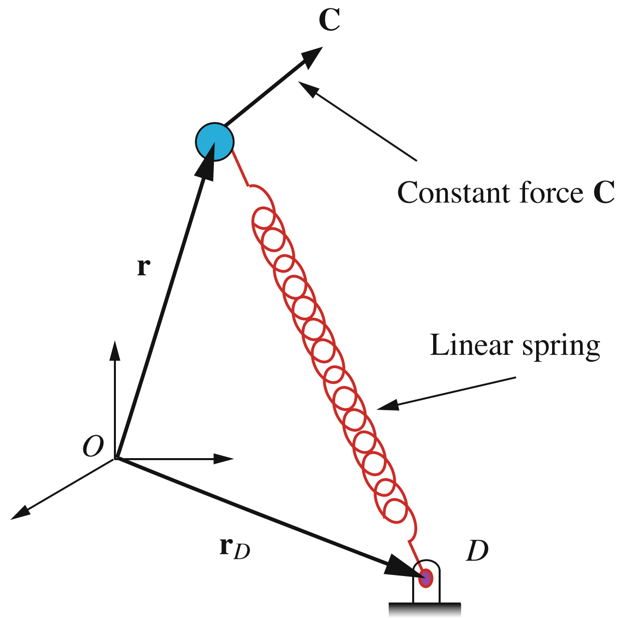
Next time we'll look at examples of conservative forces, especially:

- spring forces
- gravitational forces

5.5 Examples of Conservative Forces

The two primary examples of conservative forces are:

- constant forces $\bar{C}$ (e.g. gravity) +
- spring forces $\bar{F}_s$.



5.5.1 Constant Forces

Let's guess the form of a potential energy function for a constant force:

$$U_c = -\bar{C} \cdot \bar{r} \ .$$

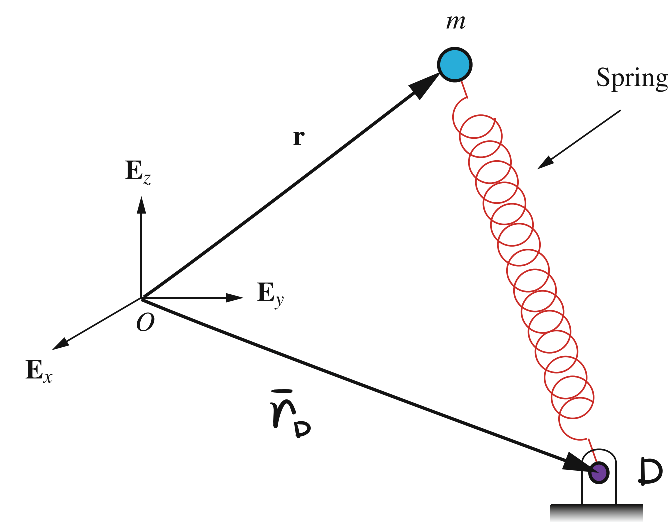
Check it by seeing if it satisfies: $\bar{P} \cdot \bar{v} = -\frac{\partial U}{\partial \bar{r}} \cdot \bar{v} = -\frac{\partial U}{\partial t}$.

$$-\frac{\partial U_c}{\partial t} = -\frac{d}{dt}(-\bar{C} \cdot \bar{r}) = \cancel{\dot{\bar{C}}} \cdot \bar{r} + \bar{C} \cdot \dot{\bar{r}} = \bar{C} \cdot \bar{v} \quad \checkmark$$

E.g. Gravity: $\bar{F}_g = -mg E_y$

$$U_g = -\bar{F}_g \cdot \bar{r} = mgy \quad \text{(Cartesian basis)}$$

5.5.2 Spring Forces



Recall that a spring force (with fixed point D) is given by

$$\bar{F}_s = -K(\|\bar{r}-\bar{r}_D\|-L)\frac{\bar{r}-\bar{r}_D}{\|\bar{r}-\bar{r}_D\|}$$

O'Reilly derives the potential energy for a spring forces $\bar{F}_s$:

$$U_s = \frac{1}{2}K\underbrace{(\|\bar{r}-\bar{r}_D\|-L)}_{\text{change in length of the spring}}^2 .$$

5.6 Energy Conservation

Consider a particle acted on by the forces:

$F_1, F_2, \ldots, F_n,$ ⟵ conservative forces
F_{nc} ⟵ sum of non-conservative forces

The conservative forces have potential energies U_i $(i \in \{1,\ldots,n\})$.

The resultant conservative force is

$$\bar{F}_c = \sum_{i=1}^{n}\bar{F}_i = -\sum_{i=1}^{n}\frac{\partial U_i}{\partial \bar{r}} = -\frac{\partial U}{\partial \bar{r}} \quad \text{where} \quad U = \sum_{i=1}^{n} U_i .$$

So the total resultant force on the particle is:

$$\boxed{\overline{F} = \overline{F}_c + \overline{F}_{nc} = \overline{F}_{nc} - \frac{\partial U}{\partial \overline{r}}} .$$

Aside: define the total energy

The total energy E is defined by:

$$E = T + U .$$

We can rewrite the work-energy theorem in terms of the total energy E. Starting with our original def.,

$$\begin{aligned} \dot{T} &= \overline{F} \cdot \overline{v} \\ &= (\overline{F}_c + \overline{F}_{nc}) \cdot \overline{v} \\ &= -\dot{U} + \overline{F}_{nc} \cdot \overline{v} \end{aligned}$$

$$\Longrightarrow \boxed{\dot{E} = \dot{T} + \dot{U} = \overline{F}_{nc} \cdot \overline{v}}$$

If the non-conservative forces do no work on the particle, i.e.,

$\overline{F}_{nc} \cdot \overline{v} = 0$, it implies $\dot{E} = 0$.

Therefore $E = T + U = \frac{1}{2} m \|\overline{v}\|^2 + U(\overline{r}) = E_0$ (a constant)

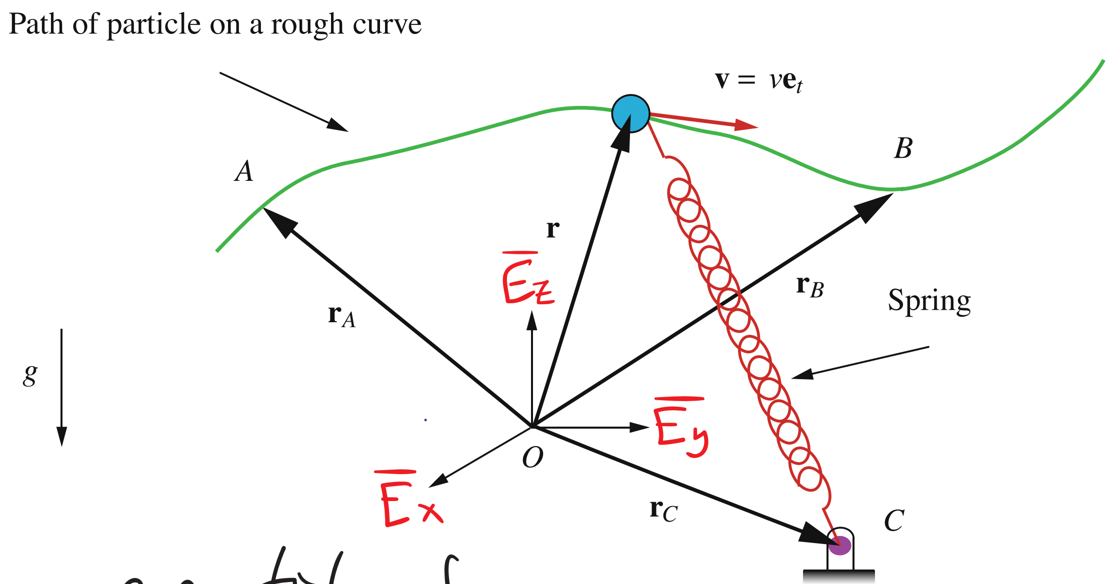
We typically use $\dot{E} = 0$ to find a single unknown in a conservative motion of a particle.

E.g. Given a particle w/ initial speed v_0 and initial position $\bar{r}_0$, find the particle's velocity at another location $\bar{r}_1$. Assume only conservative forces act on the particle.

$$\left.\begin{aligned} E_0 &= \frac{1}{2} m v_0^2 + U(\bar{r}_0) \\ E_1 &= \frac{1}{2} m v_1^2 + U(\bar{r}_1) \end{aligned}\right\} \text{from energy conservation} \Rightarrow E_0 = E_1$$

$$\Rightarrow \quad v_1 = \left(v_0^2 + \frac{2}{m}\left(U(\bar{r}_0) - U(\bar{r}_1)\right)\right)^{1/2}.$$

5.7 A Particle Moving on a Rough Curve (example)



Given a particle of mass m; moving on the fixed, rough space curve above; attached to the linear spring with spring constant K and unstretched length L; in a gravitational field; find:
(a) the work done by friction from point A to point B +
(b) if the curve was smooth, $\bar{v}(t_B)$ given $\bar{v}(t_A)$.

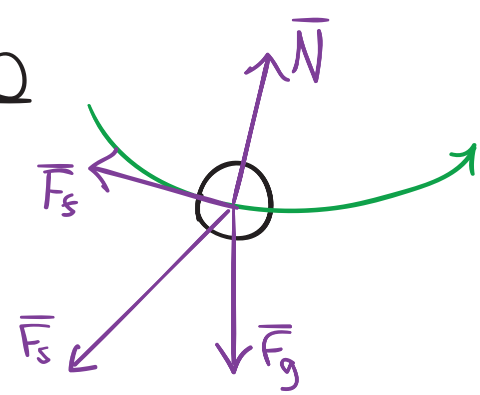
5.7.1 Kinematics

In the Serret-Frenet basis,

$$\bar{v} = v\bar{e}_t \qquad \bar{a} = \dot{v}\bar{e}_t + \kappa v^2 \bar{e}_n \qquad T = \frac{1}{2}mv^2$$

5.7.2 Forces

FBD



$$\overline{F}_g = -mg\overline{E}_z$$

$$\overline{F}_s = -K(\|\bar{r}-\bar{r}_c\| - L)\frac{\bar{r}-\bar{r}_c}{\|\bar{r}-\bar{r}_c\|}$$

$$\overline{N} = N_n\bar{e}_n + N_b\bar{e}_b$$

So the resultant force is:

$$
\begin{aligned}
\bar{F} &= \bar{F}_g + \bar{F}_s + \bar{N} + \bar{F}_f \\
&= -mg\,\bar{E}_z - K(\|\bar{r}-\bar{r}_c\| - L)\frac{\bar{r}-\bar{r}_c}{\|\bar{r}-\bar{r}_c\|} + N_n\,\bar{e}_n + N_b\,\bar{e}_b + \bar{F}_f .
\end{aligned}
$$

5.7.2 Work done by friction

From the work-energy theorem: $\dot{T} = \bar{F}\cdot\bar{v}$.

But $\bar{N} \perp \bar{v}$ so $\bar{N}\cdot\bar{v} = 0$.

The spring + gravitational forces are conservative, so the other form of the work-energy theorem is helpful:

$$\dot{E} = \bar{F}_{nc}\cdot\bar{v} = \bar{F}_f\cdot\bar{v} .$$

Integrating,

$$\int_{E_A}^{E_B} dE = \int_{t_A}^{t_B} \bar{F}_f\cdot\bar{v}\,dt$$

$$\underbrace{E_B - E_A}_{\text{this is easy to compute}} = \underbrace{\int_{t_A}^{t_B} \bar{F}_f\cdot\bar{v}\,dt}_{\text{not so much, but this is what we want to know}} = W_{AB_f}$$

So we can compute W_{AB_f}, the work of the friction force, without computing the complicated integral.

$$
\begin{aligned}
W_{AB_f} &= E_B - E_A = (T_A + U_A) - (T_B + U_B) \\
&= \tfrac{1}{2}m(v_B^2 - v_A^2) + mg\,\bar{E}_z\cdot(\bar{r}_B - \bar{r}_A) + \tfrac{1}{2}K\left((\|\bar{r}_B-\bar{r}_c\| - L)^2 - (\|\bar{r}_A-\bar{r}_c\| - L)^2\right)
\end{aligned}
$$

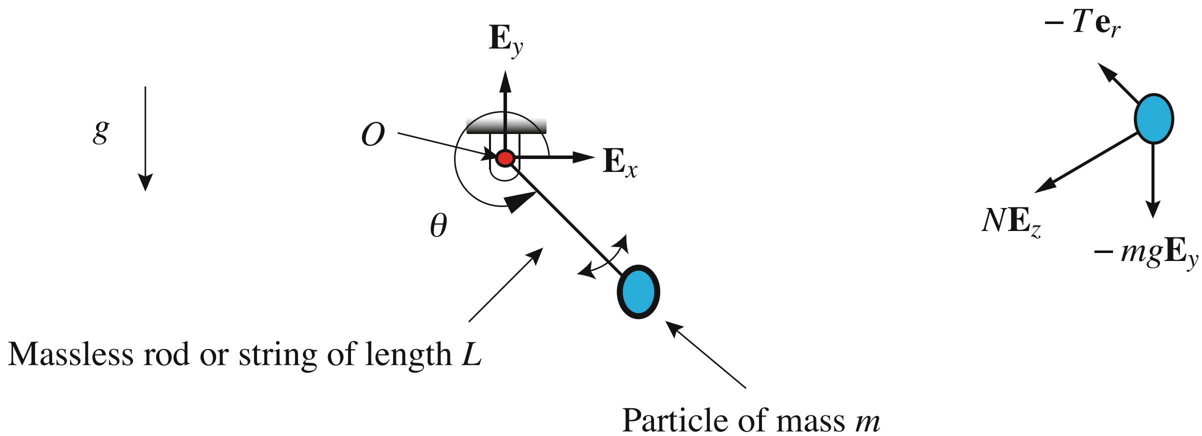
What if the curve was smooth?

Energy is conserved. Therefore $\dot{E}=0$, $W_{AB_f}=0$, and $E_B = E_A$. From the equation above for W_{AB_f},

$$v_B = \left(v_A^2 - 2g\bar{E}_z \cdot (\bar{r}_B - \bar{r}_A) - \frac{K}{m}\left((\|\bar{r}_B - \bar{r}_c\| - L)^2 - (\|\bar{r}_A - \bar{r}_c\| - L)^2 \right) \right)^{1/2}.$$

Notice that v_B doesn't depend on the path taken between A and B.

The Planar Pendulum (example)



In this problem, which we analyzed in Section 2.4, the only non-conservative forces are the tension in the string/rod and the normal force. However, neither of these forces is in the direction of motion.

Therefore, energy is conserved, $\dot{E}=0$, and we can use this to determine, for instance, the velocity of the particle at a given point in the motion. We only need to know what the constant energy is, which is often computed from knowledge of the system's kinetic + potential energies at some point.

Chapter 6: Momenta, Impulses, + Collisions

Topics:
- linear + angular momenta of a single particle
- conservation of momentum
- impact

6.1 Linear Momentum + Its Conservation

Consider a particle of mass m, position $\bar{r}$, and velocity $\bar{v}$. Recall the definition od linear momentum:

$$\boxed{\bar{G} = m\bar{v}} \quad .$$

6.1.1 Linear Impulse + Linear Momentum

The integral form of the "balance of linear momentum" $\bar{F} = m\bar{a}$ is:

$$\boxed{\bar{G}(t_1) - \bar{G}(t_2) = \underbrace{\int_{t_0}^{t_1} \bar{F}\, dt}_{\text{Linear impulse}}}$$

6.1.2 Conservation of Linear Momentum

Let $\bar{c}$ be a vector. $\bar{G}$ is conserved in the direction of $\bar{c}$ iff

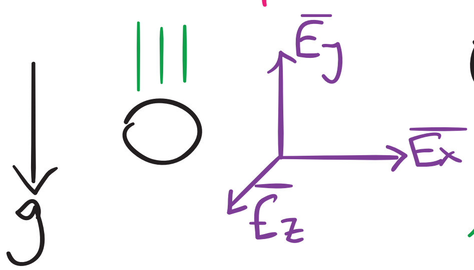
$$\frac{d}{dt}(\bar{G}\cdot\bar{c}) = 0 \quad .$$

This implies: $\dot{\bar{G}} \cdot \bar{c} + \bar{G} \cdot \dot{\bar{c}} = 0 \implies$

$$\bar{F} \cdot \bar{c} + \bar{G} \cdot \dot{\bar{c}} = 0 \qquad (\bar{F} \triangleq \dot{\bar{G}})$$

If $\bar{c}$ is a constant vector, $\bar{F} \cdot \bar{c} = 0$, i.e. linear momentum is conserved in the constant direction $\bar{c}$ iff there is no resultant force in that direction.

6.1.3 Example: Particle in a gravitational field



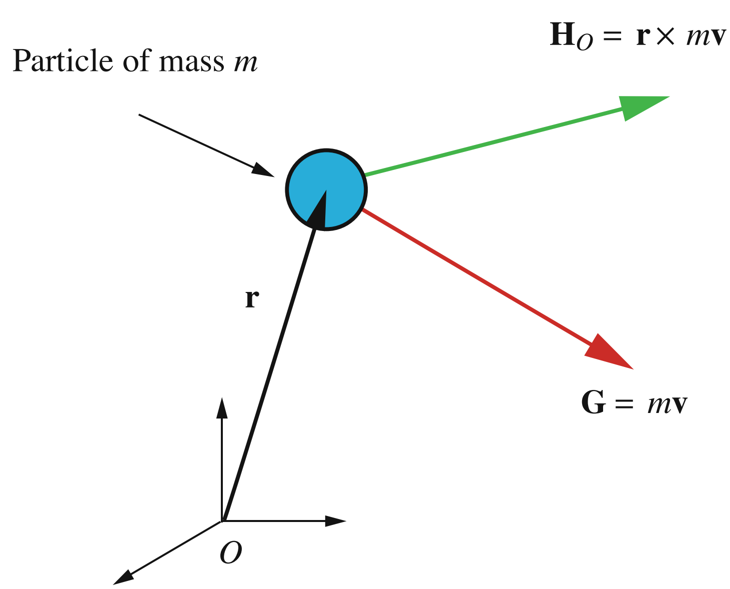
Q: In which directions is linear momentum conserved?

A: In any direction perpendicular to $\bar{E}_y$.

6.2 Angular Momentum + Its Conservation

Let the angular momentum about the point O, $\bar{H}_0$, of a particle of mass m be defined as:

$$\boxed{\begin{aligned} \bar{H}_0 &\triangleq \bar{r} \times \bar{G} \\ &= \bar{r} \times m\bar{v} \end{aligned}}$$



In Cartesian coordinates,

$$\overline{H}_o = \det\begin{bmatrix} \overline{E}_x & \overline{E}_y & \overline{E}_z \\ x & y & z \\ m\dot{x} & m\dot{y} & m\dot{z} \end{bmatrix}$$

$$= m(y\dot{z} - z\dot{y})\overline{E}_x + m(z\dot{x} - x\dot{z})\overline{E}_y + m(x\dot{y} - y\dot{x})\overline{E}_z$$

In cylindrical polar coordinates

$$\overline{H}_o = \det\begin{bmatrix} \overline{e}_r & \overline{e}_\theta & \overline{E}_z \\ r & 0 & z \\ m\dot{r} & mr\dot{\theta} & m\dot{z} \end{bmatrix}$$

$$= -mzr\dot{\theta}\,\overline{e}_r + m(z\dot{r} - r\dot{z})\overline{e}_\theta + mr^2\dot{\theta}\,\overline{E}_z$$

When the motion is planar, this simplifies to:

$$\overline{H}_o = mr^2\dot{\theta}\,\overline{E}_z \quad .$$

6.2.1 Angular Momentum Theorem

How does the angular momentum evolve in time?

$$\dot{\overline{H}}_o = \frac{d}{dt}(\overline{r} \times m\overline{v}) = \overline{v} \times m\overline{v} + \overline{r} \times \overline{F} = \overline{r} \times \overline{F}$$

The final result we call the angular momentum theorem

$$\boxed{\dot{\overline{H}}_o = \overline{r} \times \overline{F}} \quad .$$

6.2.2 Conservation of Angular Momentum

The angular momentum in the direction of a vector $\bar{c}$ is conserved iff

$$\frac{d}{dt}\left(\bar{H}_o \cdot \bar{c}\right) = 0 \quad .$$

This implies $\dot{\bar{H}}_o \cdot \bar{c} + \bar{H}_o \cdot \dot{\bar{c}} = 0 \implies$

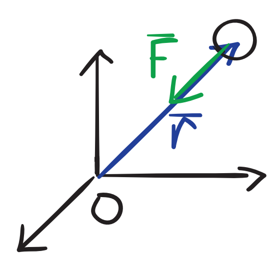
$$(\bar{r} \times \bar{F}) \cdot \bar{c} + \bar{H}_o \cdot \dot{\bar{c}} = 0 .$$

If $\bar{c}$ is constant, this reduces to

$$(\bar{r} \times \bar{F}) \cdot \bar{c} = 0$$

In this class, often we can choose $\bar{E}_z = \bar{c}$.

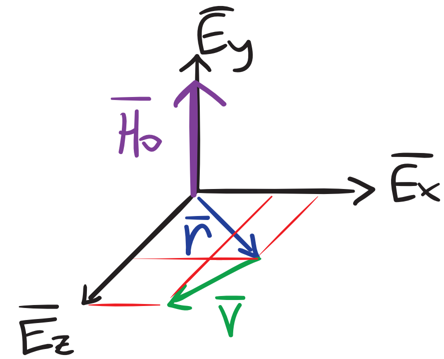
6.2.3 Central Force Problems



When the resultant force $\bar{F}$ is parallel to $\bar{r}$: $\dot{\bar{H}}_o = \bar{r} \times \bar{F} = \bar{0}$.

Therefore, $\bar{H}_o$ is conserved.

Let $\bar{H}_o = h\bar{h}$.



We can set-up these problems such that $\bar{E}_z = \bar{h}$, $\bar{r} = r\,\bar{e}_r$, + $\bar{v} = \dot{r}\bar{e}_r + r\dot{\theta}\,\bar{e}_\theta$ by choosing $\bar{H}_o = h\bar{E}_z = \bar{r}(t_o) \times m\bar{v}(t_o)$ (w/ initial cond. $\bar{r}(t_o)$ + $\bar{v}(t_o)$)

6.3 Collision of Particles

In this section we model the collision of particles. In order to lend the model some realism, we have to allow the particles to "deform," which is ad hoc.

The following theory is called "frictionless, oblique, central impact of two particles."

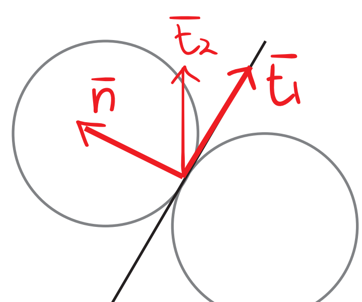
6.3.1 The Model and the Impact Stages

We model two masses m_1 and m_2 as particles with position vectors locating the centers of mass.

Of interest are four distinct time intervals:

I: $t < t_0$ — just before impact @ t_0
II: $t_0 < t < t_1$ — during the compression phase of impact
III: $t_1 \leq t < t_2$ — during the restitution phase of impact
IV: $t_2 \leq t$ — just after loss of contact

We use the basis $(\bar{n}, \bar{t}_1, \bar{t}_2)$:



6.3.2 Linear Impact During Impulses

Forces during II have subscript "d" and forces during III have subscript "r"!

The forces of m_1 on m_2 are: $\overline{F}_{2d} = F_{2d}\,\overline{n}$ + $\overline{F}_{2r} = F_{2r}\,\overline{n}$.
The forces of m_2 on m_1 are: $\overline{F}_{1d} = F_{1d}\,\overline{n}$ + $\overline{F}_{1r} = F_{1r}\,\overline{n}$.

These forces have no tangential components!

All other forces on m_1 + m_2 have resultant forces $\overline{R}_1$ + $\overline{R}_2$, respectively. We assume that the impulses during II + III are dominated by the inter-particle forces.

We can define the coefficient of restitution e as

$$e \triangleq \frac{\int_{t_1}^{t_2} \overline{F}_{1r} \cdot \overline{n}\, d\tau}{\int_{t_0}^{t_1} \overline{F}_{1d} \cdot \overline{n}\, d\tau} = \frac{\int_{t_1}^{t_2} \overline{F}_{2r} \cdot \overline{n}\, d\tau}{\int_{t_0}^{t_1} \overline{F}_{2d} \cdot \overline{n}\, d\tau}.$$

So, if $e=1$, the compression and restitution impulses are equal. If $e=0$, there is no restitution impulse. The former is called a perfectly elastic collision, the latter a perfectly inelastic collision. In general, $0 \le e \le 1$, and e is determined experimentally.

Let "primed" velocities denote velocities after impact. It can be shown that:

$$\boxed{e = \frac{\bar{v}_2' \cdot \bar{n} - \bar{v}_1' \cdot \bar{n}}{\bar{v}_1 \cdot \bar{n} - \bar{v}_2 \cdot \bar{n}}} \;.$$

This is a very useful equation.

6.3.3 Linear Momenta

The integral form of the balance of linear momentum for each particle gives:

$$m_1\bar{v}_1' - m_1\bar{v}_1 = \int_{t_0}^{t_1} \left(\bar{F}_{1d}(\tau) + \bar{R}_1(\tau)\right) d\tau + \int_{t_1}^{t_2} \left(\bar{F}_{1r} + \bar{R}_1(\tau)\right) d\tau$$

$$m_2\bar{v}_2' - m_2\bar{v}_2 = \int_{t_0}^{t_1} \left(\bar{F}_{2d}(\tau) + \bar{R}_2(\tau)\right) d\tau + \int_{t_1}^{t_2} \left(\bar{F}_{2r} + \bar{R}_2(\tau)\right) d\tau$$

Using e and assuming that the effects of $\bar{R}_i$ are negligible during impact, and that $\bar{F}_{1d} = -\bar{F}_{2d}$ & $\bar{F}_{1r} = -\bar{F}_{2r}$,

$$m_1\bar{v}_1' - m_1\bar{v}_1 = (1+e)\int_{t_0}^{t_1} F_{1d}(\tau)\, d\tau$$

$$m_2\bar{v}_2' - m_2\bar{v}_2 = -(1+e)\int_{t_0}^{t_1} F_{1d}(\tau)\, d\tau \;.$$

If we take the dot-product of these equations in the directions $\{\bar{n}, \bar{t}_1, \bar{t}_2\}$, we find that:

$$\boxed{\begin{gathered} \bar{v}_1' \cdot \bar{t}_1 = \bar{v}_1 \cdot \bar{t}_1 \qquad \bar{v}_1' \cdot \bar{t}_2 = \bar{v}_1 \cdot \bar{t}_2 \\ \bar{v}_2' \cdot \bar{t}_1 = \bar{v}_2 \cdot \bar{t}_1 \qquad \bar{v}_2' \cdot \bar{t}_2 = \bar{v}_2 \cdot \bar{t}_2 \\ m_2 \bar{v}_2' \cdot \bar{n} + m_1 \bar{v}_1' \cdot \bar{n} = m_2 \bar{v}_2 \cdot \bar{n} + m_1 \bar{v}_1 \cdot \bar{n} \\ m_2 \bar{v}_2' \cdot \bar{n} - m_2 \bar{v}_2 \cdot \bar{n} = -(1+e) \int_{t_0}^{t_1} \bar{F}_{1d}(\tau) \cdot \bar{n} \, d\tau . \end{gathered}}$$

I.e.:

in the $\bar{t}_1$ + $\bar{t}_2$-directions, each particle's momentum is separately conserved.

In the $\bar{n}$-direction, the system's momentum is conserved.

With the six boxed equations above, we can solve for the six components of unknown velocities $\bar{v}_1'$ + $\bar{v}_2'$, provided we know $\bar{v}_1$, $\bar{v}_2$, and the linear impulse of $\bar{F}_{1d}$ during the collision. This last one is often unknown, so we instead use e from experimental data.

6.3.4 Postimpact velocities

It is convenient to solve the above system of equations for the postimpact velocities, for which we often solve in typical problems.

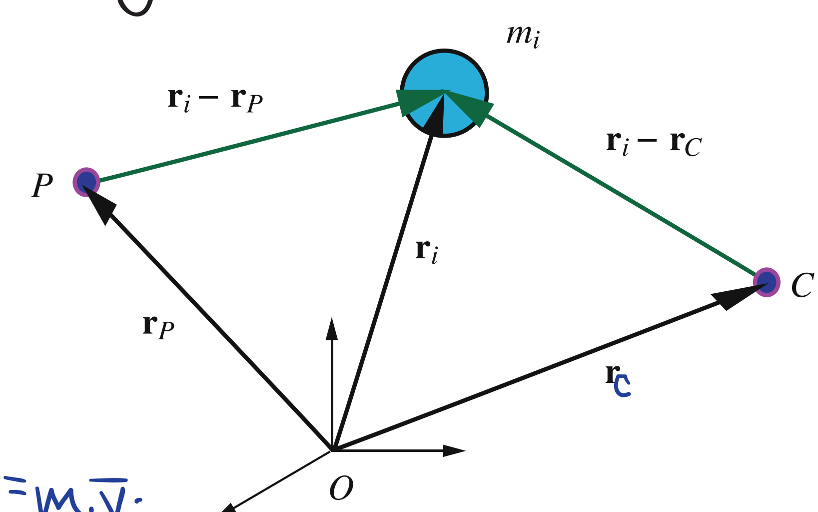
$$\boxed{\begin{aligned} \bar{V}_1' &= (\bar{V}_1 \cdot \bar{t}_1)\bar{t}_1 + (\bar{V}_1 \cdot \bar{t}_2)\bar{t}_2 + \frac{1}{m_1 + m_2}\Big((m_1 - e m_2)\bar{V}_1 \cdot \bar{n} + (1+e) m_2 \bar{V}_2 \cdot \bar{n}\Big)\bar{n} \\ \bar{V}_2' &= (\bar{V}_2 \cdot \bar{t}_1)\bar{t}_1 + (\bar{V}_2 \cdot \bar{t}_2)\bar{t}_2 + \frac{1}{m_1 + m_2}\Big((m_2 - e m_1)\bar{V}_2 \cdot \bar{n} + (1+e) m_1 \bar{V}_1 \cdot \bar{n}\Big)\bar{n} \end{aligned}}$$

Chapter 7: Dynamics of a System of Particles

Topics: – linear momenta
angular momenta +
Kinetic energy for a system of particles
– center of mass
– Conservation of kinematical quantities

7.1 Preliminaries

Consider a system of n particles, each of mass m_i, with $i \in \mathbb{Z}$. The position vector of m_i is denoted $\bar{r}_i$. We use the following figure throughout this section. The following notation is used for each particle.



Velocity: $\bar{v}_i = \dot{\bar{r}}_i$

Acceleration: $\bar{a}_i = \dot{\bar{v}}_i$

Linear Momentum: $\bar{G}_i = m_i \bar{v}_i$

Angular Momentum about point P: $\bar{H}_{P_i} = (\bar{r}_i - \bar{r}_P) \times \bar{G}_i$

Kinetic Energy: $T_i = \frac{1}{2} m_i \bar{v}_i \cdot \bar{v}_i$

7.2 The Center of Mass, Momenta, + Kinetic Energy

7.2.1 The Center of Mass

The center of mass C of a system of particles is the point described by the position vector

$$\bar{r} = \frac{1}{m}\sum_{i=1}^{n} m_i \bar{r}_i \qquad \text{where}$$

$$m = \sum_{i=1}^{n} m_i \, .$$

The velocity of the center of mass is

$$\bar{v} = \dot{\bar{r}} = \frac{1}{m}\sum_{i=1}^{n} m_i \bar{v}_i = \frac{1}{m}\sum_{i=1}^{n} G_i \, .$$

From these expressions, we can write the identities

$$\sum_{i=1}^{n} m_i(\bar{r}-\bar{r}_i) = \bar{0} \qquad \sum_{i=1}^{n} m_i(\bar{v}-\bar{v}_i) = \bar{0}$$

which we will use momentarily.

7.2.2 Linear Momentum

The linear momentum $\bar{G}$ of a system of particles is the sum of the linear momenta of the particles. The following demonstration is instructive.

$$\bar{G} = m\dot{\bar{r}} \quad \text{(by def., with some position } \bar{r}\text{)}$$
$$= \sum_{i=1}^{n} m_i \dot{\bar{r}}_i \quad \text{(assuming } \bar{r} \text{ locates } C\text{)}$$
$$= \sum_{i=1}^{n} \bar{G}_i \quad \text{(by def. of lin. mom.)}$$

This final expression is the definition of the linear momentum of a system of particles. We found it with the assumption that the linear momentum of the summed masses could be described as the sum of the masses multiplied by the velocity of a single point: the center of mass C. Therefore, our assumption was valid.

7.2.3 Angular Momentum

The angualar momentum $\bar{H}_P$ of a system of particles about a point P is:

$$\bar{H}_P \triangleq \sum_{i=1}^{n} \bar{H}_{P_i} = \sum_{i=1}^{n} (\bar{r}_i - \bar{r}_P) \times m_i \bar{v}_i$$

that is, the sum of the angular momentum of each particle about P.

It is simple to show (using the identities from 7.2.1) that

$$\bar{H}_P = \bar{H}_C + (\bar{r} - \bar{r}_P) \times \bar{G} \quad \text{where} \quad \bar{H}_C = \sum_{i=1}^{n} (\bar{r}_i - \bar{r}) \times m_i \bar{v}_i .$$

7.2.4 Kinetic Energy

The kinetic energy of a system of particles is defined to be

$$T \triangleq \sum_{i=1}^{n} T_i = \frac{1}{2}\sum_{i=1} m_i \bar{v}_i \cdot \bar{v}_i \, .$$

This can be rewritten as

$$T = \underbrace{\frac{1}{2} m \bar{v} \cdot \bar{v}}_{\text{kinetic energy of the center of mass}} + \underbrace{\frac{1}{2}\sum_{i=1}^{n} m_i (\bar{v}_i - \bar{v}) \cdot (\bar{v}_i - \bar{v})}_{\text{additional term including the velocities of the particles relative to C}} \, .$$

7.3 Kinetics of a System of Particles

The resultant force $\overline{F}$ on a system of particles is

$$\overline{F} \triangleq \sum_{i=1}^{n} \overline{F_i}$$

where $\overline{F_i}$ is the resultant force on the ith particle.

Newton's second Law (Euler's first law/balance of linear momentum) for each particle is

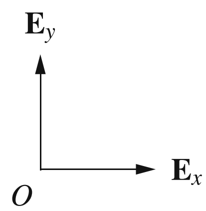
$$\overline{F_i} = m_i \overline{a_i} \, .$$

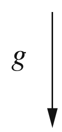
So $\overline{F} = \sum_{i=1}^{n} m_i \overline{a_i} = m \overline{a}$, by the definition of the center of mass.

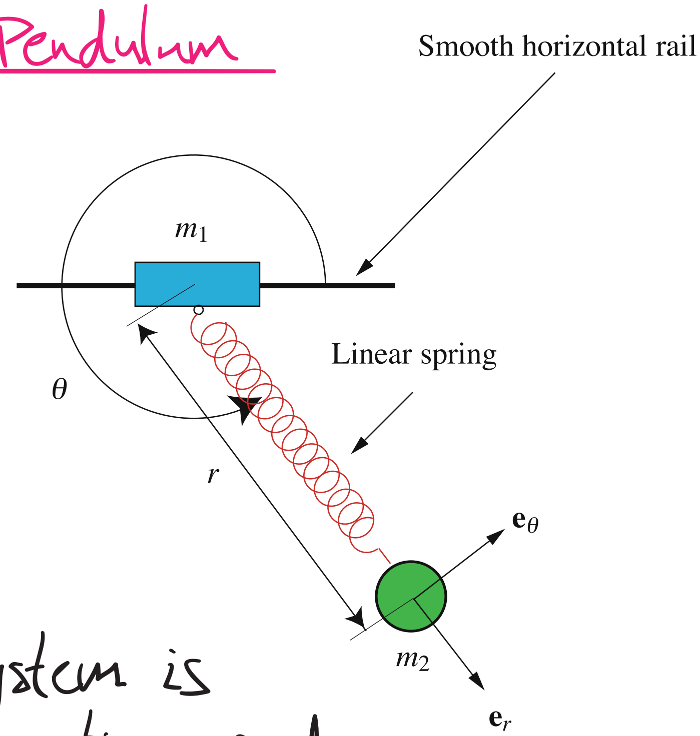
I.e. the resultant force on a system of particles is equal to the total mass of the system multiplied by the acceleration of the center of mass.

This is a useful fact. Solving the coupled equations of motion for every particle in a system is often very difficult.

7.5 The Cart + the Pendulum (Example)







The figure shows a two-particle system of masses m_1 + m_2. Show that the linear momentum of the system is conserved in the $\overline{E}_x$-direction, and explore what this means for the motion of the particles.

7.5.1 Kinematics

Position: $\overline{r}_1 = x\overline{E}_x + y_0\overline{E}_y + z_0\overline{E}_z$ | $\overline{r}_2 = \overline{r}_1 + r\overline{e}_r$

(center of mass)
$$\overline{r} = \frac{m_1\overline{r}_1 + m_2\overline{r}_2}{m_1 + m_2}$$
$$= \overline{r}_1 + \frac{m_2}{m_1+m_2} r\overline{e}_r$$
$$= \overline{r}_2 - \frac{m_1}{m_1+m_2} r\overline{e}_r .$$

Velocity: $\overline{v}_1 = \dot{x}\overline{E}_x$ | $\overline{v}_2 = \dot{x}\overline{E}_x + \dot{r}\overline{e}_r + r\dot{\theta}\overline{e}_\theta$

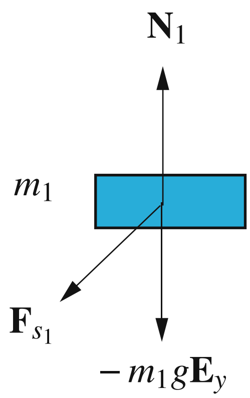
$$\overline{v} = \dot{\overline{r}} = \dot{x}\overline{E}_x + \frac{m_2}{m_1+m_2}\left(\dot{r}\overline{e}_r + r\dot{\theta}\overline{e}_\theta\right).$$

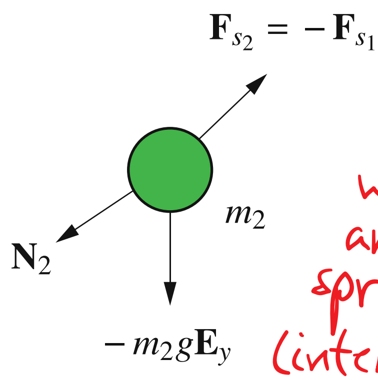
Acceleration: $\bar{a}_1 = \ddot{x}\bar{E}_x$

$$\bar{a}_2 = \ddot{x}\bar{E}_x + (\ddot{r} - r\dot{\theta}^2)\bar{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\bar{e}_\theta$$

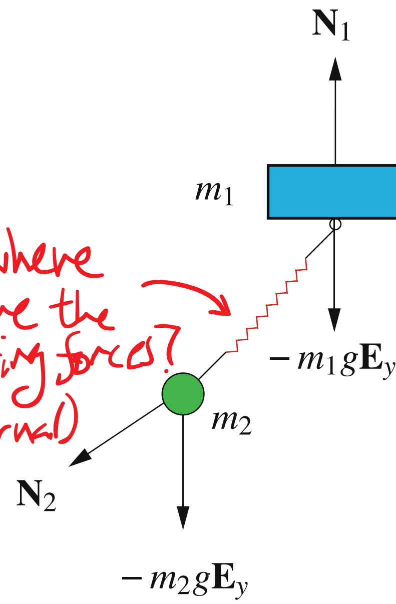
7.5.2 Forces

FBDs:





where are the spring forces? (internal)



Write what we know about the forces:

$$\bar{F}_{g_1} = -m_1 g\bar{E}_y \qquad \bar{F}_{g_2} = -m_2 g\bar{E}_y$$

$$\bar{F}_{s_1} = -\bar{F}_{s_2} = -K\left(\|\bar{r}_1 - \bar{r}_2\| - L\right)\frac{\bar{r}_1 - \bar{r}_2}{\|\bar{r}_1 - \bar{r}_2\|} = K(r-L)\bar{e}_r$$

$$\bar{N}_1 = N_{1y}\bar{E}_y + N_{1z}\bar{E}_z \qquad \bar{N}_2 = N_{2z}\bar{E}_z \; .$$

7.5.3 Balance Laws

The balances of linear momentum for the particles are:

$$-m_1 g\bar{E}_y + N_{1y}\bar{E}_y + N_{1z}\bar{E}_z + K(r-L)\bar{e}_r = m_1\ddot{x}\bar{E}_x$$

$$-m_2 g\bar{E}_y + N_{2z}\bar{E}_z - K(r-L)\bar{e}_r = m_2\ddot{x}\bar{E}_x + m_2(\ddot{r} - r\dot{\theta}^2)\bar{e}_r + m_2(r\ddot{\theta} + 2\dot{r}\dot{\theta})\bar{e}_\theta$$

With minimal rearranging of the six scalar equations, from which we want $r, x, \theta, N_{1y}, N_{1z}, + N_{2z}$ we find that

$$\bar{N}_1 = (m_1 g - K(r-L)\sin\theta)\bar{E}_y \qquad \bar{N}_2 = \bar{0}$$

$$K(r-L)\cos\theta = m_1\ddot{x}$$
$$-K(r-L) - m_2 g\sin\theta = m_2\ddot{x}\cos\theta + m_2(\ddot{r} - r\dot{\theta}^2)$$
$$-m_2 g\cos\theta = -m_2\ddot{x}\sin\theta + m_2(r\ddot{\theta} + 2\dot{r}\dot{\theta})$$

Given $r(t_0), \theta(t_0), x(t_0), \dot{r}(t_0), \dot{\theta}(t_0), + \dot{x}(t_0)$, $r(t), \theta(t)$, + $x(t)$ can be found by solving the coupled ODEs. Such a solution is beyond the scope of this class.

7.5.4 Analysis

Now let's consider the balance of linear momentum for the system:

$$-(m_1+m_2)g\bar{E}_y + \bar{N}_1 + \bar{N}_2 = (m_1+m_2)\ddot{x}\bar{E}_x + m_2(\ddot{r} - r\dot{\theta}^2)\bar{e}_r + m_2(r\ddot{\theta} + 2\dot{r}\dot{\theta})\bar{e}_\theta.$$

Therefore $\bar{F}\cdot\bar{E}_x = 0$, and linear momentum is conserved in the x-direction.
Therefore, the x-component of the velocity of the center of mass is constant.

Writing out the $\bar{E}_x$-component of linear momentum,

$$\bar{G} \cdot \bar{E}_x = (m_1 + m_2)\bar{v} \cdot \bar{E}_x$$
$$= (m_1 + m_2)\dot{x} + m_2(\dot{r}\cos\theta - r\dot{\theta}\sin\theta)$$

If we write $G_0 = \bar{G} \cdot \bar{E}_x$ and solve for $\dot{x}$,

$$\dot{x} = \frac{1}{m_1 + m_2}\left(G_0 - m_2(\dot{r}\cos\theta - r\dot{\theta}\sin\theta)\right).$$

In the next lecture we will return to this analysis to show that energy is conserved.

7.6 Conservation of Angular Momentum

In 7.2 we noted that the angular momentum of a system of particles about some point P is

$$\bar{H}_P = \bar{H}_c + (\bar{r} - \bar{r}_P) \times \bar{G} \qquad \text{where}$$

$$\bar{H}_c = \sum_{i=1}^{n} (\bar{r}_i - \bar{r}) \times m_i \bar{v}_i \,.$$

Taking the time-derivative, it can be shown that

$$\dot{\bar{H}}_P = \sum_{i=1}^{n} (\bar{r}_i - \bar{r}_P) \times \bar{F}_i - \bar{v}_P \times \bar{G}$$

This is called the angular momentum theorem for a system of particles.

We define the resultant moment of the system relative to P as

$$\bar{M}_P = \sum_{i=1}^{n} (\bar{r}_i - \bar{r}_P) \times \bar{F}_i \,.$$

We can use this to rewrite $\dot{\bar{H}}_P$:

$$\dot{\bar{H}}_P = \bar{M}_P - \bar{v}_P \times \bar{G}$$

so, not just the moment!

Two special cases of the angular momentum theorem:

If P is a fixed point O and $\bar{r}_0 = \bar{0}$:

$$\dot{\bar{H}}_0 = \bar{M}_0 \quad \text{where} \quad \bar{M}_0 = \sum_{i=1}^{n} \bar{r}_i \times \bar{F}_i .$$

If P is the center of mass C: $\bar{v} \times \bar{G} = \bar{0}$ and

$$\dot{\bar{H}}_C = \bar{M}_C \quad \text{where} \quad \bar{M}_C = \sum_{i=1}^{n} (\bar{r}_i - \bar{r}) \times \bar{F}_i .$$

To summarize an important point:
If P is fixed or C, the time rate of change of the angular momentum about P is the resultant moment about P. If P is moving and not C, this is not true.

Finally, we consider when $\bar{H}_P$ is conserved in the direction of a vector $\bar{c}$:

$$\frac{d}{dt}\left(\bar{H}_P \cdot \bar{c}\right) = 0 .$$

This implies the necessary and sufficient condition for the conservation of $\bar{H}_P$ in the direction of $\bar{c}$:

$$\left(\bar{M}_P - \bar{v}_P \times \bar{G}\right) \cdot \bar{c} + \bar{H}_P \cdot \dot{\bar{c}} = 0 .$$

7.8 Work, Energy, and Conservative Forces

Recalling the work-energy theorem for a single particle:

$$\dot{T}_i = \bar{F}_i \cdot \bar{v}_i \, ,$$

and defining the total kinetic energy of a system of particles as

$$T = \sum_{i=1}^{n} T_i \, ,$$

we can derive the work-energy theorem for a system of particles:

$$\dot{T} = \sum_{i=1}^{n} \bar{F}_i \cdot \bar{v}_i \, .$$

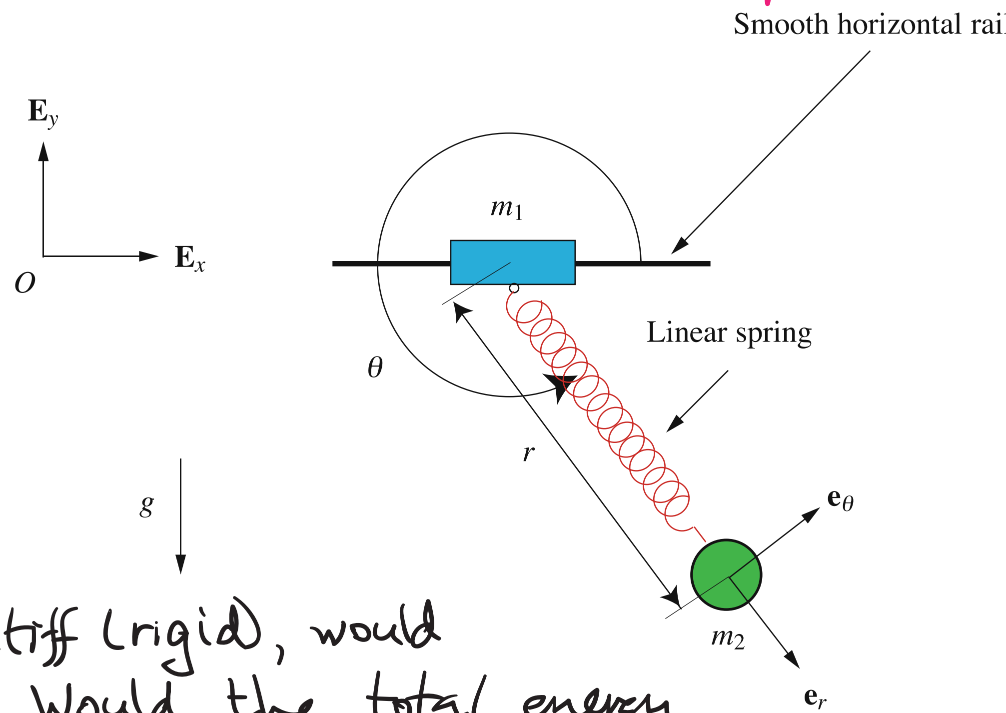
Similar to the development in chapter 5, we can rewrite this by separating conservative and non-conservative forces such that

$$\dot{E} = \sum_{i=1}^{n} \bar{F}_{nc_i} \cdot \bar{v}_i$$

where E is the total energy and $\bar{F}_{nci}$ is the resultant of the nonconservative forces on particle i.

The second form is usually more useful in solving problems.

7.8.1 The Cart + Pendulum Revisited (Example)



Part I

Is the energy of the system conserved?

Part II

If the spring was infinitly stiff (rigid), would it do work? Would the total energy of the system be conserved?

Part I solution

The work-energy theorem gives

$$\dot{T} = (\bar{F}_{s1} - m_1 g \bar{E}_y + \bar{N}_1) \cdot \bar{v}_1 + (\bar{F}_{s2} - m_2 g \bar{E}_y + \bar{N}_2) \cdot \bar{v}_2$$

The normal forces are perpendicular to the velocities, $\bar{N}_1 \cdot \bar{v}_1 = \bar{N}_2 \cdot \bar{v}_2 = 0$, so they do no work.

Furthermore, the spring powers combine to give

$$\bar{F}_{s1} \cdot \bar{v}_1 + \bar{F}_{s2} \cdot \bar{v}_2 = -K(\|\bar{r}_1 - \bar{r}_2\| - L) \frac{\bar{r}_1 - \bar{r}_2}{\|\bar{r}_1 - \bar{r}_2\|} \cdot (\bar{v}_1 - \bar{v}_2)$$

$$= -\frac{d}{dt}\left(\frac{1}{2} K (\|\bar{r}_1 - \bar{r}_2\| - L)^2\right)$$

In summary,

$$\dot{T} = -\frac{d}{dt}\underbrace{\left(\frac{1}{2}k(\|\bar{r}_1-\bar{r}_2\|-L)^2 + m_1 g \bar{E}_y\cdot\bar{r}_1 + m_2 g \bar{E}_y\cdot\bar{r}_2\right)}_{U}.$$

Recognizing $U=\sum_i U_i$,

$$\dot{T} = -\dot{U} \Rightarrow \dot{T}+\dot{U}=0 \Rightarrow \dot{E}=0.$$

So the total energy of the system is conserved.

<u>Part II solution</u>

Kinematics: $\bar{r}_2-\bar{r}_1 = L\bar{e}_r \qquad \bar{v}_2-\bar{v}_1 = L\dot{\theta}\bar{e}_\theta$

$(\bar{r}_2-\bar{r}_1)\cdot(\bar{v}_2-\bar{v}_1)=0$.

Work-energy theorem:

$$\dot{T} = (S\bar{e}_r - m_1 g\bar{E}_y + \bar{N}_1)\cdot\bar{v}_1 + (-S\bar{e}_r - m_2 g \bar{E}_y + \bar{N}_2)\cdot\bar{v}_2$$

where $S\bar{e}_r$ is the tension force in the rod. As before, we can write this as

$$\begin{aligned}\dot{E} \triangleq \dot{T}+\dot{U} &= S\bar{e}_r\cdot(\bar{v}_1-\bar{v}_2)\\ &= S\bar{e}_r\cdot(L\dot{\theta}\bar{e}_\theta)\\ &= 0\end{aligned}$$

so energy is conserved.

Part III: Dynamics of a Single Rigid Body

Chapter 8 Planar Kinematics of Rigid Bodies

Until now, we have considered only "particle" masses as models of bodies. In Part III, we consider a new model: rigid bodies.

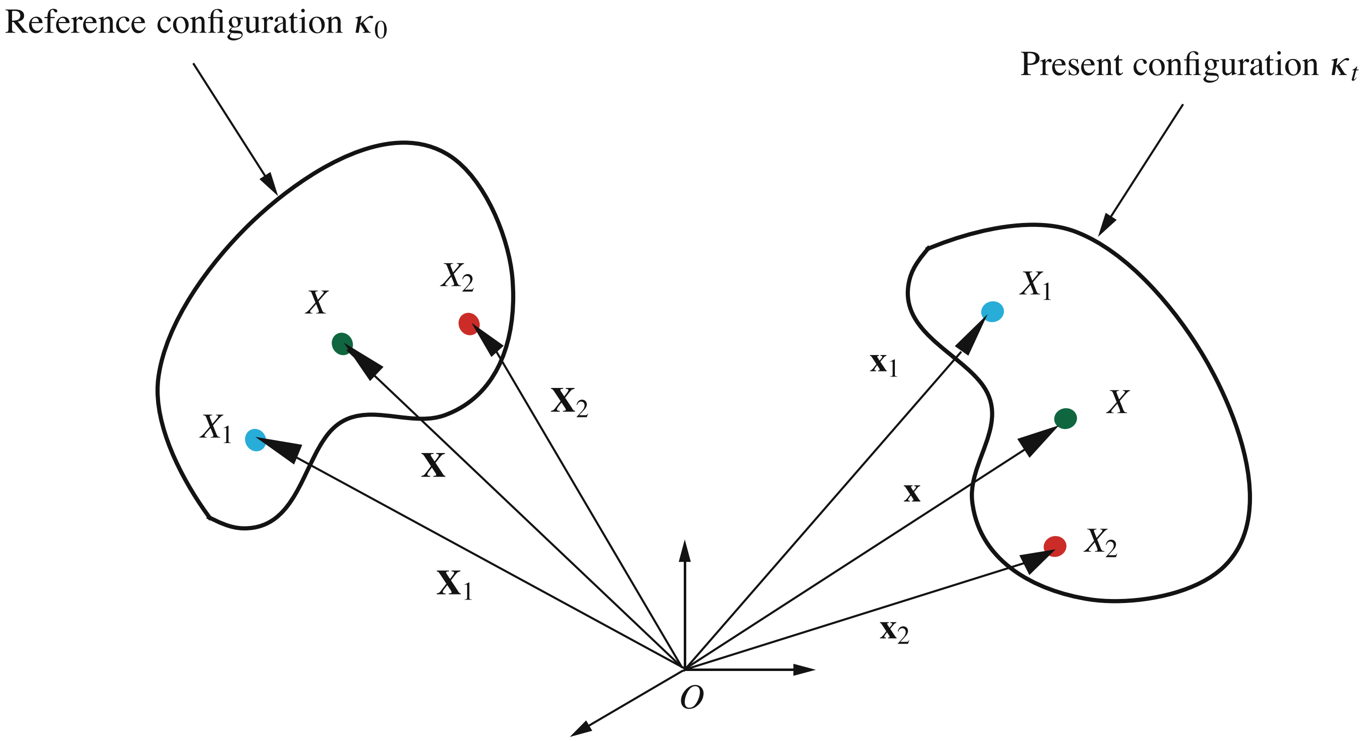
8.1 Motion of a Rigid Body

A body $\mathcal{B}$ is a collection of points representing particles. A point is denoted X. The vector $\bar{x}$ describes the pos. of point X at time t. The present configuration $\bar{\kappa}_t$ of $\mathcal{B}$ is a smooth bijection (function). It maps points of $\mathcal{B}$ to vectors in $\mathbb{E}^3$.

Let $\bar{\kappa}_0$ be a reference configuration and $\bar{X} = \bar{\kappa}_0(X)$ be the corresponding vector-valued function of time of the position of X in $\mathbb{E}^3$.

One can define the motion of $\mathcal{B}$ as a function of $\bar{X}$ and t:

$$\bar{x} = \chi(\bar{X}, t) \quad .$$



So the position vector $\bar{x}$ of a point X is a function of the initial position vector $\bar{X}$ of X and the time t.

8.1.2 Rigidity

The above is true for any body, including bodies that deform. In the field of continuum mechanics, this is useful. We will not consider this most-general case in this course. We are concerned with rigid-body motion $\bar{x} = x_R(\bar{X}, t)$, which simplifies the situation.

In a rigid body, the following two physical ideas are modeled by the math:

- the distance between any two points is constant &
- the orientation between any two points is constant.

The first is expressed mathematically, for points X_1 and X_2, as

$$\| \overline{x}_1 - \overline{x}_2 \| = \| \overline{X}_1 - \overline{X}_2 \| .$$

The second is expressed by restricting motion to be such that the following linear transformation has certain properties:

$$\overline{x}_1 - \overline{x}_2 = Q \circ (\overline{X}_1 - \overline{X}_2)$$

where $\overline{Q}$ is a proper-orthogonal or "rotation" matrix having the following properties:

$$Q\,Q^T = I \quad + \quad \det Q = 1 .$$

The first of these restricts the nine components of Q to three independent parameters. The most common parameterization is Euler angles. We will typically consider only cases that require a single parameter (planar rotation).

Because Q is a rotation matrix,

$$0 = \frac{d}{dt} I = \frac{d}{dt}\left(QQ^T\right) = \dot{Q}Q^T + Q\dot{Q}^T .$$

Therefore,

$$\dot{Q}Q^T = -Q\dot{Q}^T = -\left(\dot{Q}Q^T\right)^T .$$

Therefore $\dot{Q}Q^T$ is skew-symmetric and can be written, for some Ω_{21}, Ω_{13}, + Ω_{32} as

$$\dot{Q}Q^T = \begin{bmatrix} 0 & -\Omega_{12} & \Omega_{13} \\ \Omega_{12} & 0 & -\Omega_{32} \\ -\Omega_{13} & \Omega_{32} & 0 \end{bmatrix} .$$

We will use this in the next section.

8.1.3 Angular Velocity + Acceleration Vectors

Recall: $(\bar{x}_1 - \bar{x}_2) = Q(\bar{X}_1 - \bar{X}_2)$. (*)

Therefore: $(\bar{X}_1 - \bar{X}_2) = Q^T(\bar{x}_1 - \bar{x}_2)$. (**)

Differentiating the equation (*) with respect to time, we can the relative velocity of the two points:

$$(\bar{v}_1 - \bar{v}_2) = \dot{Q}(t)(\bar{X}_1 - \bar{X}_2)$$

where $\bar{v}_1 = \dot{\bar{x}}_1$ and $\bar{v}_2 = \dot{\bar{x}}_2$. Subsituting (**) into this equation and recalling our expression for $\dot{Q}Q^T$ from section 8.1.2, we get:

$$\begin{aligned}(\bar{v}_1 - \bar{v}_2) &= \dot{Q}Q^T(\bar{x}_1 - \bar{x}_2) \\ &= \begin{bmatrix} 0 & -\Omega_{21} & \Omega_{13} \\ \Omega_{21} & 0 & -\Omega_{32} \\ -\Omega_{13} & \Omega_{32} & 0 \end{bmatrix}(\bar{x}_1 - \bar{x}_2).\end{aligned}$$

$$\boxed{\bar{v}_1 - \bar{v}_2 = \bar{\omega} \times (\bar{x}_1 - \bar{x}_2)}$$

where $\bar{\omega} = \Omega_{32}\bar{E}_x + \Omega_{13}\bar{E}_y + \Omega_{21}\bar{E}_z$ is called the angular velocity vector. $\bar{\omega}$ is the same for relating any two points in a body, and is a function of time t.

We can find the relative acceleration in the usual way of time-differentiating the relative velocity,

$$\bar{a}_1 - \bar{a}_2 = \dot{\bar{v}}_1 - \dot{\bar{v}}_2$$
$$= \dot{\bar{\omega}} \times (\bar{x}_1 - \bar{x}_2) + \bar{\omega} \times (\bar{v}_1 - \bar{v}_2)$$

$$\boxed{\bar{a}_1 - \bar{a}_2 = \bar{\alpha} \times (\bar{x}_1 - x_2) + \bar{\omega} \times (\bar{\omega} \times (\bar{x}_1 - \bar{x}_2))}$$

where $\bar{\alpha} = \dot{\bar{\omega}}$ is the angular acceleration vector.

8.1.4 Fixed-Axis Rotation

All the above is general (rotation about all axes simultaneously). In this class, we often work with planar rotation problems, for which we align the $\bar{E}_z$ basis vector perpendicular to the plane of rotation. In this case,

$$Q = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

where θ is the counterclockwise rotation of the body about $\bar{E}_z$.

It is easy to show that $\dot{Q}Q^T = \dot{\theta} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and

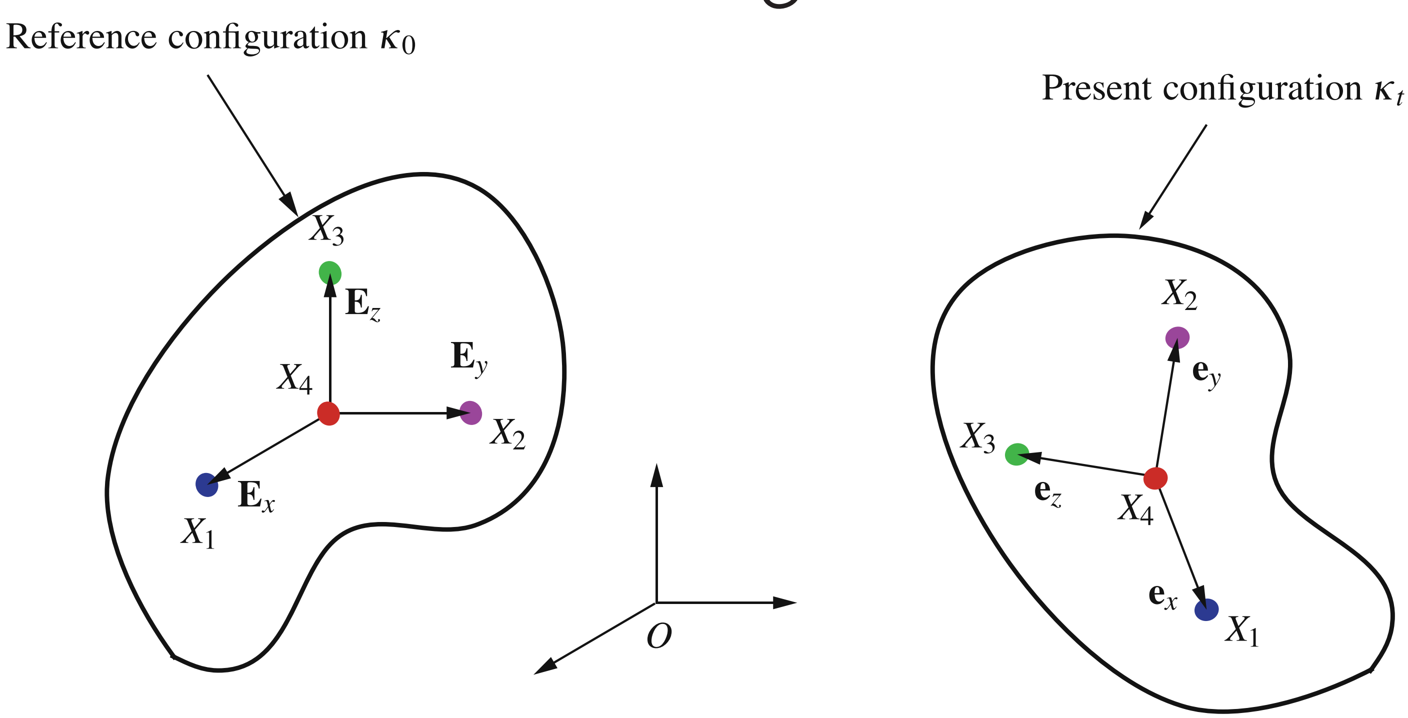
$$\boxed{\bar{\omega} = \dot{\theta}\bar{E}_z} \quad + \quad \boxed{\bar{\alpha} = \ddot{\theta}\bar{E}_z}.$$

8.2 Kinematical Relations + a Corotational Basis

Up to this point, we have used a fixed Cartesian basis. We now introduce + explore a convenient basis called a corotational (body-fixed) basis.

8.2.1 The Corotational Basis

We define the corotational basis $(\bar{e}_x, \bar{e}_y, \bar{e}_z)$ that rotates with the body as follows.



First, we choose the four points on the body X_1, X_2, X_3, + X_4 such that

$$\bar{E}_x = \bar{X}_1 - \bar{X}_4, \quad \bar{E}_y = \bar{X}_2 - \bar{X}_4, \quad \bar{E}_z = \bar{X}_3 - \bar{X}_4$$

form a fixed, right-handed, Cartesian basis.

Because Q preserves the distance and orientation between points, in all configurations the following vectors form a ("corotational") basis:

$$\bar{e}_x = \bar{x}_1 - \bar{x}_4 \ , \quad \bar{e}_y = \bar{x}_2 - \bar{x}_4 \ , \quad \bar{e}_z = \bar{x}_3 - \bar{x}_4 \ .$$

We have already derived expressions for relative velocities and accelerations, which can now be used to derive expressions for the time-rate-of-change of $\bar{e}_i$.

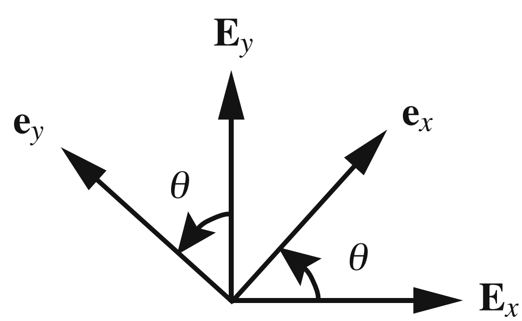
$$\dot{\bar{e}}_x = \bar{v}_1 - \bar{v}_4 = \bar{\omega} \times \bar{e}_x \qquad \dot{\bar{e}}_y = \bar{v}_2 - \bar{v}_4 = \bar{\omega} \times \bar{e}_y \qquad \dot{\bar{e}}_z = \bar{v}_3 - \bar{v}_4 = \bar{\omega} \times \bar{e}_z$$

$$\ddot{\bar{e}}_x = \bar{a}_1 - \bar{a}_4 = \bar{\alpha} \times \bar{e}_x + \bar{\omega} \times (\bar{\omega} \times \bar{e}_x)$$

and similarly for $\ddot{\bar{e}}_y$ + $\ddot{\bar{e}}_z$. We will use these soon.

8.2.2 The Corotational Basis for a Fixed-Axis

For a fixed axis of rotation, the rotation matrix Q is, in the Cartesian basis,



$$Q = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

where θ is as in the figure.

This yields the following relations:

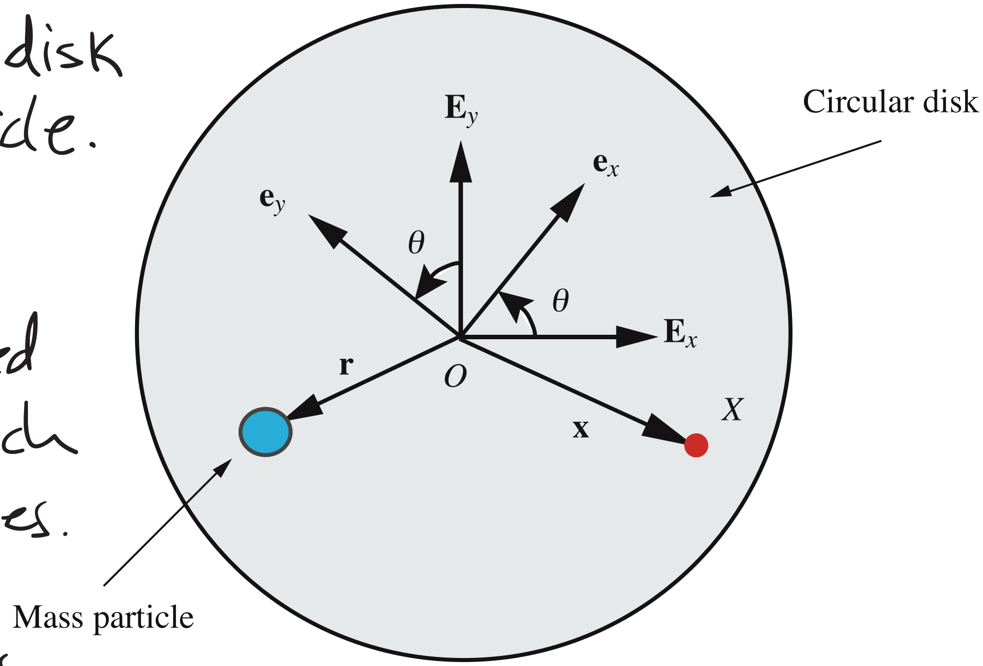
$$\begin{bmatrix} \overline{e_x} \\ \overline{e_y} \\ \overline{e_z} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \overline{E_x} \\ \overline{E_y} \\ \overline{E_z} \end{bmatrix}.$$

Previously, we showed that, for the fixed-axis case, $\overline{\omega} = \dot{\theta}\,\overline{E_z}$ and $\overline{\alpha} = \ddot{\theta}\,\overline{E_z}$, which we can use to find that

$$\begin{aligned} \dot{\overline{e_x}} &= \overline{\omega} \times \overline{e_x} = \dot{\theta}\,\overline{e_y} \\ \dot{\overline{e_y}} &= \overline{\omega} \times \overline{e_y} = -\dot{\theta}\,\overline{e_x} \\ \dot{\overline{e_z}} &= \overline{\omega} \times \overline{e_z} = \overline{0} \end{aligned}.$$

8.2.3 A Particle Moving on a Rigid Body (example)

Consider the disk and mass particle.



The center of the disk is fixed point about which the disk rotates.

The disk rotates about $\overline{E}_z$ with $\overline{\omega} = \dot{\theta}\,\overline{E}_z = \omega\,\overline{E}_z$ and angular acceleration $\overline{\alpha} = \ddot{\theta}\,\overline{E}_z = \alpha\,\overline{E}_z$.

Suppose the position of the particle is

$$\overline{r} = 10t^2\,\overline{e}_x + 20t\,\overline{e}_y$$

and the position of the point X on the disk is

$$\overline{X} = x\,\overline{e}_x + y\,\overline{e}_y$$

where x and y are constant scalars.

Find the velocities of the particle and the point X.

Velocity of the particle:

$$\dot{\bar{r}} = 20t\,\bar{e}_x + 10t^2\dot{\bar{e}}_x + 20\bar{e}_y + 20t\,\dot{\bar{e}}_y$$
$$= 20t\,\bar{e}_x + 10t^2\omega\bar{e}_y + 20\bar{e}_y - 20t\omega\bar{e}_x$$
$$= (20t - 20t\omega)\bar{e}_x + (20 + 10t^2\omega)\bar{e}_y,$$

Velocity of point $\underline{\bar{X}}$:

$$\dot{\bar{X}} = x\dot{\bar{e}}_x + y\dot{\bar{e}}_y$$
$$= x\omega\bar{e}_y - y\omega\bar{e}_x$$
$$= -y\omega\bar{e}_x + x\omega\bar{e}_y.$$

Notice that $\dot{\bar{X}} = \bar{\omega} \times \bar{X} = \det\begin{bmatrix} \bar{e}_x & \bar{e}_y & \bar{e}_z \\ 0 & 0 & \omega \\ x & y & 0 \end{bmatrix}$
because the origin is fixed.

But $\dot{\bar{r}} \neq \bar{\omega} \times \bar{r}$. Why?
Because $\bar{r}$ is the position vector of a particle that is moving independently from the disk.

8.4 Center of Mass and Linear Momentum

In this section we define the center of mass C and the linear momentum $\bar{G}$ of a body.
Let $\mathcal{R}$ denote the region of space occupied by

the body in its present configuration. Let $\mathcal{R}_0$ denote the region occupied in the body's reference configuration.

Let the density of the material of the body be $\rho(\bar{x},t)$ in its present configuration and $\rho_0(\bar{X})$ in its reference configuration.

8.4.1 The Center of Mass

The position vectors of the center of mass of the body is

$$\bar{x}^* = \frac{\int_{\mathcal{R}} \bar{x} \rho \, dv}{\int_{\mathcal{R}} \rho \, dv}$$

for the present configuration and

$$\bar{X}^* = \frac{\int_{\mathcal{R}_0} \bar{X} \rho_0 \, dV}{\int_{\mathcal{R}_0} \rho_0 \, dV} .$$

for the reference configuration.

We assume that mass m is conserved, so

$$dm = \rho_0 \, dV = \rho \, dv$$

$$m = \int_{\mathcal{R}_0} \rho_0 \, dV = \int_{\mathcal{R}} \rho \, dv .$$

So we often write:

$$\boxed{\overline{\mathbf{x}} = \frac{1}{m}\int_{\mathcal{R}} \mathbf{x}\,\rho\, dv} \qquad \boxed{\overline{\mathbf{X}} = \frac{1}{m}\int_{\mathcal{R}_0} \mathbf{X}\,\rho_0\, dV}.$$

Using the results of Section 8.1.2, we can write an equation relating the center of mass C and another point Y:

present conf. pos. vec. of point Y

ref. conf. pos. vec. of point Y

$$\overline{\mathbf{x}} - \mathbf{y} = \mathbf{Q}\circ(\overline{\mathbf{X}} - \mathbf{Y})$$

Differentiating w.r.t. time to get the relative velocity:

$$\overline{\mathbf{v}} - \dot{\mathbf{y}} = \boldsymbol{\omega}\times(\overline{\mathbf{x}} - \mathbf{y})$$

and once more to get the relative acceleration:

$$\overline{\mathbf{a}} - \ddot{\mathbf{y}} = \boldsymbol{\alpha}\times(\overline{\mathbf{x}} - \mathbf{y}) + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times(\overline{\mathbf{x}} - \mathbf{y})).$$

$\overline{\mathbf{v}}$ is the velocity of the center of mass

$\overline{\mathbf{a}}$ is the acceleration of the center of mass.

8.4.2 The Linear Momentum

Definition: the linear momentum of a rigid body (using the definitions above) is

$$\boxed{\bar{G} = \int_R \bar{v} \rho \, dv .}$$

This can be written in the following convenient way:

$$\bar{G} = \int_R \bar{v} \rho \, dv$$

$$= \int_R \frac{d\bar{x}}{dt} \rho \, dv$$

$$= \frac{d}{dt} \int_R \bar{x} \rho \, dv$$

$$= \frac{d}{dt} (m \bar{\bar{x}})$$

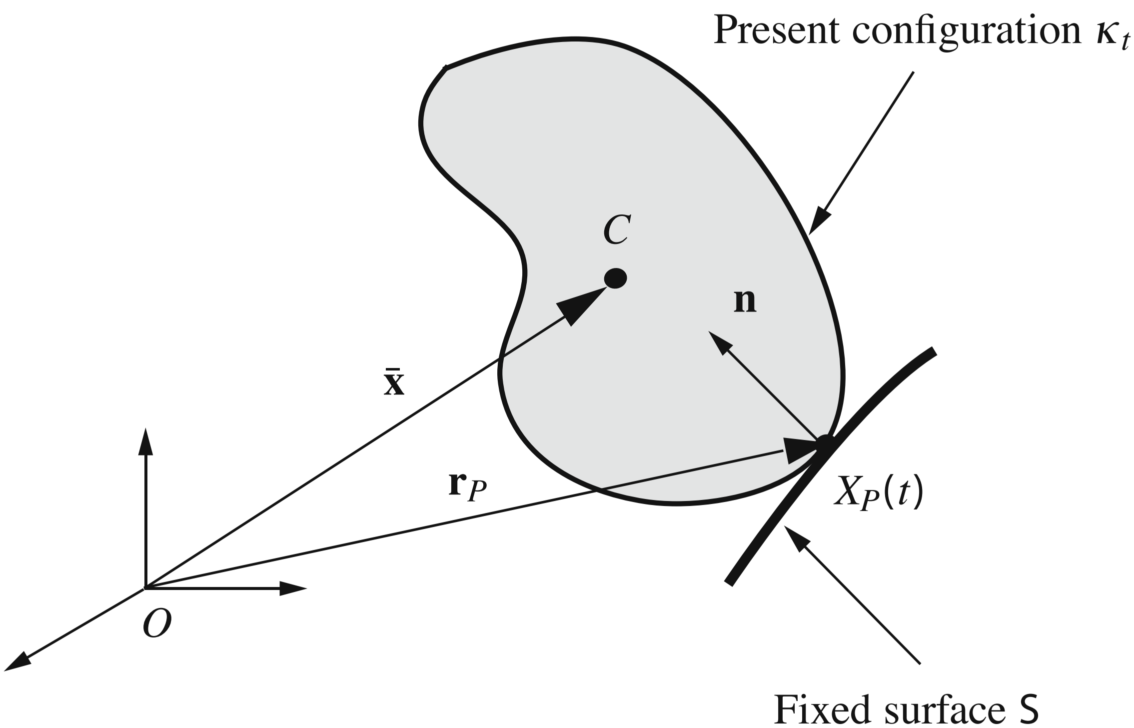
$$\boxed{\bar{G} = m \bar{\bar{v}}}$$

So the linear momentum of the rigid body is its mass times the velocity of the center of mass.

This result is identical to that for a system of particles.

8.5 Kinematics of Rolling and Sliding

Often, we would like to analyze a body $\mathcal{B}$ that is in contact with another body's surface $\mathcal{S}$ at a single point. The material point $P = X_P(t)$ of $\mathcal{B}$ that is in contact varies with time. We denote the position vector of P at time t, $\bar{r}_P$ and the velocity, $\bar{v}_P$. The unit normal vector to $\mathcal{S}$ at P is denoted $\bar{n}$.



Because P is a material point on $\mathcal{B}$, it has velocity and acceleration:

$$\bar{v}_P = \bar{v} + \bar{\omega} \times (\bar{r}_P - \bar{x})$$

$$\bar{a}_P = \bar{a} + \bar{\alpha} \times (\bar{r}_P - \bar{x}) + \bar{\omega} \times (\bar{\omega} \times (\bar{r}_P - \bar{x}))$$

If the rigid body is sliding on the fixed surface:

$$\overline{v}_P \cdot \overline{n} = \overline{0} \quad ,$$

which implies the sliding condition:

$$\overline{\dot{x}} \cdot \overline{n} = -(\overline{\omega} \times (\overline{r}_P - \overline{x})) \cdot \overline{n} \quad .$$

If the rigid body is rolling on the fixed surface:

$$\overline{v}_P = \overline{0} \quad ,$$

which implies the rolling condition:

$$\overline{\dot{x}} = -\overline{\omega} \times (\overline{r}_P - \overline{x}) \quad .$$

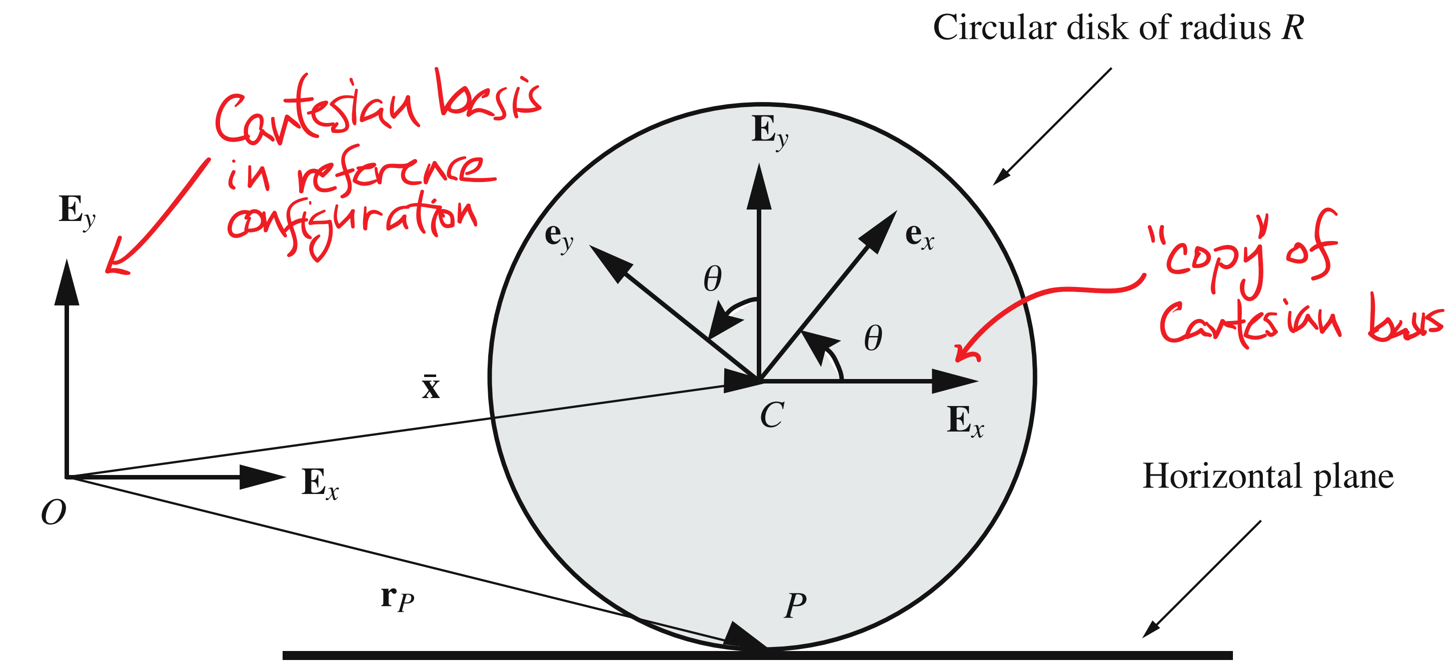
Finally, we note that the acceleration of P for a rolling rigid body is not necessarily $\overline{0}$!

8.6 Kinematics of a Rolling Circular Disk

A common problem in rigid body dynamics is the rolling circular disk of radius R.

First, we define the corotational basis:

$\overline{e}_x = \cos\theta\, \overline{E}_x + \sin\theta\, \overline{E}_y$, $\overline{e}_x = \cos\theta\, \overline{E}_y - \sin\theta\, \overline{E}_x$, $\overline{e}_z = \overline{E}_z$,



Because this is fixed-axis (planar) rotation,

$$\bar{\omega} = \dot{\theta}\bar{E}_z \quad + \quad \bar{\alpha} = \ddot{\theta}\bar{E}_z \, .$$

The center of mass position vector is written

$$\bar{x} = x\bar{E}_x + y\bar{E}_y + z\bar{E}_z \, .$$

Also, $\bar{r}_P = \bar{x} - R\bar{E}_y$. (In this problem $\bar{n} = \bar{E}_y$.)

Using the rolling condition, we find that

$$\bar{v} = -\bar{\omega} \times (\bar{r}_P - \bar{x}) = -\dot{\theta}\bar{E}_z \times (-R\bar{E}_y)$$
$$= -R\dot{\theta}\bar{E}_x$$

Therefore, $\bar{a} = \dot{\bar{v}} = -R\ddot{\theta}\bar{E}_x$.

The velocity of P is zero. Let's calculate its acceleration.

$$\overline{a}_p = \overline{a} + \overline{\alpha} \times (\overline{r}_p - \overline{x}) + \overline{\omega} \times (\overline{\omega} \times (\overline{r}_p - \overline{x}))$$
$$= \ddot{x}\overline{E}_x + \ddot{\theta}\overline{E}_z \times (-R\overline{E}_y) + \dot{\theta}\overline{E}_z \times (\dot{\theta}\overline{E}_z \times (-R\overline{E}_y))$$
$$= R\dot{\theta}^2\overline{E}_y \, .$$

Wait, what? How can $\overline{v}_p = \overline{0}$ and $\overline{a}_p \neq \overline{0}$?
In fact, if we calculate $\dot{\overline{r}}_p$ and summarize what we know:

$$\dot{\overline{r}}_p = \overline{v} \neq \overline{0} = \overline{v}_p$$
$$\dot{\overline{v}}_p = \overline{0} \neq R\dot{\theta}^2 = \overline{a}_p \quad .$$

This is because $\overline{v}_p$ and $\overline{a}_p$ refer not to the velocity and acceleration of the trajectory of point P, but to the velocity and acceleration of the material point on the disk that happens to be at point P at time t.

Turning to the velocity and acceleration of an arbitrary point on the body $\underline{X}$, we write:

$$\overline{X} - \overline{x} = x_1\overline{e}_x + y_1\overline{e}_y \, ,$$

where x_1 + y_1 are constants.

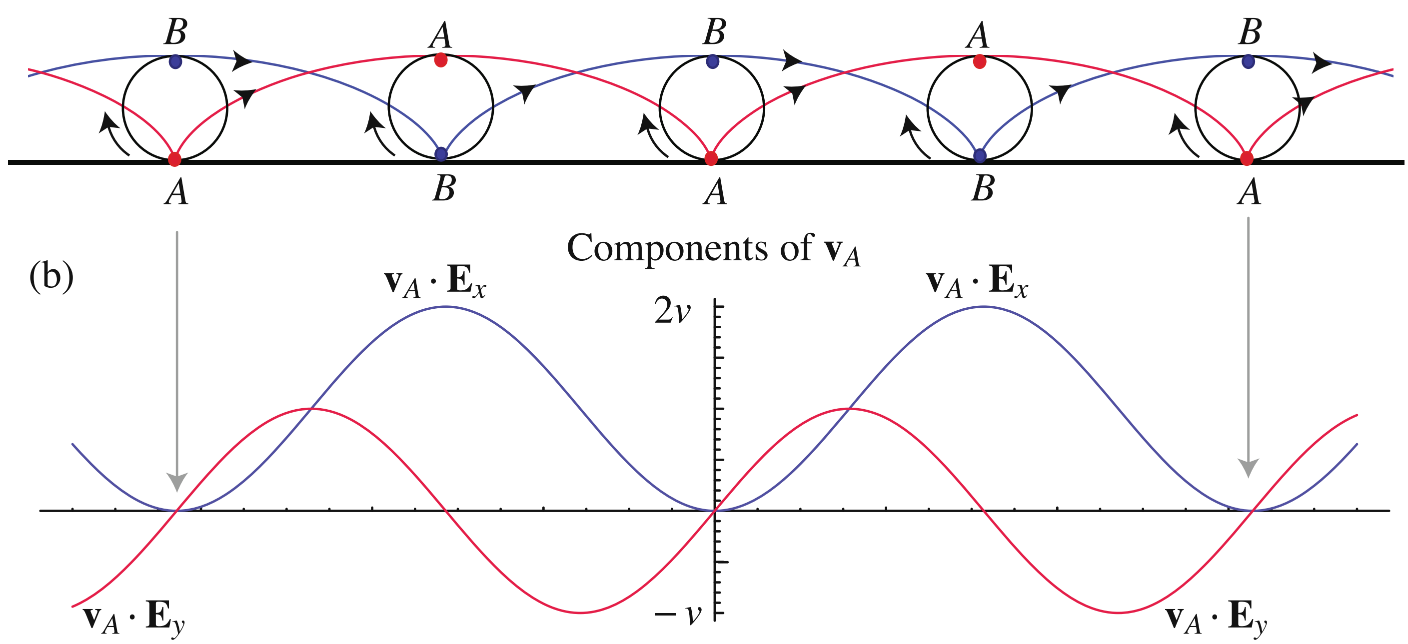
Velocity:

$$\bar{v} = \overline{\underline{v}} + \bar{\omega} \times (\bar{x} - \overline{\underline{x}})$$
$$= -R\dot{\theta}\bar{E}_x + \dot{\theta}\bar{E}_z \times (x_1 \bar{e}_x + y_1 \bar{e}_y)$$
$$= -R\dot{\theta}\bar{E}_x + \dot{\theta}(x_1 \bar{e}_y - y_1 \bar{e}_x)$$

Acceleration:

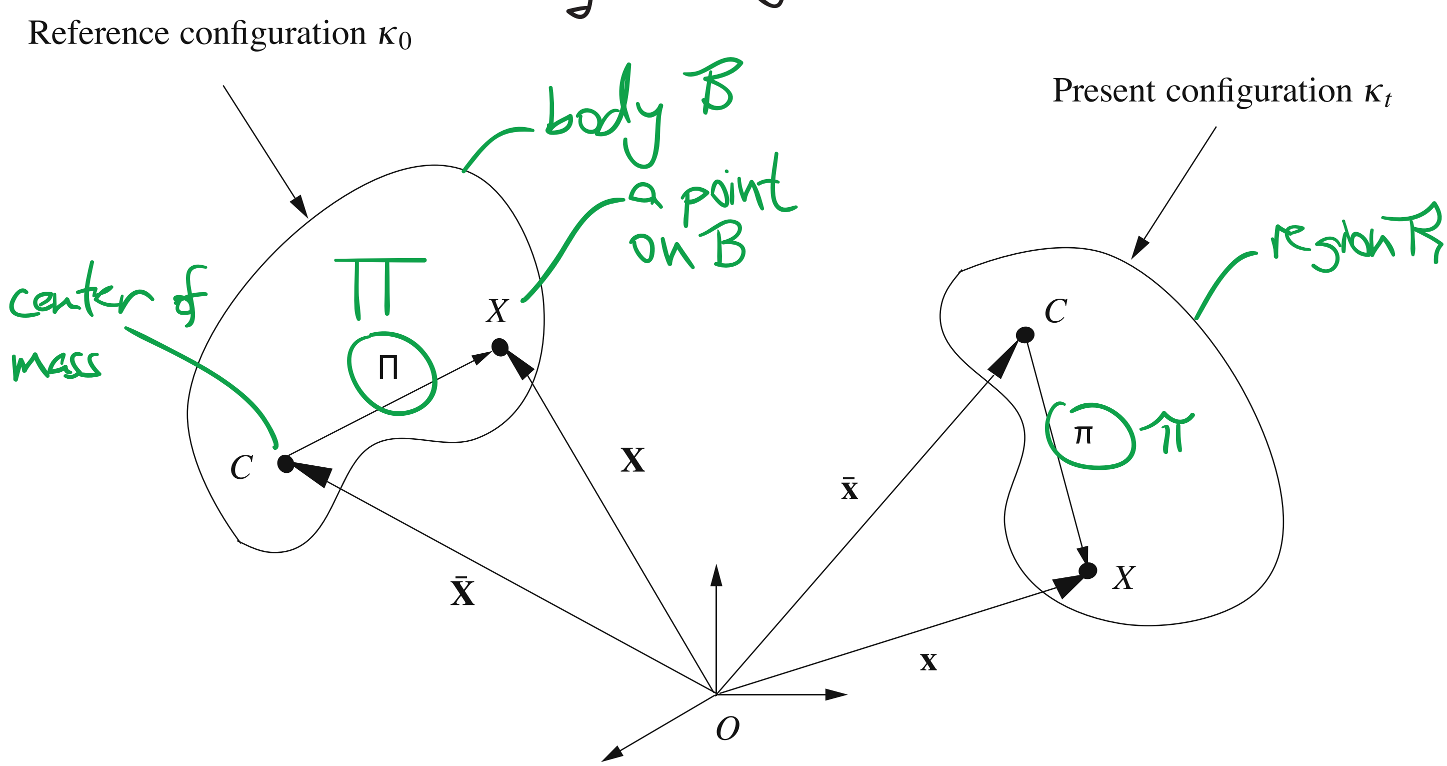
$$\bar{a} = \overline{\underline{a}} + \alpha \times (\bar{x} - \overline{\underline{x}}) + \bar{\omega} \times (\bar{\omega} \times (\bar{x} - \overline{\underline{x}}))$$
$$= -R\ddot{\theta}\bar{E}_x + \ddot{\theta}\bar{E}_z \times (x_1 \bar{e}_x + y_1 \bar{e}_y) + \dot{\theta}\bar{E}_z \times (\dot{\theta}E_z \times (x_1 e_x + y_1 e_y))$$
$$= -R\ddot{\theta}\bar{E}_x + \ddot{\theta}(x_1 \bar{e}_y - y_1 \bar{e}_x) - \dot{\theta}^2 (x_1 \bar{e}_x + y_1 \bar{e}_y)$$

Let's examine two points A and B on a disk rolling at constant horizontal translational velocity.



8.7 Angular Momenta

Before we can discuss the balance laws of Chapter 9, we must introduce the angular momentum of a rigid body.



Definition: the angular momentum of $\mathcal{B}$ about its center of mass C is

$$\bar{H} = \int_{\mathcal{R}} (\bar{x} - \bar{\bar{x}}) \times \bar{v} \rho \, dv .$$

Definition: the angular momentum of $\mathcal{B}$ about the fixed point O is

$$\bar{H}_o = \int_{\mathcal{R}} \bar{x} \times \bar{v} \rho \, dv .$$

These equations can be related as follows.

$$\bar{H}_o = \int_R \bar{X} \times \bar{v} \rho \, dv$$

$$= \int_R (\bar{X} - \bar{\bar{X}} + \bar{\bar{X}}) \times \bar{v} \rho \, dv$$

$$= \int_R (\bar{X} - \bar{\bar{X}}) \times \bar{v} \rho \, dv + \int_R \bar{\bar{X}} \times \bar{v} \rho \, dv$$

$$= \bar{H} + \bar{\bar{X}} \times \int_R \bar{v} \rho \, dv$$

linear momentum of B

$$\boxed{\bar{H}_o = \bar{H} + \bar{\bar{X}} \times \bar{G}} \, .$$

See O'Reilly p. 153 for an expression for the angular momentum about an arbitrary point.

8.8 Inertia tensor

Expressing the relative position vectors in bases:

$$\bar{\bar{\Pi}} = \Pi_x \bar{E}_x + \Pi_y \bar{E}_y + \Pi_z \bar{E}_z$$

$$\bar{\pi} = \Pi_x \bar{e}_x + \Pi_y \bar{e}_y + \Pi_z \bar{e}_z \, .$$

Also, we write the angular velocity of the body:

$$\bar{\omega} = \omega_x \bar{e}_x + \omega_y \bar{e}_y + \omega_z \bar{e}_z \, .$$

8.8.1 The Inertia Tensor

If we consider the angular momentum of $\mathcal{B}$ about its center of mass $\bar{H}$, we can derive the following important relationship:

$$\boxed{\bar{H} = I\,\bar{\omega}}$$

where

I is the inertia tensor, which is written as a matrix in the corotational basis as:

$$I = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{xy} & I_{yy} & I_{yz} \\ I_{xz} & I_{yz} & I_{zz} \end{bmatrix}.$$

The following components are called moments of inertia:

$$I_{xx} = \int_{\mathcal{R}} (\Pi_y^2 + \Pi_z^2)\,\rho\,dv = \int_{\mathcal{R}_0} (\Pi_y^2 + \Pi_z^2)\,\rho_0\,dV,$$

$$I_{yy} = \int_{\mathcal{R}} (\Pi_x^2 + \Pi_z^2)\,\rho\,dv = \int_{\mathcal{R}_0} (\Pi_x^2 + \Pi_z^2)\,\rho_0\,dV,$$

$$I_{zz} = \int_{\mathcal{R}} (\Pi_x^2 + \Pi_y^2)\,\rho\,dv = \int_{\mathcal{R}_0} (\Pi_x^2 + \Pi_y^2)\,\rho_0\,dV,$$

The remaining components are called products of inertia:

$$I_{xy} = -\int_{\mathcal{R}} \Pi_x\Pi_y\rho\,dv = -\int_{\mathcal{R}_0} \Pi_x\Pi_y\rho_0\,dV,$$

$$I_{xz} = -\int_{\mathcal{R}} \Pi_x\Pi_z\rho\,dv = -\int_{\mathcal{R}_0} \Pi_x\Pi_z\rho_0\,dV,$$

$$I_{yz} = -\int_{\mathcal{R}} \Pi_y\Pi_z\rho\,dv = -\int_{\mathcal{R}_0} \Pi_y\Pi_z\rho_0\,dV.$$

$\mathbf{I}$ is positive-definite, and so its eigenvalues are positive. Its components depend on the basis chosen. If the basis $(\bar{E}_x, \bar{E}_y, \bar{E}_z)$ is chosen such that $\bar{E}_x, \bar{E}_y, + \bar{E}_z$ are the eigenvectors of $\mathbf{I}$, then $\{\bar{E}_x, \bar{E}_y, \bar{E}_z\}$ and $\{\bar{e}_x, \bar{e}_y, \bar{e}_z\}$ are called the principal axes of the body in its reference and present configurations, respectively.

In this case, in the corotational basis,

$$\mathbf{I} = \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix}.$$

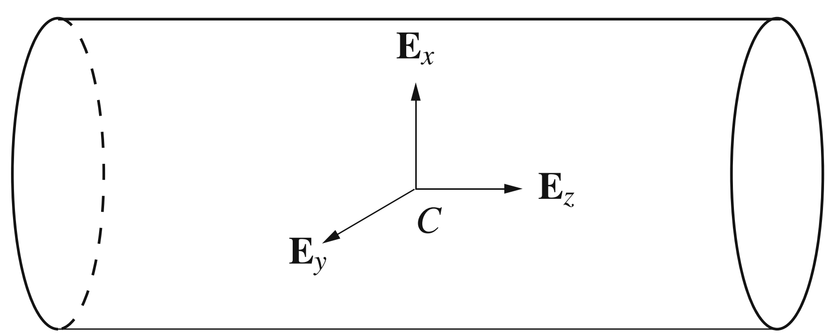
Therefore, we always try to choose $\bar{E}_x, \bar{E}_y, + \bar{E}_z$ as the principal axes.

8.8.3 A Circular Cylinder (example)

What is the inertia tensor for the homogeneous cylinder of mass m, radius R, and Length L?

Choose the principle axes, as shown.

Cylinder of radius R and length L



If $I_{xy} = I_{xz} = I_{yz} = 0$, (and they are) we have correctly chosen the principal axes.

Computing the remaining integrals,

$$I = \begin{bmatrix} \frac{1}{4}mR^2 + \frac{1}{12}mL^2 & 0 & 0 \\ 0 & \frac{1}{4}mR^2 + \frac{1}{12}mL^2 & 0 \\ 0 & 0 & \frac{1}{2}mR^2 \end{bmatrix}.$$

If we set $R=0$, we have the inertia tensor for a "slender rod." With $L=0$, we have the inertia tensor for a "thin disk."

8.8.4 The Parallel Axis Theorem + Practical Notes

The Parallel Axis Theorem is commonly used to find the inertia tensor of a point that is not the center of mass. However, we circumvent the need for it by expressing the angular momentum about an arbitrary point A:

$$\overline{H}_A = \int_R (\overline{x} - \overline{x}_A) \times \overline{v} \rho \, dv \qquad \text{(def.)}$$

$$= \overline{H} \times (\overline{x} - \overline{x}_A) \times \overline{G} \quad .$$

We will use this in Chapter 9. Note that this approach is more general than the PAT, which only applies to points on the body.

To find the moments of inertia, we often refer to a table of common shapes of bodies, like that in the cover of Hibbeler.

Chapter 9: Kinetics of a Rigid Body

9.1 Balance Laws for a Rigid Body

Before we get to the balance laws, we need to discuss forces and moments on a rigid body.

9.1.1 Resultant Forces and Moments

Given a set of n forces $\{\overline{F}_1, \overline{F}_2, \ldots, \overline{F}_i, \ldots, \overline{F}_n\}$ acting on a body $\mathcal{B}$ at material points $\{\underline{X}_1, \ldots, \underline{X}_i, \ldots \underline{X}_n\}$, the resultant force is

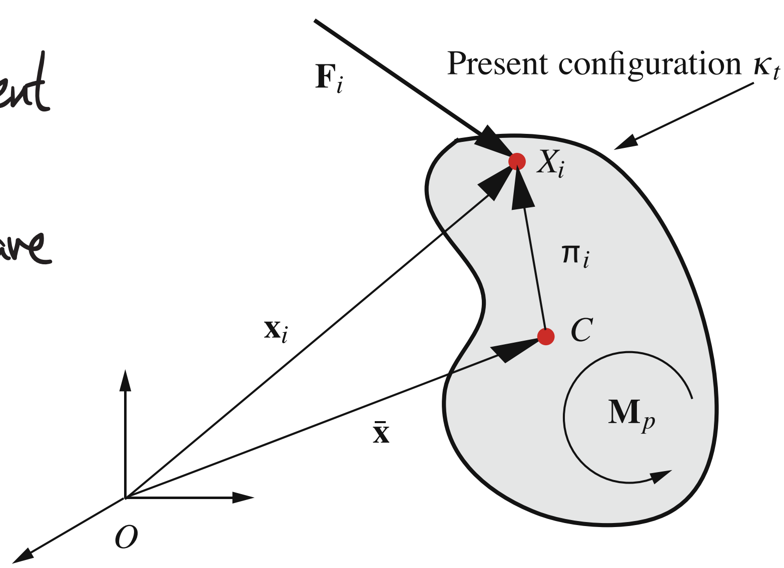
$$\overline{F} = \sum_{i=1}^{n} \overline{F}_i \quad .$$

Similarly, the resultant moment $\overline{M}_0$ relative to the fixed point O is the sum of individual moments about O acting on $\mathcal{B}$. We denote the resultant moment relative to the center of mass $\overline{M}$.

Given an external moment $\overline{M}_p$ not induced by an $\overline{F}_i$, the resultant moments are

$$\overline{M}_0 = \overline{M}_p + \sum_{i=1}^{n} \overline{x}_i \times \overline{F}_i$$

$$\overline{M} = \overline{M}_p + \sum_{i=1}^{n} (\overline{x}_i - \overline{\overline{x}}) \times \overline{F}_i$$



9.1.2 Euler's Laws

Euler's Laws are the momentum balance laws for rigid bodies. The first law is the balance of linear momentum:

$$(\star) \qquad \boxed{\overline{F} = \dot{\overline{G}} = m\dot{\overline{v}}} \; .$$

The second law is the balance of angular momentum:

$$(\star\star) \qquad \boxed{\overline{M}_O = \dot{\overline{H}}_O} \; .$$

relative to the fixed point O

Together, the Euler's Laws give six scalar equations. Another form of the second law is:

$$(\star\star\star) \qquad \boxed{\overline{M} = \dot{\overline{H}}} \; .$$

relative to the center of mass C

For rigid bodies with a fixed point (i.e. "pinned"), we typically use $(\star\star)$. For others we often use $(\star\star\star)$.

Recall from Section 8.8,

$$\begin{aligned}
\overline{H} = \mathbb{I}\,\overline{\omega} &= (I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z)\,\overline{e_x} \\
&+ (I_{xy}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z)\,\overline{e_y} \\
&+ (I_{xz}\omega_x + I_{yz}\omega_y + I_{zz}\omega_z)\,\overline{e_z} \; .
\end{aligned}$$

Taking the time-derivative,

$$\dot{\overline{H}} = \underline{I}\,\dot{\overline{\omega}} = \overset{\circ}{\overline{H}} + \overline{\omega} \times \overline{H}, \quad \text{where}$$

$$\begin{aligned}\overset{\circ}{\overline{H}} = &\left(I_{xx}\dot{\omega}_x + I_{xy}\dot{\omega}_y + I_{xz}\dot{\omega}_z\right)\overline{e_x} \\ &+ \left(I_{xy}\dot{\omega}_x + I_{yy}\dot{\omega}_y + I_{yz}\dot{\omega}_z\right)\overline{e_y} \\ &+ \left(I_{xz}\dot{\omega}_x + I_{yz}\dot{\omega}_y + I_{zz}\dot{\omega}_z\right)\overline{e_z}\end{aligned}$$

is the corotational rate of $\overline{H}$. This gives, in general, a very complicated set of equations.

9.1.3 The Fixed-Axis of Rotation Case

We worked-out the kinematics in Chapter 8:

$$\overline{e_x} = \cos\theta\,\overline{E_x} + \sin\theta\,\overline{E_y} \quad \Big| \quad \overline{e_y} = \cos\theta\,\overline{E_y} - \sin\theta\,\overline{E_x} \quad \Big| \quad \overline{e_z} = \overline{E_z}$$

$$\dot{\overline{e_x}} = \dot{\theta}\,\overline{e_y} \quad \Big| \quad \dot{\overline{e_y}} = -\dot{\theta}\,\overline{e_x} \quad \Big| \quad \overline{\omega} = \dot{\theta}\,\overline{E_z} = \omega \overline{E_z}$$

The angular moment and its time-derivative are:

$$\overline{H} = I_{xz}\omega\,\overline{e_x} + I_{yz}\omega\,\overline{e_y} + I_{zz}\omega\,\overline{E_z} \quad \text{and}$$

$$\dot{\overline{H}} = \left(I_{xz}\dot{\omega} - I_{yz}\omega^2\right)\overline{e_x} + \left(I_{yz}\dot{\omega} + I_{xz}\omega^2\right)\overline{e_y} + I_{zz}\dot{\omega}\,\overline{E_z}.$$

The kinetics are simply Euler's equations using the above kinematics:

$$\overline{F} = m\dot{\overline{v}} \quad \Big| \quad \overline{M} = \dot{\overline{H}}.$$

The first equation gives the motion of the center of mass and reaction forces.

The $\bar{e}_x$ and $\bar{e}_y$ scalar equations from the second equation ($\bar{M} = \dot{\bar{H}}$), give the reaction moment $\bar{M}_c$ that keeps the body rotating about the $\bar{E}_z$-axis ($\bar{M}_c$ is often just $\bar{0}$).

The $\bar{e}_z = \bar{E}_z$ scalar equation from $\bar{M} = \dot{\bar{H}}$ gives the differential equation for $\theta(t)$.

9.1.4 The Four Steps for solving problems! ← this

We follow four steps that are similar to the four we used for particles.

1. Kinematics

Pick: - an origin for the Cartesian basis in a reference configuration. (O)
- a coordinate system to work in $(\bar{E}_x, \bar{E}_y, \bar{E}_z)$
- a corotational basis $(\bar{e}_x, \bar{e}_y, \bar{e}_z)$

Establish expressions for $\bar{H}$ or $\bar{H}_o$, $\bar{\bar{x}}$, $\bar{\bar{v}}$, and $\bar{\bar{a}}$.

2. Forces and moments

Draw a free-body diagram show external forces $\bar{F}_i$ and moments $\bar{M}_p$.

Write what is known about each force and moment.

3. Euler's Laws

Write out the six scalar equations from

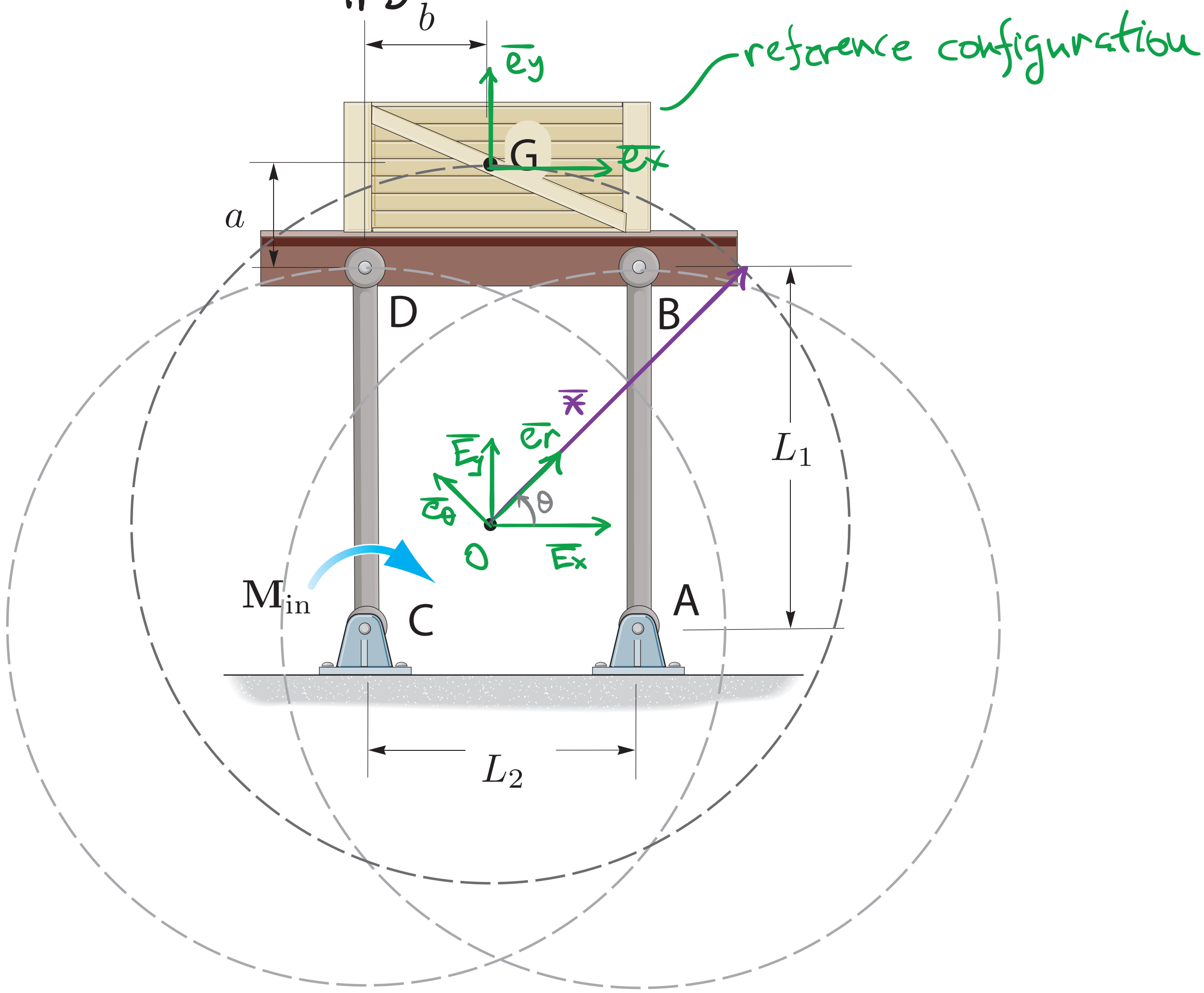
$$\bar{F} = m\bar{\bar{a}} \quad \Big| \quad \bar{M} = \dot{\bar{H}} \quad \text{or} \quad \bar{M}_o = \dot{\bar{H}}_o$$

4. Analysis

Solve for what is needed, using the six scalar equations and sometimes additional kinematic equations.

Hibbeler 17-55 (but more-general), the O'Reilly way

Given an applied moment $\overline{M}_{in}$ at C, find the angular acceleration $\ddot{\theta}(\theta)$ of the links and $\overline{T}_1(\theta,\dot{\theta})$ and $\overline{T}_2(\theta,\dot{\theta})$, the pin forces at D and B. Assume massless links and no slipping of the box.



Kinematics

The motion of points G, D, and B follow the circular paths shown. We choose the origin O of our Cartesian coordinate system to be at the center of the circle that G follows. This is convenient because a polar coordinate basis is natural in this case. The corotational basis is colinear with the Cartesian basis because the orientation of the box doesn't change throughout its motion. The position vector of G is

$$\overline{x} = x\overline{E}_x + y\overline{E}_y = r\,\overline{e}_r = L_1\overline{e}_r\,.$$

Either differentiating or using the results of § 2.2,

$$\bar{v} = \dot{r}\,\bar{e}_r + r\dot{\theta}\,\bar{e}_\theta = L_1\dot{\theta}\,\bar{e}_\theta \qquad \text{and}$$

$$\bar{a} = -L_1\dot{\theta}^2\,\bar{e}_r + L_1\ddot{\theta}\,\bar{e}_r \quad .$$

We will also need the position vectors of D and B.

$$\bar{x}_D = \bar{x} - b\,\bar{E}_x - a\bar{E}_y = \bar{x} - b(\cos\theta\,\bar{e}_r - \sin\theta\,\bar{e}_\theta) - a(\sin\theta\,\bar{e}_r + \cos\theta\,\bar{e}_\theta) = (L_1 - b\cos\theta - a\sin\theta)\bar{e}_r + (b\sin\theta - a\cos\theta)\bar{e}_\theta$$

$$\bar{x}_B = \bar{x} + (L_2 - b)\bar{E}_x - a\bar{E}_y = (L_1 + (L_2 - b)\cos\theta - a\sin\theta)\,\bar{e}_r + ((b - L_2)\sin\theta - a\cos\theta)\,\bar{e}_\theta$$

In a moment, we will apply Euler's laws. In anticipation of that, let's compute the angular momentum of the box about O:

$$\bar{H}_O \triangleq \bar{H} + \bar{x} \times \bar{G}$$

Cartesian basis

$$\bar{H} \triangleq \mathbb{I}\bar{\omega} = \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ \omega \end{bmatrix} = I_{zz}\,\cancel{\omega}^{\,0}\,\bar{E}_z = 0\bar{E}_z$$

$$\bar{G} \triangleq m\bar{v} = mL_1\dot{\theta}\,\bar{e}_\theta \quad .$$

Combining these expressions:

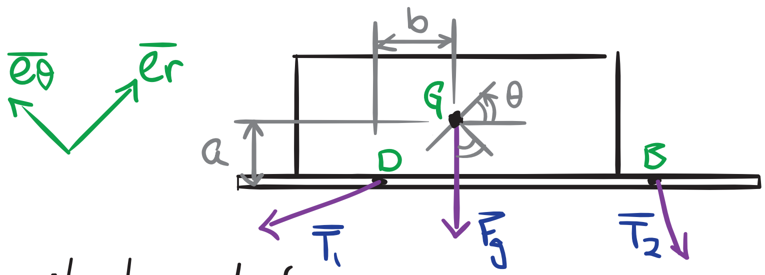
$$\begin{aligned} \bar{H}_O &= L_1\bar{e}_r \times mL_1\dot{\theta}\,e_\theta \\ &= \det\begin{bmatrix} \bar{e}_r & \bar{e}_\theta & \bar{E}_z \\ L_1 & 0 & 0 \\ 0 & mL_1\dot{\theta} & 0 \end{bmatrix} \\ &= mL_1^2\dot{\theta}\,\bar{E}_z \quad . \end{aligned}$$

And we'll need the time-derivatives of the linear and angular momenta:

$$\dot{\bar{G}} = m\bar{a} = m(-L_1\dot{\theta}^2\,\bar{e}_r + L_1\ddot{\theta}\,\bar{e}_r)$$

$$\dot{\bar{H}}_O = mL_1^2\,\ddot{\theta}\,\bar{E}_z \quad .$$

Forces + Moments on the box



The bars exert the forces:

$$\bar{T}_1 = T_{1r}\bar{e}_r + T_{1\theta}\bar{e}_\theta \quad \&$$
$$\bar{T}_2 = T_{2r}\bar{e}_r + T_{2\theta}\bar{e}_\theta \ .$$

The gravitational force is:

$$\bar{F}_g = -mg\sin\theta\,\bar{e}_r - mg\cos\theta\,\bar{e}_\theta$$

Resultant force: $\bar{F} = \bar{F}_g + \bar{T}_1 + \bar{T}_2 = (-mg\sin\theta + T_{1r} + T_{2r})\bar{e}_r + (-mg\cos\theta + T_{1\theta} + T_{2\theta})\bar{e}_\theta$.

Moments about O:

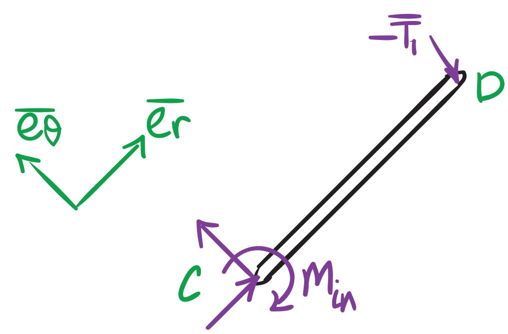
$$\bar{M}_g = \bar{x} \times \bar{F}_g = L_1\bar{e}_r \times (-mg\sin\theta\,\bar{e}_r - mg\cos\theta\,\bar{e}_\theta) = -mgL_1\cos\theta\,\bar{E}_z$$

$$\bar{M}_1 = \bar{x}_D \times \bar{T}_1 = \big((aT_{1r} - bT_{2r})\cos\theta - (bT_{1r} + aT_{2r})\sin\theta\big)E_z$$

$$\bar{M}_2 = \bar{x}_B \times \bar{T}_2 = \big(L_1T_{2\theta} + (aT_{1\theta} + (L_2-b)T_{2\theta})\cos\theta - (bT_{1\theta} - L_2T_{1\theta} + aT_{2\theta})\sin\theta\big)\bar{E}_z.$$

Resultant moment: $\bar{M}_O = \bar{M}_g + \bar{M}_1 + \bar{M}_2$. $\bar{M}_{in}$ doesn't enter because it's applied to the link CD, not the box.

Forces + Moments on Link CD



The resultant moment about C is:

$$\bar{M}_c = \bar{M}_{in} + L_1\bar{e}_r \times (-\bar{T}_1) = \bar{M}_{in} - L_1T_{1\theta}\bar{E}_z.$$

Introducing the resultant force doesn't help us because we don't know or care about the reaction at C.

Euler's Laws on the links

The link CD is massless, so its angular momentum is zero and

$$\bar{M}_c = \bar{0} \Rightarrow \bar{M}_{in} - L_1T_{1\theta}\bar{E}_z = \bar{0} \Rightarrow T_{1\theta} = \frac{1}{L_1}\bar{M}_{in}\cdot\bar{E}_z \ .$$

Similarly: $T_{2\theta} = 0$.

Euler's Laws on the box

First Law: $\bar{F} = \dot{\bar{G}} = m\bar{a}$ which, written in the polar coord's, is

$$\begin{bmatrix} -mg\sin\theta + T_{1r} + T_{2r} \\ -mg\cos\theta + T_{1\theta} + T_{2\theta} \end{bmatrix} = \begin{bmatrix} m(-L_1\dot{\theta}^2) \\ mL_1\ddot{\theta} \end{bmatrix}$$

Second Law: $\bar{M}_0 = \dot{\bar{H}}_0 \Rightarrow \bar{M}_g + \bar{M}_1 + \bar{M}_2 = mL_1^2\ddot{\theta}\,\bar{E}_z$

Analysis

The second scalar equation from the First Law gives:

$$\begin{aligned} \ddot{\theta} &= \frac{1}{mL_2}(-mg\cos\theta + T_{1\theta} + \cancelto{0}{T_{2\theta}}) \\ &= \frac{1}{mL_1}\left(-mg\cos\theta + \frac{1}{L_1}M_{in}\right) \\ &= \frac{1}{mL_1^2}(-mgL_1\cos\theta + M_{in}) \end{aligned}$$

ANS

We know $T_{1\theta} = \frac{1}{L_1}\bar{M}_{in}\cdot\bar{E}_z$ and $T_{2\theta} = 0$ and $\ddot{\theta}(\theta)$. We still want $T_{1r}(\theta,\dot{\theta})$ and $T_{2r}(\theta,\dot{\theta})$, so we need two equations with T_{1r}, T_{2r}, θ, + $\dot{\theta}$ the only unknowns.

The first scalar equation of the First Law is one. The Second Law gives only one (nontrivial) scalar equation. They are linear and easy to solve:

$\bar{T}_1(\theta,\dot{\theta}) = T_{1r}\bar{e}_r + T_{1\theta}\bar{e}_\theta$ where $T_{1r}(\theta,\dot{\theta}) =$ ANS

$$\frac{mL_1^3\ddot{\theta} + (b-L_2)M_{in}\sin\theta + agL_1m\sin^2\theta + (mgL_1^2 - aM_{in} + bL_1mg\sin\theta)\cos\theta - mL_1^2(b\cos\theta + a\sin\theta)\dot{\theta}^2}{(a+b)L_1\cos\theta + (a-b)L_1\sin\theta}$$

$\bar{T}_2(\theta,\dot{\theta}) = T_{2r}\bar{e}_r + T_{2\theta}\bar{e}_\theta$ where $T_{2r}(\theta,\dot{\theta}) =$ ANS

$$\frac{-mL_1^3\ddot{\theta} - (b-L_2)M_{in}\sin\theta - bgL_1m\sin^2\theta + (-mgL_1^2 - aM_{in} + aL_1mg\sin\theta)\cos\theta + mL_1^2(-a\cos\theta + b\sin\theta)\dot{\theta}^2}{(a+b)L_1\cos\theta + (a-b)L_1\sin\theta}$$

Note that there was a more-convenient point about which to apply Euler's Laws: G. Because the box has no rotation $\bar{\alpha} = \bar{0}$, and $\bar{M} = 0$. This simplifies the solution for $\bar{T}_1(\theta,\dot{\theta})$ and $\bar{T}_2(\theta,\dot{\theta})$, mostly because the position vectors in the moment equations are easier.

9.2 Work-Energy Theorem and Energy Conservation

9.2.1 Koenig's Decomposition

The definition of the kinetic energy of a rigid body is

$$\boxed{T \triangleq \frac{1}{2}\int_{\mathcal{R}} \bar{v}\cdot\bar{v}\,\rho\, dv} \quad ,$$

where $\mathcal{R}$ is the region of space occupied by the body, ρ is its density, and $\bar{v}$ is the velocity of a material point in the body.

In practice, however, we use Koenig's decomposition of this expression for a rigid body:

$$\boxed{T = \underbrace{\frac{1}{2}m\bar{v}\cdot\bar{v}}_{\text{translational}} + \underbrace{\frac{1}{2}\bar{H}\cdot\bar{\omega}}_{\text{rotational}}} \quad ,$$

where m is the body's mass, $\bar{v}$ is the velocity of the center of mass, $\bar{H}$ is the angular momentum about the center of mass, and $\bar{\omega}$ is the angular velocity.

So a rigid body's kinetic energy is the sum of the translational and rotational kinetic energy of its center of mass.

9.2.2 The Work-Energy Theorem

Starting with the Koenig decomposition of T, the first form of the work-energy theorem for a rigid body can be derived:

$$\boxed{\dot{T} = \bar{F}\cdot\bar{v} + \bar{M}\cdot\bar{\omega}} \quad ,$$

where $\bar{F}$ is the resultant force on the body and $\bar{M}$ is the resultant moment about the center of gravity. This is a natural extension of the work-energy theorem for a single particle

There is a second form of the theorem that is often useful:

$$\boxed{\dot{T} = \sum_{i=1}^{n} \bar{F}_i \cdot \bar{v}_i + \bar{M}_p \cdot \bar{\omega}},$$

where $\bar{F}_i$ are the n forces applied to a body from §9.1, $\bar{v}_i$ are the velocities of the points where the $\bar{F}_i$ are applied, and $\bar{M}_p$ is the externally applied moment (not a result of the $\bar{F}_i$).

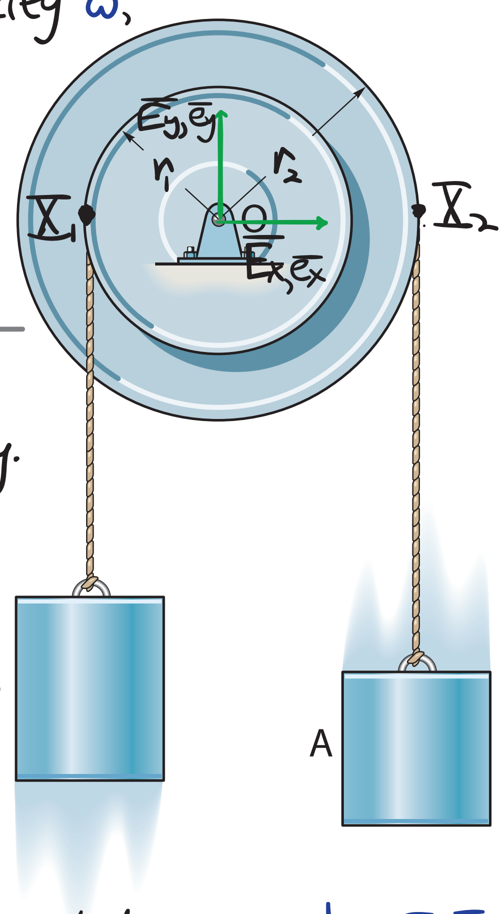
So we can see from this form of the theorem that:

I. The mechanical power of a force $\bar{P}$ applied to a body at a point X on the body is $\bar{P} \cdot \bar{v}$, where $\bar{v}$ is the velocity of point X.

II. The mechanical power of an applied moment $\bar{L}$ is $\bar{L} \cdot \bar{\omega}$.

Something like Hibbeler 18-7 (example)

Given the double spool's angular velocity ω, what is the system's kinetic energy? Is the energy of the system conserved? Assume that the ropes do not slip. The moment of inertia of the spool about O in the $\bar{E}_z$-direction is I_{zz}.



The kinetic energy of the system is the sum of the kinetic energy of each body.

The kinetic energy of the spool is

$$T_S = \frac{1}{2} m \cancelto{\bar{0}}{\bar{v}_S} \cdot \cancelto{\bar{0}}{\bar{v}_S} + \frac{1}{2} \bar{H} \cdot \bar{\omega}$$
$$= \frac{1}{2} (I \bar{\omega}) \cdot \bar{\omega}$$
$$= \frac{1}{2} I_{zz} \omega^2 \quad .$$

Since B and A are translating and not rotating, $T_A = \frac{1}{2} m_A \bar{v}_A \cdot \bar{v}_A$ and $T_B = \frac{1}{2} m_B \bar{v}_B \cdot \bar{v}_B$.

So all we need to find are the velocities of A and B.

If we assume the ropes aren't slack, the tangential velocity of each mass is the same as the corresponding ropes at each point, which is the same as the corresponding radius of the spool's vel. Therefore, if we find the velocity of a point on each radius, we will have the velocity of each mass.

Using the important equation relating the velocities of any two points on a rigid body, using O as one of the points because it has a known (zero) velocity, we can find the velocity $\bar{v}_1$ of a point on the radius $\bar{X}_1$:

$$\bar{v}_1 - \cancelto{0}{\bar{v}} = \bar{\omega} \times (\bar{X}_1 - \cancelto{0}{\bar{x}})$$

$$\bar{v}_1 = \bar{\omega} \times \bar{X}_1$$

$$= \omega \bar{E}_z \times (-r_1 \bar{e}_x)$$

$$= \det \begin{bmatrix} \bar{e}_x & \bar{e}_y & \bar{E}_z \\ 0 & 0 & \omega \\ -r_1 & 0 & 0 \end{bmatrix} \Rightarrow$$

$$\bar{v}_B = \bar{v}_1 = -r_1 \omega \bar{e}_y .$$

Similarly for $\bar{v}_2$ of point $\bar{X}_2$:

$$\bar{v}_2 - \cancel{\bar{v}} = \bar{\omega} \times (\bar{X}_2 - \cancel{\bar{x}})$$

$$\bar{v}_2 = \bar{\omega} \times \bar{X}_2$$

$$= \omega \bar{E}_z \times (r_2 \bar{e}_x)$$

$$= \det \begin{bmatrix} \bar{e}_x & \bar{e}_y & \bar{E}_z \\ 0 & 0 & \omega \\ r_2 & 0 & 0 \end{bmatrix}$$

$$\bar{v}_A = \bar{v}_2 = r_2 \omega \bar{e}_y .$$

Finally: $T_A = \frac{1}{2} m_A \bar{v}_A \cdot \bar{v}_A = \frac{1}{2} m_A r_2^2 \omega^2$ and $T_B = \frac{1}{2} m_B r_1^2 \omega^2$. The total is:

$$T = T_S + T_A + T_B = \frac{1}{2} I_{zz} \omega^2 + \frac{1}{2} m_A r_2^2 \omega^2 + \frac{1}{2} m_B r_1^2 \omega^2$$

$$= \frac{1}{2}(I_{zz} + m_A r_2^2 + m_B r_1^2)\omega^2 . \longleftarrow$$ ANS

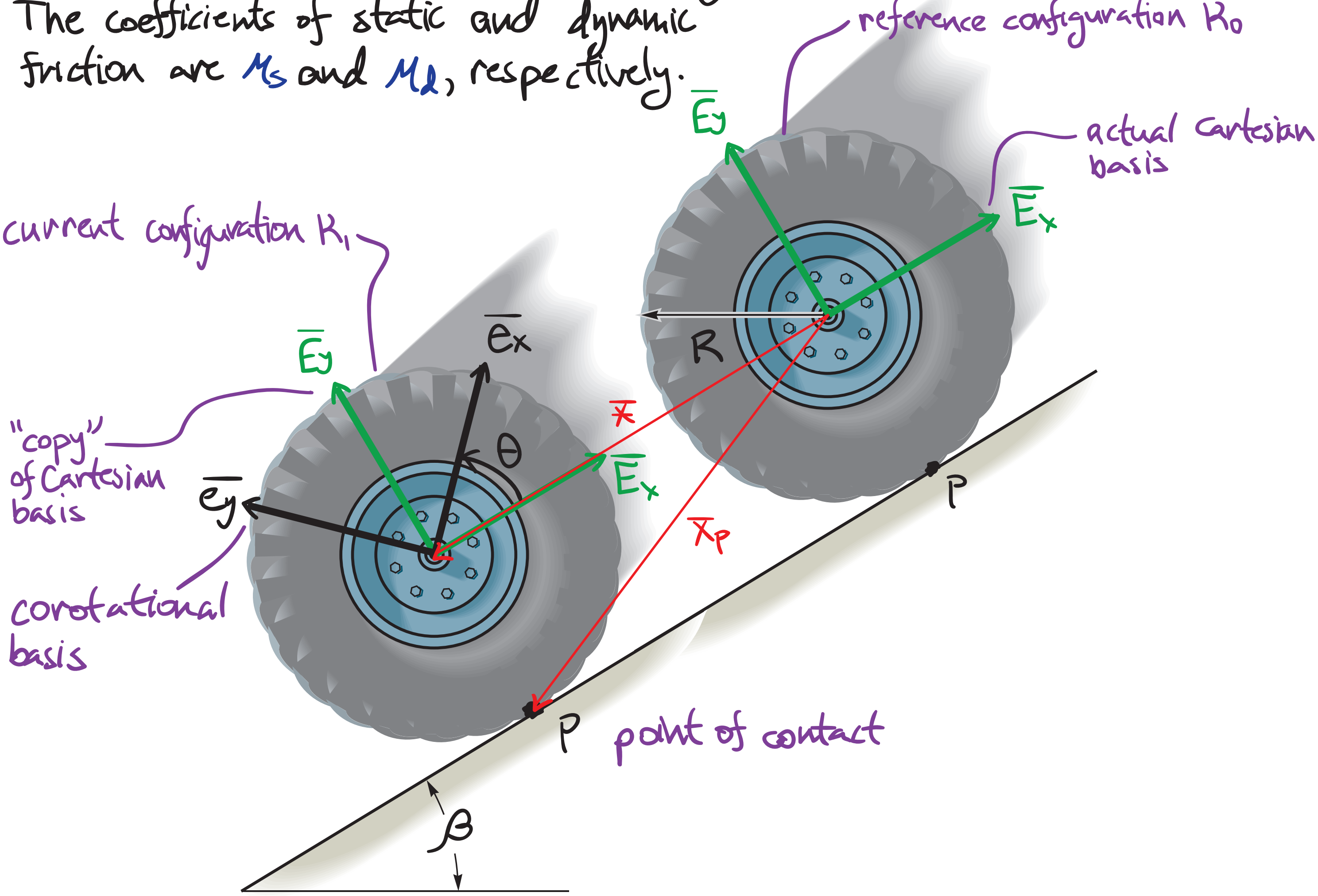
What about energy conservation? The only forces or moments external to the system are gravity and the pin reaction, which we now assume has no friction..

The gravitational force is conservative, as it always is. The pin reaction force does no work because the point at which they are applied has zero velocity (doesn't move).

Therefore, the system's energy is conserved. ANS

Example Based-On Hibbeler 17-94

Given the wheel with moment of inertia about its axle I_{zz} rolling down the incline, find the angular acceleration of the wheel if it doesn't slip and the range of β for which this is valid. The coefficients of static and dynamic friction are μ_s and μ_d, respectively.



Kinematics

Since the wheel isn't pinned, we're probably going to be interested primarily in the motion of the center of mass. We place the origin of the Cartesian basis at the center of mass in the reference configuration $\mathcal{K}_0$. The corotational basis has angle θ with respect to the Cartesian basis at some later ("present") configuration $\mathcal{K}_1$.

Position of the center of mass: $\bar{x} = x\bar{E}_x$.
Velocity of the center of mass: $\bar{v} = \dot{\bar{x}} = \dot{x}\bar{E}_x$.
Acceleration of the center of mass: $\bar{a} = \dot{\bar{v}} = \ddot{x}\bar{E}_x$.

Since we will have a moment equation (we have a rigid body, after all, so there's always a moment equation), we will want to know the angular velocity and acceleration.

Angular velocity: $\bar{\omega} = \omega \bar{E}_z = \dot{\theta}\, \bar{E}_z$.

Angular acceleration: $\bar{\alpha} = \alpha \bar{E}_z = \ddot{\theta}\, \bar{E}_z$.

When the wheel is rolling without slipping, we can relate the translation and rotation of the wheel with a convenient constraint. Even if the wheel is slipping, we can relate the velocities of the center of mass and the instantaneous point of contact P with the surface:

$$
\begin{aligned}
\bar{v} - \bar{v}_P &= \bar{\omega} \times (\bar{x} - \bar{x}_P) \\
\bar{v} &= \bar{v}_P + \bar{\omega} \times (\bar{x} - \bar{x}_P) \\
&= \bar{v}_P + \omega \bar{E}_z \times (\bar{x} - (\bar{x} - R\bar{E}_y)) \\
&= \bar{v}_P + \omega \bar{E}_z \times (R\bar{E}_y) \\
&= \bar{v}_P - R\omega \bar{E}_x \quad .
\end{aligned}
$$

If there is no slipping, $\bar{v}_P = \bar{0}$, + we get the familiar kinematic constraint:

$$\bar{v} = -R\omega \bar{E}_x . \quad \text{And} \quad \bar{a} = \dot{\bar{v}} = -R\alpha \bar{E}_x .$$

If there is slipping, but no loss of contact, we get

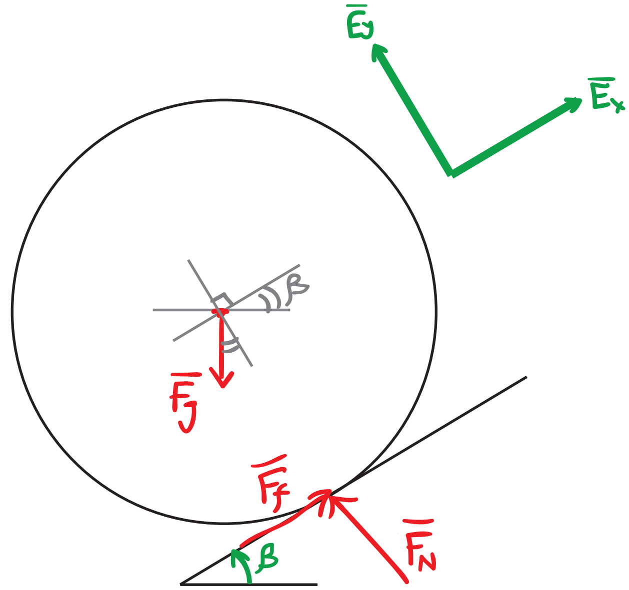
$$\bar{v} = (v_p - R\omega)\bar{E}_x .$$

Finally, we can write the linear momentum and the angular momentum about the center of mass, and their time-derivatives:

$$\bar{G} = m\bar{v} \qquad \bar{H} = I\bar{\omega} = I_{zz}\,\omega \bar{E}_z$$

$$\dot{\bar{G}} = m\bar{a} \qquad \dot{\bar{H}} = I_{zz}\alpha \bar{E}_{zz} .$$

Forces and Moments



Gravity: $\bar{F}_g = -mg(\sin\beta\,\bar{E}_x + \cos\beta\,\bar{E}_y)$

Normal: $\bar{F}_N = F_N \bar{E}_y$

Friction: $\bar{F}_f = F_f \bar{E}_x$.

Moment of friction: $\bar{M}_f = -R\bar{E}_y \times \bar{F}_f$
$= F_f R \bar{E}_z$.

Unpacking the friction force, we have two situations: (1) when the magnitude of the friction force

$$\|\bar{F}_f\| \leq \mu_s \|\bar{F}_N\| \qquad \text{(no slipping)}$$

and (2) when the friction force is

$$\bar{F}_f = -\mu_d \|\bar{F}_N\| \frac{\bar{v}_P}{\|\bar{v}_P\|} . \qquad \text{(slipping)}$$

Since we are trying to find the conditions for no slipping, we will use the first inequality.

Euler's Laws

Assuming no slipping (which we will then have to check), the first Law gives:

$$\bar{F} = m\bar{a}$$

$$\bar{F}_g + \bar{F}_N + \bar{F}_f = m(-R\alpha\bar{E}_x)$$

$$-mg(\sin\beta\,\bar{E}_x + \cos\beta\,\bar{E}_y) + F_N\bar{E}_y + F_f\bar{E}_x = -mR\alpha\bar{E}_x .$$

The second Law (about the center of mass) gives:

$$\bar{M} = \dot{\bar{H}}$$

$$F_f R\bar{E}_z = I_{zz}\alpha\bar{E}_z .$$

Analysis

We have three unknowns: F_f, F_N, and α. The first law gave us two (nontrivial) scalar equations and the second law gave us one. They are linear, so they are easy to solve.

$$F_N = mg\cos\beta$$

$$F_f = \frac{mg\sin\beta}{1 + mR^2/I_{zz}}$$

$$\alpha = \frac{mgR\sin\beta}{I_{zz} + mR^2} .$$ ANS

But we assumed that the wheel rolled without slipping, so we need to show the conditions under which this is true:

$$\|F_f\| \leq \mu_s \|F_N\|$$

$$\frac{mg\sin\beta}{1 + mR^2/I_{zz}} \leq \mu_s\, mg\cos\beta \qquad (0 \leq \beta \leq \pi/2)$$

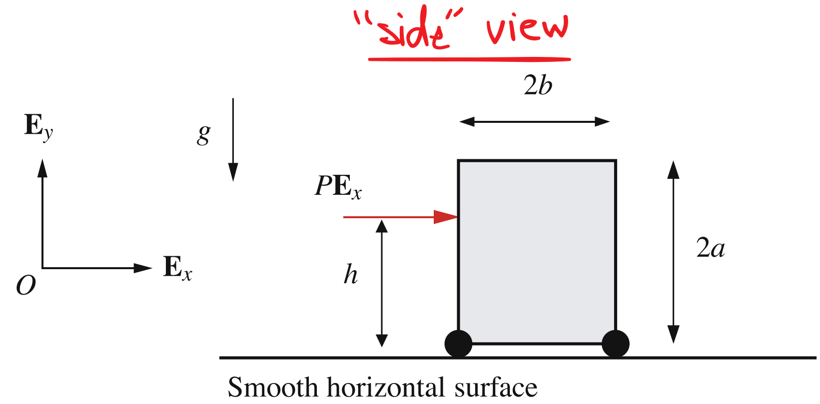
$$\tan\beta \leq \mu_s\,(1 + mR^2/I_{zz})$$

$$\beta \leq \arctan\left(\mu_s\,(1 + mR^2/I_{zz})\right) .$$ ANS

So it slips if $$\beta > \arctan\left(\mu_s\,(1 + mR^2/I_{zz})\right) .$$

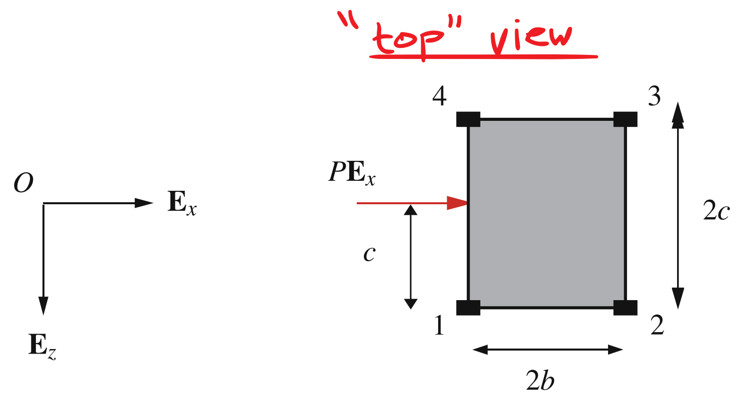
9.3.1 The Overturning Cart

The cart of mass m slides on a smooth horizontal surface.



Over what range of applied force P will the cart not tip? Ignore the mass of the wheels.

What assumptions are required to obtain the answer?



Kinematics

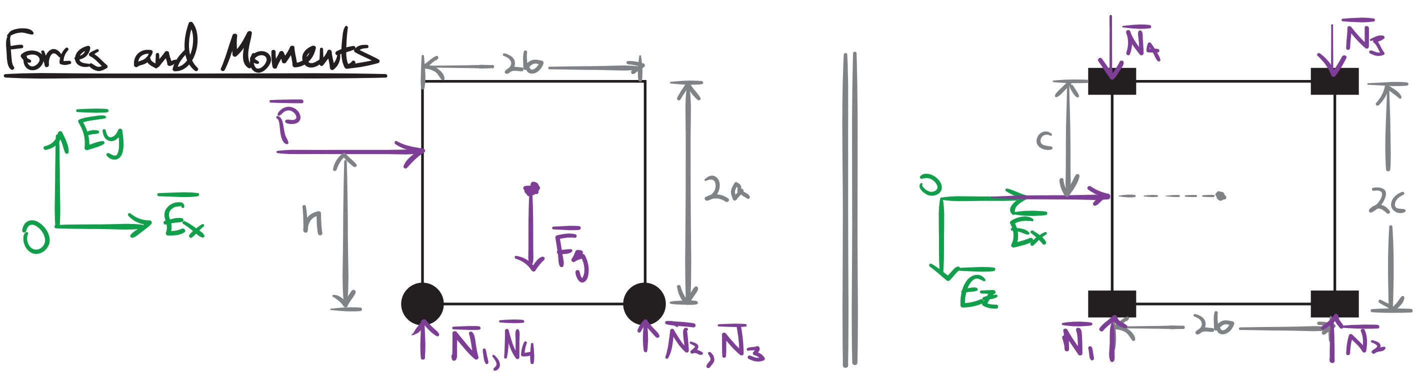
The center of mass has the following position, velocity, and acceleration:

$$\bar{x} = x\,\bar{E}_x + y_0\,\bar{E}_y + z_0\,\bar{E}_z$$
$$\bar{v} = \dot{x}\,\bar{E}_x$$
$$\bar{a} = \ddot{x}\,\bar{E}_x$$

The momenta are: $\bar{G} = m\bar{v} = m\dot{x}\bar{E}_x$ and $\bar{H} = I\bar{\omega} = \bar{0}$

The time rate of change of these momenta are: $\dot{\bar{G}} = m\ddot{x}\bar{E}_x$ and $\dot{\bar{H}} = \bar{0}$.

Forces and Moments



Gravity: force: $\bar{F}_g = -mg\bar{E}_y$, moment about $\bar{x}$: $\bar{M}_g = \bar{0}$.

Reaction forces: forces: $\bar{N}_1 = N_{1y}\bar{E}_y + N_{1z}\bar{E}_y$, moment:

$$\bar{M}_1 = (-b\bar{E}_x - a\bar{E}_y + c\bar{E}_z)\times\bar{N}_1$$
$$= (-aN_{1z} - cN_{1y})\bar{E}_x + N_{1z}b\bar{E}_y - N_{1y}b\bar{E}_z .$$

Similarly, if we write out the position vectors of each wheel relative to the center of mass, we get

$$\bar{N}_i = N_{iy}\bar{E}_y + N_{iz}\bar{E}_z \quad \text{and}$$

$$\bar{M}_2 = (\bar{x}_2 - \bar{x})\times\bar{N}_2 = (-cN_{2y} - aN_{2z})\bar{E}_x - bN_{2z}\bar{E}_y + bN_{2y}\bar{E}_z$$
$$\bar{M}_3 = (cN_{3y} - aN_{3z})\bar{E}_x - bN_{3z}\bar{E}_y + bN_{3y}\bar{E}_z$$
$$\bar{M}_4 = (cN_{4y} - aN_{4z})\bar{E}_x + bN_{4z}\bar{E}_y - bN_{4y}\bar{E}_z .$$

Applied force: $\bar{P} = P\bar{E}_x$, moment: $\bar{M}_P = (\bar{x}_P - \bar{x})\times\bar{P}$

$$= (-b\bar{E}_x + (h-a)\bar{E}_y + 0\bar{E}_z)\times P\bar{E}_x$$
$$= (a-h)P\bar{E}_z .$$

Euler's Laws

First Law in Cartesian coordinates: $\bar{F} = \dot{\bar{G}}$

$$\begin{bmatrix} P \\ -mg + N_{1y} + N_{2y} + N_{3y} + N_{4y} \\ N_{1z} + N_{2z} + N_{3z} + N_{4z} \end{bmatrix} = \begin{bmatrix} m\ddot{x} \\ 0 \\ 0 \end{bmatrix} .$$

Second Law in Cartesian coordinates: $\bar{M} = \dot{\bar{H}}$

$$\begin{bmatrix} c(-N_{1y} - N_{2y} + N_{3y} + N_{4y}) - a(N_{1z} + N_{2z} + N_{3z} + N_{4z}) \\ b(N_{1z} - N_{2z} - N_{3z} + N_{4z}) \\ b(-N_{1y} + N_{2y} + N_{3y} - N_{4y}) + (a-h)P \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We have six scalar equations.

Analysis

We have eight unknown reaction forces and the unknown acceleration $\ddot{x}$ and only six equations. This is an indeterminate system, so we must make some assumptions. We're only interested in the $\bar{E}_y$-direction because we want to know the conditions under which the wheels lose contact. Therefore, let's ignore the equations with $\bar{E}_z$-component reaction forces. Also, the $\bar{E}_x$-component of the First Law doesn't involve reaction forces, so we ignore it as well.

This leaves the $\bar{E}_y$ scalar equation of the First Law and the $\bar{E}_z$ scalar equation of the Second Law. With four unknown reaction forces in these equations, we must make two assumptions.

Let's assume that the front wheels have the same $\bar{E}_y$ reaction and the rear wheels have the same (but different than the front, in general) $\bar{E}_y$ reaction. That is:

$$N_{1y} = N_{4y} \quad \text{and} \longleftarrow \text{ANS}$$
$$N_{2y} = N_{3y} \quad . \longleftarrow \text{ANS}$$

Now we can easily solve for the reactions:

$$N_{1y} = N_{4y} = \frac{1}{4}\left(mg - \frac{(h-a)}{b}P\right)$$
$$N_{2y} = N_{3y} = \frac{1}{4}\left(mg + \frac{(h-a)}{b}P\right) \quad .$$

The threshold for losing contact is $N_{1y}=N_{4y}<0$ or $N_{2y}=N_{3y}<0$ for the rear and front wheels, respectively. This implies, for the cart not to tip,

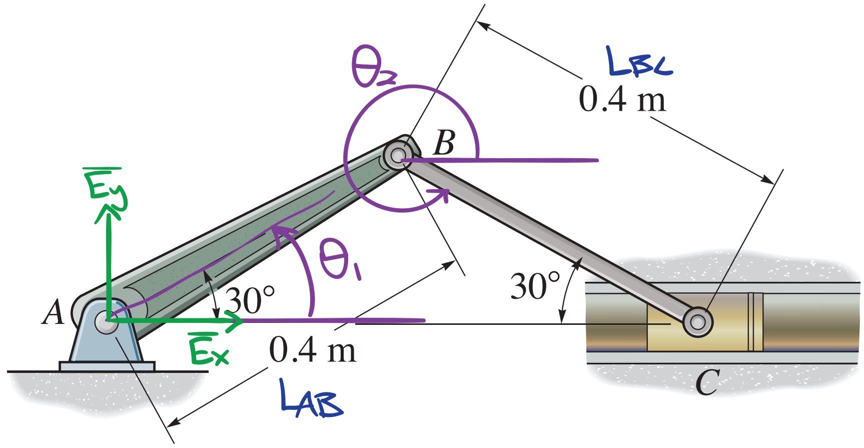
$$0 \leq \frac{1}{4}\left(mg - \frac{(h-a)}{b}P\right) \quad + \quad 0 \leq \frac{1}{4}\left(mg + \frac{(h-a)}{b}P\right)$$

$$\frac{(h-a)}{b}P \leq mg \qquad\qquad -\frac{(h-a)}{b}P \leq mg$$

$$P \leq \frac{mgb}{h-a} \qquad\qquad P \geq -\frac{mgb}{h-a}$$

$\Rightarrow$ The cart will not tip if $\boxed{-\frac{mgb}{h-a} \leq P \leq \frac{mgb}{h-a}}$. $\longleftarrow$ ANS

Hibbeler 16-111 the O'Reilly-way



Given $\bar{\omega}_{AB}$ and $\bar{\alpha}_{AB}$, find $\bar{v}_C$ and $\bar{a}_C$ as functions of θ_1 and θ_2.

Kinematics (this problem only has kinematics)

Relative position vectors can be written for each link:

$$\bar{r}_{BA} \triangleq \bar{r}_B - \bar{r}_A = \bar{r}_B = L_{AB}\cos\theta_1 \bar{E}_x + L_{AB}\sin\theta_1 \bar{E}_y$$
$$= L(\cos\theta_1 \bar{E}_x + \sin\theta_1 \bar{E}_y)$$

$(L = L_{AB} = L_{BC})$

$$\bar{r}_{CB} \triangleq \bar{r}_C - \bar{r}_B = L(\cos\theta_1 + \cos\theta_2)\bar{E}_x - L(\cos\theta_1 \bar{E}_x + \sin\theta_1 \bar{E}_y)$$
$$= L(\cos\theta_2 \bar{E}_x - \sin\theta_1 \bar{E}_y).$$

Since we want the motion of point C, we can write its position vector and take time-derivatives. Starting with the definitions of the relative position vectors,

$$\bar{r}_C = \bar{r}_B + \bar{r}_{CB} = (\bar{r}_A + \bar{r}_{BA}) + \bar{r}_{CB} = \bar{r}_{BA} + \bar{r}_{CB}$$
$$= L(\cos\theta_1 + \cos\theta_2)\bar{E}_x .$$

Now, we just take time-derivatives.

$$\bar{v}_C = L(-\dot{\theta}_1 \sin\theta_1 - \dot{\theta}_2 \sin\theta_2)\bar{E}_x = -L(\omega_{AB}\sin\theta_1 + \omega_{BC}\sin\theta_2)\bar{E}_x \quad (1)$$

where $\bar{\omega}_{AB} = \dot{\theta}_1 \bar{E}_z = \omega_{AB}\bar{E}_z$ and $\bar{\omega}_{BC} = \dot{\theta}_2 \bar{E}_z = \omega_{BC}\bar{E}_z$.

$$\bar{a}_c = -L\left(\ddot{\theta}_1 \sin\theta_1 + \dot{\theta}_1^2 \cos\theta_1 + \ddot{\theta}_2 \sin\theta_2 + \dot{\theta}_2^2 \cos\theta_2\right)$$
$$= -L\left(\alpha_{AB} \sin\theta_1 + \omega_{AB}^2 \cos\theta_1 + \alpha_{BC} \sin\theta_2 + \omega_{BC}^2 \cos\theta_2\right) \qquad (2)$$

where $\bar{\alpha}_{AB} = \ddot{\theta}_1 \bar{E}_z = \alpha_{AB} \bar{E}_z$ and $\bar{\alpha}_{BC} = \ddot{\theta}_2 \bar{E}_z = \alpha_{BC} \bar{E}_z$.

This is great, but we don't know ω_{BC} and α_{BC}. We need two more equations without more unknowns. We can find these from an analysis of the motion of point B, which lies on both bodies.

Position of B: $\bar{r}_B = L(\cos\theta_1 \bar{E}_x + \sin\theta_1 \bar{E}_y)$

Velocity of B: $\bar{v}_B = L\dot{\theta}_1(-\sin\theta_1 \bar{E}_x + \cos\theta_1 \bar{E}_y) = L\omega_{AB}(-\sin\theta_1 \bar{E}_x + \cos\theta_1 \bar{E}_y)$

Acceleration of B: $\bar{a}_B = L\left(\ddot{\theta}_1(-\sin\theta_1 \bar{E}_x + \cos\theta_1 \bar{E}_y) + \dot{\theta}_1^2(-\cos\theta_1 \bar{E}_x - \sin\theta_1 \bar{E}_y)\right)$
$= L\left((-\alpha_{AB} \sin\theta_1 - \omega_{AB}^2 \cos\theta_1)\bar{E}_x + (\alpha_{AB} \cos\theta_1 - \omega_{AB}^2 \sin\theta_1)\bar{E}_y\right)$

θ_1, ω_{AB}, and α_{AB} are given, so we know $\bar{r}_B$, $\bar{v}_B$, and $\bar{a}_B$. We can relate the motion of the point B on the body BC to the point C. This relationship will give a velocity and an acceleration eq in term of the unknowns we already had: $\bar{v}_c$, $\bar{a}_c$, ω_{BC}, α_{BC}, so we will be able to solve for all our unknowns. The relative velocity is

$$\bar{v}_c - \bar{v}_B = \bar{\omega}_{BC} \times \bar{r}_{CB} \qquad \text{(O'Reilly p. 136)}$$
$$= \omega_{BC} \bar{E}_z \times L(\cos\theta_2 \bar{E}_x - \sin\theta_1 \bar{E}_y)$$
$$= \det \begin{bmatrix} \bar{E}_x & \bar{E}_y & \bar{E}_z \\ 0 & 0 & \omega_{BC} \\ L\cos\theta_2 & -L\sin\theta_1 & 0 \end{bmatrix}$$
$$= L\omega_{BC} \sin\theta_1 \bar{E}_x + L\omega_{BC} \cos\theta_2 \bar{E}_y \implies (+\bar{v}_B \text{ to both sides})$$

$$\bar{v}_c = L\omega_{BC}(\sin\theta_1 \bar{E}_x + \cos\theta_2 \bar{E}_y) + \overbrace{L\omega_{AB}(-\sin\theta_1 \bar{E}_x + \cos\theta_1 \bar{E}_y)}^{\bar{v}_B}$$
$$= L\left((\omega_{BC} - \omega_{AB}) \sin\theta_1 \bar{E}_x + (\omega_{BC} \cos\theta_2 + \omega_{AB} \cos\theta_1)\bar{E}_y\right) \qquad (3)$$

The relative acceleration is similar:

$$\bar{a}_c - \bar{a}_B = \bar{\alpha}_{BC} \times \bar{r}_{cB} + \bar{\omega}_{BC} \times (\bar{\omega}_{BC} \times \bar{r}_{cB})$$ (O'Reilly p.137)

$$= \alpha_{BC} \bar{E}_z \times L(\cos\theta_2 \bar{E}_x - \sin\theta_1 \bar{E}_y) + \omega_{BC}\bar{E}_z \times (L\omega_{BC}\sin\theta_1 \bar{E}_x + L\omega_{BC}\cos\theta_2 \bar{E}_y)$$

$$= \alpha_{BC} L(\sin\theta_1 \bar{E}_x + \cos\theta_2 \bar{E}_y) + \omega_{BC}^2 L(-\cos\theta_2 \bar{E}_x + \sin\theta_1 \bar{E}_y)$$

$$= L((\alpha_{BC}\sin\theta_1 - \omega_{BC}^2\cos\theta_2)\bar{E}_x + (\alpha_{BC}\cos\theta_2 + \omega_{BC}^2\sin\theta_1)\bar{E}_y)$$

$$\Rightarrow \quad \bar{a}_c = L((\alpha_{BC}\sin\theta_1 - \omega_{BC}^2\cos\theta_2)\bar{E}_x + (\alpha_{BC}\cos\theta_2 + \omega_{BC}^2\sin\theta_1)\bar{E}_y) \qquad (4)$$
$$+ L((-\alpha_{AB}\sin\theta_1 - \omega_{AB}^2\cos\theta_1)\bar{E}_x + (\alpha_{AB}\cos\theta_1 - \omega_{AB}^2\sin\theta_1)\bar{E}_y)$$

The vector equations (1)–(4) yield eight scalar equations, two of which aren't independant, and six unknowns. These may be solved in any way you like. However, we show that:

(1) and (3) give:

$$\omega_{BC}\cos\theta_2 + \omega_{AB}\cos\theta_1 = 0$$

$$\Rightarrow \quad \omega_{BC} = -\frac{\cos\theta_1}{\cos\theta_2}\omega_{AB} \qquad (5)$$

(2) and (4) give:

$$\alpha_{BC}\cos\theta_2 + \omega_{BC}^2\sin\theta_1 = -\alpha_{AB}\cos\theta_1 + \omega_{AB}^2\sin\theta_1$$

$$\Rightarrow \quad \alpha_{BC} = -\alpha_{AB}\frac{\cos\theta_1}{\cos\theta_2} + (\omega_{AB}^2 - \omega_{BC}^2)\frac{\sin\theta_1}{\cos\theta_2} \qquad (6)$$

(5) and (6) can be used in (1) and (2) to give:

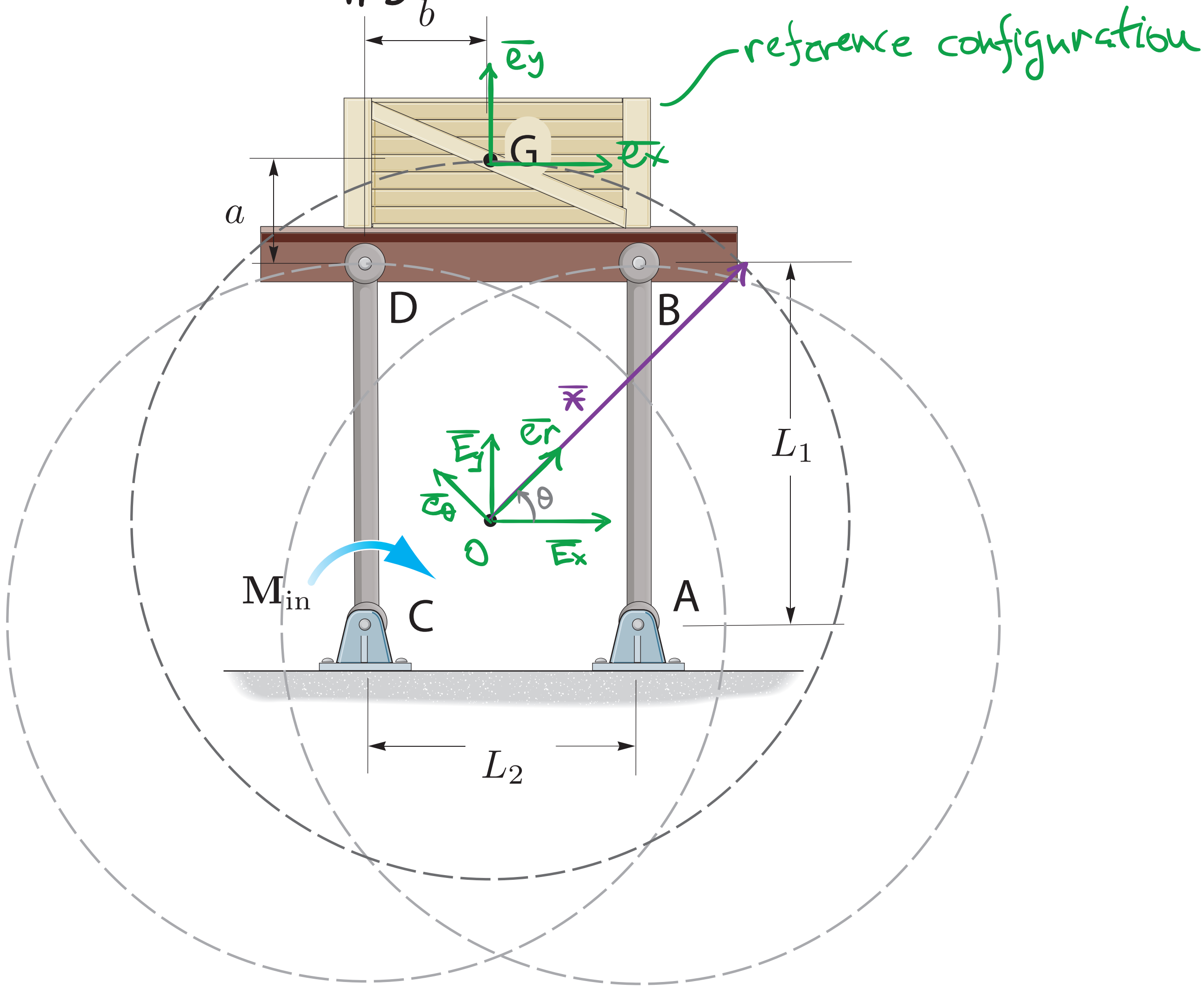
$$\bar{v}_c = -L(\omega_{AB}\sin\theta_1 + \omega_{BC}\sin\theta_2)\bar{E}_x$$
$$= -L(\omega_{AB}\sin\theta_1 - \cos\theta_1\tan\theta_2\,\omega_{AB})\bar{E}_x$$
$$= -L\omega_{AB}(\sin\theta_1 - \cos\theta_1\tan\theta_2)\bar{E}_x$$ ANS

$$\bar{a}_c = -L(\alpha_{AB}\sin\theta_1 + \omega_{AB}^2\cos\theta_1 + \alpha_{BC}\sin\theta_2 + \omega_{BC}^2\cos\theta_2)$$
$$= -L(\alpha_{AB}\sin\theta_1 + \omega_{AB}^2\cos\theta_1 + (-\alpha_{AB}\cos\theta_1 + (\omega_{AB}^2 - \omega_{BC}^2)\sin\theta_1)\tan\theta_2 + \omega_{BC}^2\cos\theta_2)$$
$$= -L(\alpha_{AB}\sin\theta_1 + \omega_{AB}^2\cos\theta_1 + (-\alpha_{AB}\cos\theta_1 + (\omega_{AB}^2 - \omega_{BC}^2)\sin\theta_1)\tan\theta_2$$
$$+ \omega_{AB}^2\cos^2\theta_1/\cos\theta_2).$$ ANS

Note that we have solved this problem in general (all $\theta_1(t)$, $\theta_2(t)$, ω_{AB}, α_{AB}). Hibbeler's solution (and question) only looks at one moment. In practice, you almost never solve it for just one moment.

Hibbeler 17-55 (but more-general), the O'Reilly way

Given an applied moment $\bar{M}_{in}$ at C, find the angular acceleration $\ddot{\theta}(\theta)$ of the links and $\bar{T}_1(\theta,\dot{\theta})$ and $\bar{T}_2(\theta,\dot{\theta})$, the pin forces at D and B. Assume massless links and no slipping of the box.



Kinematics

The motion of points G, D, and B follow the circular paths shown. We choose the origin O of our Cartesian coordinate system to be at the center of the circle that G follows. This is convenient because a polar coordinate basis is natural in this case. The corotational basis is colinear with the Cartesian basis because the orientation of the box doesn't change throughout its motion. The position vector of G is

$$\bar{x} = x\bar{E}_x + y\bar{E}_y = r\,\bar{e}_r = L_1\bar{e}_r .$$

Either differentiating or using the results of § 2.2,

$$\bar{v} = \dot{r}\,\bar{e}_r + r\dot{\theta}\,\bar{e}_\theta = L_1\dot{\theta}\,\bar{e}_\theta \qquad \text{and}$$

$$\bar{a} = -L_1\dot{\theta}^2\,\bar{e}_r + L_1\ddot{\theta}\,\bar{e}_r \quad .$$

We will also need the position vectors of D and B.

$$\bar{x}_D = \bar{x} - b\,\bar{E}_x - a\bar{E}_y = \bar{x} - b(\cos\theta\,\bar{e}_r - \sin\theta\,\bar{e}_\theta) - a(\sin\theta\,\bar{e}_r + \cos\theta\,\bar{e}_\theta) = (L_1 - b\cos\theta - a\sin\theta)\bar{e}_r + (b\sin\theta - a\cos\theta)\bar{e}_\theta$$

$$\bar{x}_B = \bar{x} + (L_2 - b)\bar{E}_x - a\,\bar{E}_y = (L_1 + (L_2 - b)\cos\theta - a\sin\theta)\,\bar{e}_r + ((b - L_2)\sin\theta - a\cos\theta)\,\bar{e}_\theta$$

In a moment, we will apply Euler's laws. In anticipation of that, let's compute the angular momentum of the box about O:

$$\bar{H}_O \triangleq \bar{H} + \bar{x}\times\bar{G}$$

Cartesian basis

$$\bar{H} \triangleq I\bar{\omega} = \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ \omega \end{bmatrix} = I_{zz}\cancelto{0}{\omega}\,\bar{E}_z = 0\bar{E}_z$$

$$\bar{G} \triangleq m\bar{v} = mL_1\dot{\theta}\,\bar{e}_\theta \quad .$$

Combining these expressions:

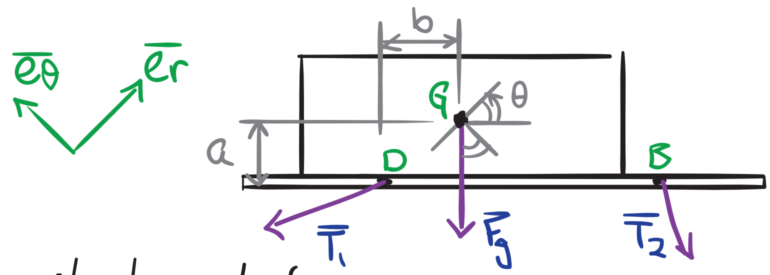
$$\begin{aligned}\bar{H}_O &= L_1\bar{e}_r \times mL_1\dot{\theta}\,\bar{e}_\theta \\ &= \det\begin{bmatrix} \bar{e}_r & \bar{e}_\theta & \bar{E}_z \\ L_1 & 0 & 0 \\ 0 & mL_1\dot{\theta} & 0 \end{bmatrix} \\ &= mL_1^2\dot{\theta}\,\bar{E}_z \quad .\end{aligned}$$

And we'll need the time-derivatives of the linear and angular momenta:

$$\dot{\bar{G}} = m\bar{a} = m(-L_1\dot{\theta}^2\,\bar{e}_r + L_1\ddot{\theta}\,\bar{e}_r)$$

$$\dot{\bar{H}}_O = mL_1^2\,\ddot{\theta}\,\bar{E}_z \quad .$$

Forces + Moments on the box



The bars exert the forces:

$$\overline{T}_1 = T_{1r}\,\overline{e}_r + T_{1\theta}\,\overline{e}_\theta \quad \&$$
$$\overline{T}_2 = T_{2r}\,\overline{e}_r + T_{2\theta}\,\overline{e}_\theta \,.$$

The gravitational force is:

$$\overline{F}_g = -mg\sin\theta\,\overline{e}_r - mg\cos\theta\,\overline{e}_\theta$$

Resultant force: $\overline{F} = \overline{F}_g + \overline{T}_1 + \overline{T}_2 = (-mg\sin\theta + T_{1r} + T_{2r})\overline{e}_r + (-mg\cos\theta + T_{1\theta} + T_{2\theta})\overline{e}_\theta$.

Moments about O:

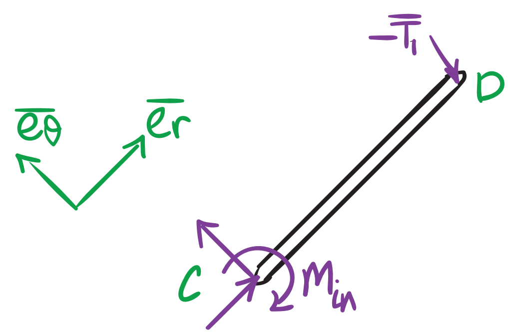
$$\overline{M}_g = \overline{x} \times \overline{F}_g = L_1\overline{e}_r \times (-mg\sin\theta\,\overline{e}_r - mg\cos\theta\,\overline{e}_\theta) = -mgL_1\cos\theta\,\overline{E}_z$$

$$\overline{M}_1 = \overline{x}_D \times \overline{T}_1 = \big((aT_{1r} - bT_{2r})\cos\theta - (bT_{1r} + aT_{2r})\sin\theta\big)E_z$$

$$\overline{M}_2 = \overline{x}_B \times \overline{T}_2 = \big(L_1T_{2\theta} + (aT_{1\theta} + (L_2 - b)T_{2\theta})\cos\theta - (bT_{1\theta} - L_2T_{1\theta} + aT_{2\theta})\sin\theta\big)\overline{E}_z .$$

Resultant moment: $\overline{M}_O = \overline{M}_g + \overline{M}_1 + \overline{M}_2$. $\overline{M}_{in}$ doesn't enter because it's applied to the link CD, not the box.

Forces + Moments on Link CD



The resultant moment about C is:

$$\overline{M}_c = \overline{M}_{in} + L_1\overline{e}_r \times (-\overline{T}_1) = \overline{M}_{in} - L_1 T_{1\theta}\,\overline{E}_z .$$

Introducing the resultant force doesn't help us because we don't know or care about the reaction at C.

Euler's Laws on the links

The link CD is massless, so its angular momentum is zero and

$$\overline{M}_c = \overline{0} \Rightarrow \overline{M}_{in} - L_1T_{1\theta}\overline{E}_z = \overline{0} \Rightarrow T_{1\theta} = \frac{1}{L_1}\overline{M}_{in}\cdot\overline{E}_z \,.$$

Similarly: $T_{2\theta} = 0$.

Euler's Laws on the box

First Law: $\bar{F} = \dot{\bar{G}} = m\bar{a}$ which, written in the polar coord's, is

$$\begin{bmatrix} -mg\sin\theta + T_{1r} + T_{2r} \\ -mg\cos\theta + T_{1\theta} + T_{2\theta} \end{bmatrix} = \begin{bmatrix} m\ -L_1\dot{\theta}^2 \\ mL_1\ddot{\theta} \end{bmatrix}$$

Second Law: $\bar{M}_O = \dot{\bar{H}}_O \Rightarrow \bar{M}_g + \bar{M}_1 + \bar{M}_2 = mL_1^2\ddot{\theta}\bar{E}_z$

Analysis

The second scalar equation from the First Law gives:

$$\begin{aligned} \ddot{\theta} &= \frac{1}{mL_2}\left(-mg\cos\theta + T_{1\theta} + \cancelto{0}{T_{2\theta}}\right) \\ &= \frac{1}{mL_1}\left(-mg\cos\theta + \frac{1}{L_1}M_{in}\right) \\ &= \frac{1}{mL_1^2}\left(-mgL_1\cos\theta + M_{in}\right) \quad \longleftarrow \text{ANS} \end{aligned}$$

We know $T_{1\theta} = \frac{1}{L_1}\bar{M}_{in}\cdot\bar{E}_z$ and $T_{2\theta} = 0$ and $\bar{\alpha}(\theta)$. We still want $T_{1r}(\theta,\dot{\theta})$ and $T_{2r}(\theta,\dot{\theta})$, so we need two equations with T_{1r}, T_{2r}, θ, + $\dot{\theta}$ the only unknowns.

The first scalar equation of the First Law is one. The Second Law gives only one (nontrivial) scalar equation. They are linear and easy to solve:

$\bar{T}_1(\theta,\dot{\theta}) = T_{1r}\bar{e}_r + T_{1\theta}\bar{e}_\theta$ where $T_{1r}(\theta,\dot{\theta}) =$ $\longleftarrow$ ANS

$$\frac{L_1(I_{zz}+mL_1^2)\alpha + (b-L_2)M_{in}\sin\theta + agL_1m\sin^2\theta + (mgL_1^2 - aM_{in} + bL_1mg\sin\theta)\cos\theta - mL_1^2(b\cos\theta + a\sin\theta)\dot{\theta}^2}{(a+b)L_1\cos\theta + (a-b)L_1\sin\theta}$$

$\bar{T}_2(\theta,\dot{\theta}) = T_{2r}\bar{e}_r + T_{2\theta}\bar{e}_\theta$ where $T_{2r}(\theta,\dot{\theta}) =$ $\longleftarrow$ ANS

$$\frac{-L_1(I_{zz}+mL_1^2)\alpha - (b-L_2)M_{in}\sin\theta - bgL_1m\sin^2\theta + (-mgL_1^2 - aM_{in} + aL_1mg\sin\theta)\cos\theta + mL_1^2(-a\cos\theta + b\sin\theta)\dot{\theta}^2}{(a+b)L_1\cos\theta + (a-b)L_1\sin\theta}$$

Note that there was a more-convenient point about which to apply Euler's Laws: G. Because the box has no rotation $\bar{\alpha} = \bar{0}$, and $\bar{M} = 0$. This simplifies the solution for $\bar{T}_1(\theta,\dot{\theta})$ and $\bar{T}_2(\theta,\dot{\theta})$, mostly because the position vectors in the moment equations are easier.