Chapter 3: Particles + Space Curves

Topics:
- Differential geometry of space curves
- Serret-Frenet basis vectors $\{\bar{e}_t, \bar{e}_n, \bar{e}_b\}$
- Rate-of-change of $\{\bar{e}_t, \bar{e}_n, \bar{e}_b\}$
- Examples of space curves
- Application to mechanics

3.1 Space Curves

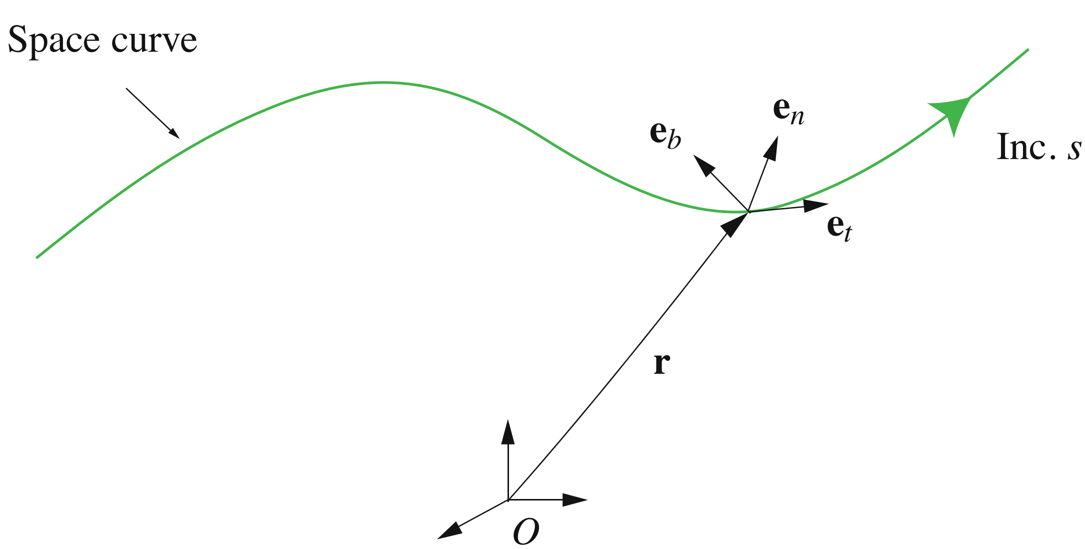
A space curve is a curved path in space. Rectilinear and circular paths are special types of space curves.

3.1.1 The Arc-Length Parameter

Position vector:

Arc-length s :

So we can parameterize the space curve $\mathcal{C}$ by s:



We define the Serret-Frenet basis vectors $\{\bar{e}_t, \bar{e}_n, \bar{e}_b\}$ for a point $P \in C$.

To define $\bar{e}_n$, let's consider that (from):

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We want $\bar{e}_n$ to be perpendicular to $\bar{e}_t$ and we want it to have unit length (i.e. $\|\bar{e}_n\|=1$). From the fact that $\|\bar{e}_t\|=1$, we have:

Therefore, $d\bar{e}_t/ds$ is perpendicular to $\bar{e}_t$! We use this fact to define $\bar{e}_n$ as follows:

where $K=\hat{K}(s)$ is the curvature of C at some point P.
We also define the radius of curvature:

When $d\bar{e}_t/ds=\bar{0}$ for some s, $K=0$, $\rho\to\infty$, and $\bar{e}_n$ is not uniquely defined. The final unit vector is defined as:

A vector in $\bar{b} \in \mathbb{E}^3$ can be written:

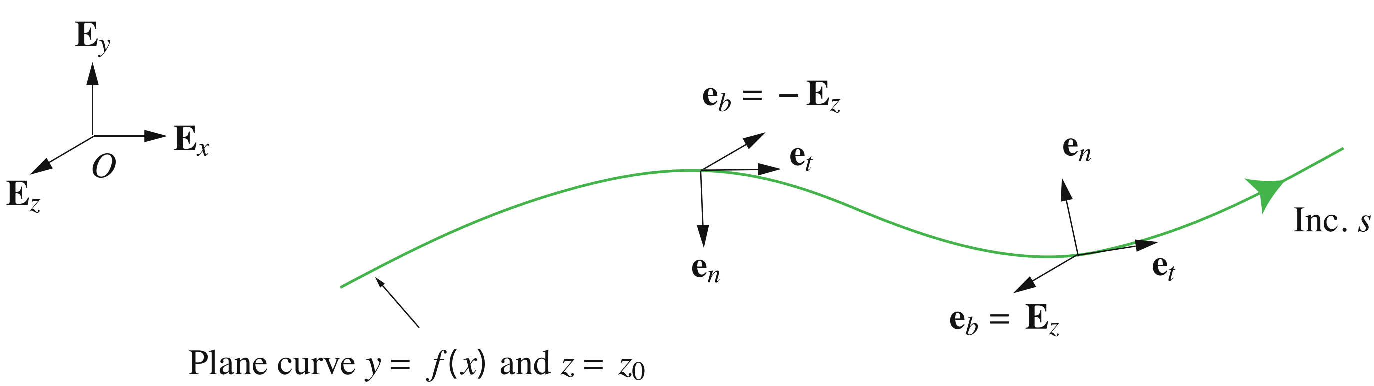
3.2 The Serret-Frenet formulae

These formulas describe the rate-of-change of $\{\bar{e}_t, \bar{e}_n, \bar{e}_b\}$ in terms of $\{\bar{e}_t, \bar{e}_n, \bar{e}_b\}$. The first one we saw above:

The others are derived in O'Reilly.

3.3 Examples of Space Curves

3.3.1 A Curve on a Plane



Position:

Arclength (assuming s + x increase in the same direction):

Serret-Frenet basis vectors:

$$\bar{e}_t = \bar{e}_t(x) = \frac{d\bar{r}}{ds} = \frac{d\bar{r}}{dx}\frac{dx}{ds} = \frac{1}{\sqrt{1+\left(\frac{df}{dx}\right)^2}}\left(\bar{E}_x + \frac{df}{dx}\bar{E}_y\right)$$

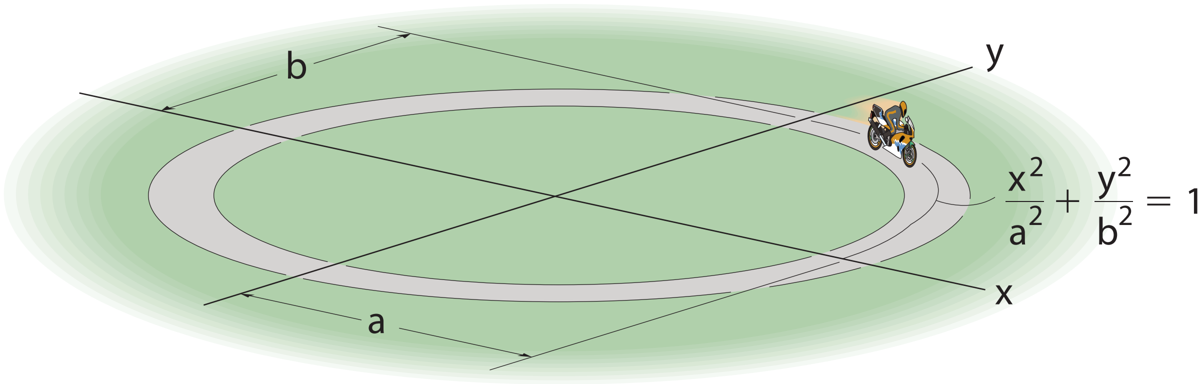
$$\bar{e}_n = \bar{e}_n(x) = \frac{\operatorname{sgn}(d^2f/dx^2)}{\sqrt{1+\left(\frac{df}{dx}\right)^2}}\left(\bar{E}_y - \frac{df}{dx}\bar{E}_x\right)$$

$$\bar{e}_b = \operatorname{sgn}(d^2f/dx^2)\,\bar{E}_z$$

Curvature and torsion:

Here K can be interpreted as a rate of rotation of $\bar{e}_t$ and $\bar{e}_n$ about $\bar{e}_b = \pm\bar{E}_z$.

Example (Hibbeler 12-158)



The motorcycle travels along the elliptical track at a constant speed v. Determine the greatest magnitude of the acceleration if $a > b$.

We have a curve on a plane: $\bar{r} = x\bar{E}_x + y\bar{E}_y$.

The acceleration is $\bar{a} = 0\,\bar{e}_t + a_n\bar{e}_n = a_n\bar{e}_n$.

More specifically, $\bar{a} = \ddot{\bar{r}} = \frac{d\dot{\bar{r}}}{dt} = \frac{d}{dt}\left(\frac{d\bar{r}}{ds}\frac{ds}{dt}\right) = \frac{d}{dt}\left(v\,\bar{e}_t\right)$

$$= \cancelto{0}{\frac{dv}{dt}}\bar{e}_t + v\frac{d\bar{e}_t}{dt} = v\frac{d\bar{e}_t}{ds}\frac{ds}{dt} = \kappa v^2\bar{e}_n$$

v is constant, so maximizing the curvature κ is tantamount to maximizing $\|\bar{a}\|$. Since the curve is symmetric about the x-axis, we can take one half of it for analysis: $\bar{r} = x\bar{E}_x + f(x)\bar{E}_y$. The results of O'Reilly, Section 3.3.1 apply:

$$\kappa = \kappa(x) = \frac{\left|\frac{d^2f}{dx^2}\right|}{\left(1 + \left(\frac{df}{dx}\right)^2\right)^{3/2}}.$$

Let's compute d^2f/dx^2 separately:

$$f(x) = b\left(1 - \frac{x^2}{a^2}\right)^{1/2}$$

$$\frac{df}{dx} = \frac{b}{a^2} x \left(1 - \frac{x^2}{a^2}\right)^{-1/2}$$

$$\frac{d^2f}{dx^2} = \frac{b}{a^2}\left(1 - \frac{x^2}{a^2}\right)^{-1/2} + \frac{b}{a^2} x \left(1 - \frac{x^2}{a^2}\right)^{-3/2}\left(-\frac{1}{2}\right)\left(-\frac{2x}{a^2}\right)$$

$$= \frac{b}{a^2}\left(1 - \frac{x^2}{a^2}\right)^{-1/2} + \frac{b}{a^4} x^2 \left(1 - \frac{x^2}{a^2}\right)^{-3/2}$$

$$K = \frac{ab}{\left(1 + \frac{b^2}{a^2}\frac{x^2}{a^2 - x^2}\right)^{3/2} |a^2 - x^2|^{3/2}} = \frac{ab}{\left(a^2 - x^2 + \frac{b^2}{a^2}x^2\right)^{3/2}}$$

$$K_{max} = \lim_{x \to a} K = \frac{ab}{\left(a^2 - a^2 + \frac{b^2}{a^2}a^2\right)^{3/2}} = \frac{ab}{b^3} = \frac{a}{b^2}$$

Therefore,

$$\bar{a}_{max} = K_{max} v^2 \bar{e}_n = \frac{a v^2}{b^2} \bar{e}_n$$

and

$$\|\bar{a}_{max}\| = \frac{a v^2}{b^2} .$$

Check units: $\frac{L}{T^2} \stackrel{?}{=} \frac{L(L/T)^2}{L^2}$ OK